Hindawi Publishing Corporation
International Journal of Mathematics and Mathematical Sciences
Volume 2007, Article ID 86909, 15 pages
doi:10.1155/2007/86909

Research Article

Best Possible Sufficient Conditions for Strong Law of Large Numbers for Multi-Indexed Orthogonal Random Elements

Kuo-Liang Su

Received 26 April 2006; Revised 18 December 2006; Accepted 5 February 2007

Recommended by Andrew Rosalsky

It will be shown and induced that the d-dimensional indices in the Banach spaces version conditions $\sum_{\mathbf{n}}(E\|X_{\mathbf{n}}\|^p/|\mathbf{n}^\alpha|^p) < \infty$ are sufficient to yield $\lim_{\min_{1\le j\le d}(n_j)\to\infty}(1/|\mathbf{n}^\alpha|)\sum_{\mathbf{k}\le\mathbf{n}}\prod_{j=1}^{d}(1-(k_j-1)/n_j)X_k = 0$ a.s. for arrays of James-type orthogonal random elements. Particularly, it will be shown also that there are the best possible sufficient conditions for multi-indexed independent real-valued random variables.

1. Introduction

The laws of large numbers for orthogonal random variables or Banach space-valued random elements are investigated by several authors. A consequence of Rademacher-Menshov theorem [2, 3] showed that the sufficient condition for a strong law of large numbers of a sequence of orthogonal real-valued random variables with 0 means and finite second moments is $\sum_{k=1}^{\infty}(\sigma_k^2/k^2)\cdot[\log_2(k+1)]^2 < \infty$. Warren and Howell [4] proposed the sufficient condition $\sum_{k=1}^{\infty}(E\|X_k\|^{1+\alpha}/k^{1+\alpha})\cdot\log^{1+\alpha}k < \infty$, $0 < \infty \le 1$, for strong convergence of the one-dimensional B-valued James-type orthogonal random variables. Móricz [5] showed that $\sum_{i=1}^{\infty}\sum_{k=1}^{\infty}(\sigma_{ik}^2/i^2k^2)\cdot[\log_2(i+1)]^2[\log_2(k+1)]^2 < \infty$ is the necessary condition for the strong convergence for arrays of quasi-orthogonal real-valued random variables. Móricz [6] obtained a sufficient condition for strong limit theorems for arrays of quasi-orthogonal real-valued random variables. Móricz et al. [7] showed that the sufficient condition for the strong convergence of $(1/m^\alpha n^\beta)\sum_{i=1}^{m}\sum_{k=1}^{n}X_{ik}$ for arrays of orthogonal, type p Banach space-valued random elements is

$$\sum_{i=1}^{\infty}\sum_{k=1}^{\infty}\left(\frac{E\|X_{ik}\|^p}{i^{\alpha p}k^{\beta p}}\right)\cdot\left[\log_2(i+1)\right]^p\left[\log_2(k+1)\right]^p < \infty. \tag{1.1}$$

In this paper, the strong laws of large numbers will be investigated for James type of orthogonality in a Banach space. In order to induce d-dimensional case, $d > 2$, we wish to investigate the strongly convergent behavior of a more general Cesaro-type means, $(1/m^{\alpha}n^{\beta})\sum_{i=1}^{m}\sum_{k=1}^{n}(1-(i-1)/m)(1-(k-1)/n)X_{ik}$, as $m,n \to \infty$, for arrays of two-dimensionally indexed orthogonal random elements in a Banach space of type p, $1 \leq p \leq 2$, and $1/2 < \alpha, \beta \leq 1$, though Su [1] showed a case of $\alpha = 1 = \beta$. In particular, it will be proven that the sufficient conditions are also the best possible even for independent real-valued random variables. The definition for an array of orthogonal random elements and the formulation of previous results and auxiliary lemmas for orthogonality are given in Section 2. The major results and their proofs are in Sections 3 and 4, respectively.

2. Preliminaries and auxiliary lemmas

The basic definitions and properties of Banach space-valued random variables (or random elements) are well established in the literature (e.g., [8]). In these preliminaries, we only introduce the concepts which are necessary and not easy to read in the literature.

Our sense of orthogonality throughout this manuscript is that of *James type orthogonality*. For elements x and y in a Banach space B, x is said to be *James orthogonal* to y (denoted by $x \perp_J y$) if $\|x\| \leq \|x+ty\|$ for all $t \in \mathcal{R}$. If B is a Hilbert space, then James type orthogonality agrees with the usual notion of orthogonality where the inner product is 0 since $\|x+ty\|^2 = (x+ty, x+ty) = \|x\|^2 + t^2\|y\|^2 + 2t(x,y) \geq \|x\|^2$ for all $t \in \mathcal{R}$ if and only if $(x,y) = 0$. However, in a Banach space where the norm is not generated by an inner product, it is possible for $x \perp_J y$ but $y \not\perp_J x$ with $(x,y) \neq 0$. For instance, let $\mathcal{R}^2 = \{(x_1,x_2) : \|(x_1,x_2)\| = |x_1| + |x_2|,\ x_1,x_2 \in \mathcal{R}\}$ and $x = (2,0)$ and $y = (2,-2)$. Then, it is clear that the usual inner product $(x,y) = 4 \neq 0$. Next, $x \perp_J y$ but $y \not\perp_J x$ since $\|x+ty\| = |2+2t| + |-2t| \geq 2 = \|x\|$ and $\|y+tx\| = |2+2t| + |-2| = 3 < \|y\| = 4$ while picking $t = -1/2$. Therefore, it is not possible to create a notation of orthogonality with a good geometrical meaning in an arbitrary Banach space without the inner product. As a result, James-type orthogonality is adopted to circumvent this shortcoming [7].

Let $\{X_{ik},\ i,k \geq 1\}$ be a double sequence of random elements in the Banach space $L^p(B)$ with zero means, that is, $E(X_{ik}) = 0$ for all i, k and finite pth moments, $E\|X_{ik}\|^p < \infty$ for all i, k, where $\|\cdot\|$ is the norm of the separable Banach space B. The following is the extended definition for arrays of orthogonal random variables in Banach spaces.

Definition 2.1. An array of random elements $\{X_{ik}\}$ is orthogonal in $L^p(B)$, $1 \leq p < \infty$, if

$$\text{(i)}\quad E\|X_{ik}\|^p < \infty \quad \forall i,k,$$

$$\text{(ii)}\quad E\left\|\sum_{i=1}^{n_1}\sum_{k=1}^{n_2} a_{\pi_1(i),\pi_2(k)}X_{\pi_1(i),\pi_2(k)}\right\|^p \leq E\left\|\sum_{i=1}^{n_1+m_1}\sum_{k=1}^{n_2+m_2} a_{\pi_1(i),\pi_2(k)}X_{\pi_1(i),\pi_2(k)}\right\|^p, \tag{2.1}$$

for all arrays $\{a_{ik}\} \subseteq \mathcal{R}$, for all n_1, n_2, m_1, and m_2, and for all permutations π_1, π_2 of the positive integers $\{1,2,\ldots,m_1+n_1\}$ and $\{1,2,\ldots,m_2+n_2\}$, respectively.

In retrospect, a separable Banach space B is of type p $(1 \leq p \leq 2)$ if and only if there is a constant $C > 0$ (depending on B only) such that $E\|\sum_{i=1}^{n} X_i\|^p \leq C\sum_{i=1}^{n} E\|X_i\|^p$ when $\{X_i\}$ are independent random elements with zero means and finite pth moments [8]. In order to obtain the desired results, a useful version of moment inequality for arrays that extend the results of Howell and Taylor [9] is listed below.

Proposition 2.2. *The following conditions are equivalent:*

(i) *B is a Banach space of type p, $1 \le p \le 2$,*

(ii) *for each array $\{X_{mn}\}$ of orthogonal random elements in $L^p(B)$, there exists a constant C (depending on $\{X_{mn}\}$ and B) such that, for all m and n,*

$$E\left\|\sum_{i=1}^{m}\sum_{k=1}^{n}X_{ik}\right\|^p \le C\sum_{i=1}^{m}\sum_{k=1}^{n}E\|X_{ik}\|^p. \tag{2.2}$$

The next lemma is from Móricz et al. [7].

Lemma 2.3. *If $\{X_{ik};\ i \ge 1, k \ge 1\}$ is an array of orthogonal (in $L^p(B)$) random elements in a Banach space B of type p for some $1 \le p \le 2$, then*

$$E\left[\left(\max_{1\le j\le m}\left\|\sum_{i=a+1}^{a+j}\sum_{k=b+1}^{b+n}X_{ik}\right\|\right)^p\right] \le C_1(\log_2 2m)^p\sum_{i=a+1}^{a+m}\sum_{k=b+1}^{b+n}E\|X_{ik}\|^p \tag{2.3}$$

for some $C_1 > 0$ and

$$E\left[\left(\max_{1\le j\le m}\max_{1\le t\le n}\left\|\sum_{i=a+1}^{a+j}\sum_{k=b+1}^{b+t}X_{ik}\right\|\right)^p\right] \le C_2(\log_2 2m)^p(\log_2 2n)^p\sum_{i=a+1}^{a+m}\sum_{k=b+1}^{b+n}E\|X_{ik}\|^p \tag{2.4}$$

for some $C_2 > 0$.

The following lemma can be derived directly from Proposition 2.2 and Lemma 2.3. Hence, the proofs are omitted.

Lemma 2.4. *Let $\{X_{ik}\}$ be an array of orthogonal (in $L^p(B)$) random elements in a Banach space B of type p for some $1 \le p \le 2$ and let $\{a_{ik}\}$ be any array of real numbers, then there exists positive constants C_3 and C_4 such that, for all m and n, $(a,b \ge 0;\ m,n \ge 1)$*

$$\text{(i)}\quad E\left\|\sum_{i=a+1}^{a+m}\sum_{k=b+1}^{b+n}a_{ik}X_{ik}\right\|^p \le C_3\sum_{i=a+1}^{a+m}\sum_{k=b+1}^{b+n}|a_{ik}|^pE\|X_{ik}\|^p,$$

$$\text{(ii)}\quad E\left[\left(\max_{1\le t\le n}\left\|\sum_{i=a+1}^{a+m}\sum_{k=b+1}^{b+t}a_{ik}X_{ik}\right\|\right)^p\right] \le C_4(\log_2 2n)^p\sum_{i=a+1}^{a+m}\sum_{k=b+1}^{b+n}|a_{ik}|^pE\|X_{ik}\|^p. \tag{2.5}$$

We also need two more crucial lemmas as follows; they are extended from Kronecker's lemma and Shiryayev [10], respectively, and will be proven in Section 4.

Lemma 2.5 (two-dimensionally indexed version of Kronecker's lemma). *Let $\{a_m\}$ and $\{b_n\}$ be sequences of positive increasing numbers, both $a_m \uparrow \infty$ and $b_n \uparrow \infty$ when $m \to \infty$ and $n \to \infty$, respectively. Let $\{x_{ij};\ i,j \ge 1\}$ be an array of positive numbers such that*

$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}(x_{ij}/a_i b_j)$ *converges. Then,*

$$\text{(i)}\quad \frac{1}{a_m}\sum_{i=1}^{m}\sum_{j=1}^{n}\frac{x_{ij}}{b_j}\longrightarrow 0 \quad \text{as } m\longrightarrow\infty,$$
$$\frac{1}{b_n}\sum_{i=1}^{m}\sum_{j=1}^{n}\frac{x_{ij}}{a_i}\longrightarrow 0 \quad \text{as } n\longrightarrow\infty, \tag{2.6}$$
$$\text{(ii)}\quad \frac{1}{a_m b_n}\sum_{i=1}^{m}\sum_{j=1}^{n}x_{ij}\longrightarrow 0 \quad \text{as } m\longrightarrow\infty \text{ or } n\longrightarrow\infty.$$

Lemma 2.6. *A sufficient and necessary condition that $\zeta_{mn}\to 0$ with probability one as $m,n\to\infty$ is that for any $\varepsilon>0$,*

$$P\Big(\sup_{i\geq m}\sup_{k\geq n}\|\zeta_{ik}\|>\varepsilon\Big)\longrightarrow 0 \quad \text{as } m\longrightarrow\infty,\ n\longrightarrow\infty. \tag{2.7}$$

3. Major results

Theorems 3.1 and 3.2 are two-dimensionally indexed versions of strong convergence for Cesaro-type means for arrays of Banach space-valued random elements and hence their proofs are more complicated than that in real cases because of the structure of spaces.

Theorem 3.1. *Let $\{X_{ik}\}$ be an array of orthogonal (in $L^p(B)$) random elements with zero means in a Banach space B of type p, for some $1\leq p\leq 2$. If $\sum_{i=1}^{\infty}\sum_{k=1}^{\infty}(E\|X_{ik}\|^p/i^{\alpha p}k^{\beta p})<\infty$, $1/2<\alpha$, $\beta\leq 1$. Then,*

$$\lim_{m,n\to\infty}\left\|\frac{1}{m^\alpha n^\beta}\sum_{i=1}^{m}\sum_{k=1}^{n}\Big(1-\frac{i-1}{m}\Big)\Big(1-\frac{k-1}{n}\Big)X_{ik}\right\|=0 \quad a.s. \tag{3.1}$$

Theorem 3.2. *Let $\{X_{ik}\}$ be an array of orthogonal (in $L^p(B)$) random elements with zero means in a Banach space B of type p, for some $1\leq p\leq 2$. If $\sum_{i=1}^{\infty}\sum_{k=1}^{\infty}(E\|X_{ik}\|^p/i^{\alpha p}k^{\beta p})\cdot[\log_2(k+1)]^p<\infty$, $1/2<\alpha$, $\beta\leq 1$. Then,*

$$\lim_{m,n\to\infty}\left\|\frac{1}{m^\alpha n^\beta}\sum_{i=1}^{m}\sum_{k=1}^{n}\Big(1-\frac{i-1}{m}\Big)X_{ik}\right\|=0 \quad a.s. \tag{3.2}$$

The generalization to d-dimensional arrays random elements of the previous two theorems can be obtained easily by the same methods [1]. Theorems 3.3 and 3.4 are to show that the sufficient conditions in the previous theorems are the best possible conditions for independent real-valued random variables, since the real line is of type p, $1\leq p\leq 2$.

Theorem 3.3. *If $\{\tau_{ik}\}$ is an array of nonnegative real numbers such that $\tau_{ik}^p/i^{\alpha(p-1)}k^{\beta(p-1)}\leq 1$ holds for indices i and k and the condition $\sum_{i=1}^{\infty}\sum_{k=1}^{\infty}(\tau_{ik}^p/i^{\alpha p}k^{\beta p})=\infty$, for some $1\leq p\leq 2$ and $1/2<\alpha$, $\beta\leq 1$, then there exists an array $\{X_{ik}\}$ of independent real-valued random* *variables such that* $E(X_{ik}) = 0$, $E|X_{ik}|^p = \tau_{ik}^p$, *and*

$$\limsup_{m+n\to\infty} \left| \frac{1}{m^\alpha n^\beta} \sum_{i=1}^{m} \sum_{k=1}^{n} \left(1 - \frac{i-1}{m}\right)\left(1 - \frac{k-1}{n}\right) X_{ik} \right| = \infty. \tag{3.3}$$

Theorem 3.4. *If* $\{\tau_{ik}\}$ *is an array of nonnegative real numbers such that* $\tau_{ik}^p \cdot \log_2^{p-1}(k+1)/i^{\alpha(p-1)}k^{\beta(p-1)} \le 1$ *holds for indices* i *and* k *and the condition* $\sum_{i=1}^{\infty}\sum_{k=1}^{\infty}(\tau_{ik}^p/i^{\alpha p}k^{\beta p}) \cdot [\log_2(k+1)]^p = \infty$, *for some* $1 \le p \le 2$ *and* $1/2 < \alpha, \beta \le 1$, *then there exists an array* $\{X_{ik}\}$ *of independent real-valued random variables such that* $E(X_{ik}) = 0$, $E|X_{ik}|^p = \tau_{ik}^p$, *and*

$$\limsup_{m+n\to\infty} \left| \frac{1}{m^\alpha n^\beta} \sum_{i=1}^{m} \sum_{k=1}^{n} \left(1 - \frac{i-1}{m}\right) X_{ik} \right| = \infty. \tag{3.4}$$

4. Proofs

Here we will verify Lemmas 2.5 and 2.6 first, then prove the case of $d = 2$, that is, Theorems 3.1, 3.2, 3.3, and 3.4. We may apply the analogous approaches for the d-dimensional cases, $d > 2$.

Proof of Lemma 2.5. (i) First, we have $\sum_{i=1}^{\infty}(x_{ij}/a_ib_j) < \infty$ for each $j \ge 1$ and $\sum_{j=1}^{\infty}(x_{ij}/a_ib_j) < \infty$ for each $i \ge 1$. Then by the one-dimensional version Kronecker's lemma, we can conclude that $(1/a_m)\sum_{i=1}^{m}(x_{ij}/b_j) \to 0$ as $m \to \infty$, for every j, and $(1/b_n)\sum_{j=1}^{n}(x_{ij}/a_i) \to 0$ as $n \to \infty$, for every i. Hence, for any $\varepsilon_1 > 0$, choose $n > N$ such that $(1/b_n)\sum_{j=1}^{n}(x_{ij}/a_i) \le \varepsilon_1/2^i$ for all i. Then, we have

$$\frac{1}{b_n}\sum_{i=1}^{m}\sum_{j=1}^{n}\frac{x_{ij}}{a_i} = \sum_{i=1}^{m}\frac{1}{b_n}\sum_{j=1}^{n}\frac{x_{ij}}{a_i} \le \sum_{i=1}^{m}\frac{\varepsilon_1}{2^i} < \varepsilon_1. \tag{4.1}$$

Similarly, for any $\varepsilon_2 > 0$, choose $m > M$ such that $(1/a_m)\sum_{i=1}^{m}(x_{ij}/b_j) \le \varepsilon_2/2^j$ for all j, then

$$\frac{1}{a_m}\sum_{i=1}^{m}\sum_{j=1}^{n}\frac{x_{ij}}{b_j} = \sum_{j=1}^{n}\frac{1}{a_m}\sum_{i=1}^{m}\frac{x_{ij}}{b_j} \le \sum_{j=1}^{n}\frac{\varepsilon_2}{2^i} < \varepsilon_2. \tag{4.2}$$

(ii) Apparently, for any $\varepsilon > 0$, when $m > M$ (say), we can conclude that

$$\frac{1}{a_m b_n}\sum_{i=1}^{m}\sum_{j=1}^{n}x_{ij} \le \frac{1}{a_m}\sum_{i=1}^{m}\sum_{j=1}^{n}\frac{x_{ij}}{b_j} < \varepsilon. \tag{4.3}$$

□

Proof of Lemma 2.6. Fix any $m \geq 1$, for any $\varepsilon > 0$, let $A^{\varepsilon}_{mn} = \{\omega : \sup_{i\geq m} \|\zeta_{in}\| > \varepsilon\}$ and $A^{\varepsilon}_{m} = \overline{\lim}_n A^{\varepsilon}_{mn} = \bigcap_{n=1}^{\infty} \bigcup_{k\geq n} A^{\varepsilon}_{mk}$. Then, $\{\omega : \sup_{i\geq m} \|\zeta_{in}\| \nrightarrow 0$, as $n \to \infty\} = \bigcup_{\varepsilon \geq 0} A^{\varepsilon}_{m} = \bigcup_{t=1}^{\infty} A^{1/t}_{m}$. However, $P(A^{\varepsilon}_{m}) = \lim_n P(\bigcup_{k\geq n} A^{\varepsilon}_{mk})$. Hence, for any fixed $m \geq 1$,

$$\begin{aligned}
0 = P\Big(\omega : \sup_{i\geq m} ||\zeta_{in}|| \nrightarrow 0 \text{ as } n \longrightarrow \infty\Big) &= P\Big(\bigcup_{\varepsilon\geq 0} A^{\varepsilon}_{m}\Big) \\
\Longleftrightarrow P\Big(\bigcup_{t=1}^{\infty} A^{1/t}_{m}\Big) &= 0 \quad \text{any fixed } m \geq 1 \\
\Longleftrightarrow P(A^{1/t}_{m}) &= 0, \quad t \geq 1, \text{ any fixed } m \geq 1 \\
\Longleftrightarrow P(A^{\varepsilon}_{m}) &= 0 \quad \text{for any } \varepsilon > 0, \text{ any fixed } m \geq 1 \\
\Longleftrightarrow P\Big(\bigcup_{k\geq n} A^{\varepsilon}_{mk}\Big) &\longrightarrow 0 \quad \text{as } n \longrightarrow \infty, \text{ any fixed } m \geq 1 \\
\Longleftrightarrow P\Big(\sup_{k\geq n} \sup_{i\geq m} ||\zeta_{ik}|| > \varepsilon\Big) &\longrightarrow 0 \quad \text{as } n \longrightarrow \infty, \text{ any fixed } m \geq 1 \\
\Longleftrightarrow P\Big(\bigcap_{m=1}^{\infty} \Big\{\sup_{k\geq n} \sup_{i\geq m} ||\zeta_{ik}|| > \varepsilon\Big\}\Big) &\longrightarrow 0 \quad \text{as } n \longrightarrow \infty \\
\Longleftrightarrow P\Big(\sup_{k\geq n} \sup_{i\geq m} ||\zeta_{ik}|| > \varepsilon\Big) &\longrightarrow 0 \quad \text{as } m \longrightarrow \infty,\ n \longrightarrow \infty.
\end{aligned} \tag{4.4}$$

□

Proof of Theorem 3.1. We need some useful arguments in the proof in [1, 7], and Lemmas 2.3 and 2.4. For positive integers u and v, for any ε,

$$P\Big[\sup_{m>2^u, n>2^v} ||\xi_{mn}|| > \varepsilon\Big] \leq \sum_{r=u}^{\infty} \sum_{s=v}^{\infty} P\Big[\max_{2^r<m\leq 2^{r+1}} \max_{2^s<n\leq 2^{s+1}} ||\xi_{mn}|| > \varepsilon\Big], \tag{4.5}$$

where

$$\xi_{mn} = \frac{1}{m^{\alpha} n^{\beta}} \sum_{i=1}^{m} \sum_{k=1}^{n} \Big(1 - \frac{i-1}{m}\Big)\Big(1 - \frac{k-1}{n}\Big) X_{ik}. \tag{4.6}$$

We have

$$\max_{2^r<m\leq 2^{r+1}} \max_{2^s<n\leq 2^{s+1}} ||\xi_{mn}|| \leq ||\xi_{2^r,2^s}|| + \sum_{j=1}^{3} A^{(j)}_{rs}, \tag{4.7}$$

where

$$\begin{aligned}
A^{(1)}_{rs} &= \max_{2^r<m\leq 2^{r+1}} ||\xi_{m,2^s} - \xi_{2^r,2^s}||, \\
A^{(2)}_{rs} &= \max_{2^s<n\leq 2^{s+1}} ||\xi_{2^r,n} - \xi_{2^r,2^s}||, \\
A^{(3)}_{rs} &= \max_{2^r<m\leq 2^{r+1}} \max_{2^s<n\leq 2^{s+1}} ||\xi_{mn} - \xi_{m,2^s} - \xi_{2^r,n} + \xi_{2^r,2^s}||.
\end{aligned} \tag{4.8}$$

Therefore,

$$P\left[\max_{2^r\le m\le 2^{r+1}}\max_{2^s\le n\le 2^{s+1}}||\xi_{mn}||>\varepsilon\right]\le P\left[||\xi_{2^r,2^s}||>\frac{\varepsilon}{4}\right]+\sum_{j=1}^{3}P\left[A_{rs}^{(j)}>\frac{\varepsilon}{4}\right]. \tag{4.9}$$

First, by Markov's inequality and Lemmas 2.3 and 2.4,

$$\begin{aligned}
&\sum_{r=u}^{\infty}\sum_{s=v}^{\infty}P\left[||\xi_{2^r,2^s}||>\frac{\varepsilon}{4}\right]\\
&\quad\le\left(\frac{4}{\varepsilon}\right)^p\Gamma_1\sum_{r=u}^{\infty}\sum_{s=v}^{\infty}\frac{1}{2^{\alpha rp}2^{\beta sp}}\sum_{i=1}^{2^r}\sum_{k=1}^{2^s}\left|1-\frac{i-1}{2^r}\right|^p\left|1-\frac{k-1}{2^s}\right|^pE||X_{ik}||^p\\
&\quad\le\left(\frac{4}{\varepsilon}\right)^p\Gamma_1\sum_{r=u}^{\infty}\sum_{s=v}^{\infty}\frac{1}{2^{\alpha rp}2^{\beta sp}}\left\{\sum_{i=1}^{2^u}\sum_{k=1}^{2^v}+\sum_{i=2^u+1}^{2^r}\sum_{k=1}^{2^v}+\sum_{i=1}^{2^u}\sum_{k=2^v+1}^{2^s}+\sum_{i=2^u+1}^{2^r}\sum_{k=2^v+1}^{2^s}\right\}E||X_{ik}||^p\\
&\quad=\left(\frac{4}{\varepsilon}\right)^p\Gamma_1\sum_{j=1}^{4}B_{uv}^{(j)},\quad\text{say, for some }\Gamma_1>0.
\end{aligned} \tag{4.10}$$

Using some basic calculation, it follows that

$$B_{uv}^{(1)}=\frac{2^{(\alpha+\beta)p}}{(2^{\alpha p}-1)(2^{\beta p}-1)}\cdot\frac{1}{2^{\alpha pu}2^{\beta pv}}\sum_{i=1}^{2^u}\sum_{k=1}^{2^v}E||X_{ik}||^p. \tag{4.11}$$

Next,

$$B_{uv}^{(2)}\le\frac{2^{(\alpha+\beta)p}}{(2^{\alpha p}-1)(2^{\beta p}-1)}\left\{\sum_{i=2^u+1}^{\infty}\sum_{k=1}^{2^v}\frac{1}{2^{\beta pv}}\frac{E||X_{ik}||^p}{i^{\alpha p}}+\sum_{i=2^u+1}^{\infty}\sum_{k=2^v+1}^{\infty}\frac{E||X_{ik}||^p}{i^{\alpha p}k^{\beta p}}\right\}. \tag{4.12}$$

Similarly,

$$\begin{aligned}
B_{uv}^{(3)}&\le\frac{2^{(\alpha+\beta)p}}{(2^{\alpha p}-1)(2^{\beta p}-1)}\left\{\sum_{i=1}^{2^u}\sum_{k=2^v+1}^{\infty}\frac{1}{2^{\alpha pu}}\frac{E||X_{ik}||^p}{k^{\beta p}}+\sum_{i=2^u+1}^{\infty}\sum_{k=2^v+1}^{\infty}\frac{E||X_{ik}||^p}{i^{\alpha p}k^{\beta p}}\right\},\\
B_{uv}^{(4)}&=\sum_{r=u}^{\infty}\sum_{s=v}^{\infty}\frac{1}{2^{\alpha rp}2^{\beta sp}}\sum_{i=2^u+1}^{2^r}\sum_{k=2^v+1}^{2^s}E||X_{ik}||^p\le\sum_{i=2^u+1}^{\infty}\sum_{k=2^v+1}^{\infty}\frac{E||X_{ik}||^p}{i^{\alpha p}k^{\beta p}}.
\end{aligned} \tag{4.13}$$

Secondly, since

$$A_{rs}^{(1)}=\max_{2^r<m\le 2^{r+1}}||\xi_{m,2^s}-\xi_{2^r,2^s}||=\max_{1\le m\le 2^r}\left\|\sum_{t=2^r+1}^{2^r+m}(\xi_{t,2^s}-\xi_{t-1,2^s})\right\|, \tag{4.14}$$

where

$$\xi_{t,2^s} - \xi_{t-1,2^s} = \sum_{i=1}^{t} \sum_{k=1}^{2^s} a_{ik}(t,s) X_{ik},$$
$$a_{ik}(t,s) = \frac{1}{2^{\beta s}} \left(1 - \frac{k-1}{2^s}\right) \left[(i-1)\left(\frac{1}{(t-1)^{\alpha+1}} - \frac{1}{t^{\alpha+1}}\right) - \left(\frac{1}{(t-1)^{\alpha}} - \frac{1}{t^{\alpha}}\right)\right], \tag{4.15}$$

we have

$$|a_{ik}(t,s)| \le \frac{1}{t^{\alpha+1} 2^{\beta s}}. \tag{4.16}$$

Hence, for some $\Gamma_2, \Gamma_2^* > 0$,

$$\begin{aligned}
&\sum_{r=u}^{\infty} \sum_{s=v}^{\infty} P\left[A_{rs}^{(1)} > \frac{\varepsilon}{4}\right] \\
&\quad \le \sum_{r=u}^{\infty} \sum_{s=v}^{\infty} P\left[\max_{1 \le m \le 2^r} \left\| \sum_{t=2^r+1}^{2^r+m} (\xi_{t,2^s} - \xi_{t-1,2^s}) \right\| > \frac{\varepsilon}{4}\right] \\
&\quad \le \left(\frac{4}{\varepsilon}\right)^p \sum_{r=u}^{\infty} \sum_{s=v}^{\infty} E\left[\left(\max_{1 \le m \le 2^r} \left\| \sum_{t=2^r+1}^{2^r+m} (\xi_{t,2^s} - \xi_{t-1,2^s}) \right\|\right)^p\right] \\
&\quad \le \left(\frac{4}{\varepsilon}\right)^p \sum_{r=u}^{\infty} \sum_{s=v}^{\infty} \Gamma_2 [\log_2 2 \cdot 2^r]^p \sum_{m=2^r+1}^{2^{r+1}} E\|\xi_{m,2^s} - \xi_{m-1,2^s}\|^p \\
&\quad \le \left(\frac{4}{\varepsilon}\right)^p \Gamma_2^* \sum_{r=u}^{\infty} \sum_{s=v}^{\infty} \frac{(r+1)^p}{2^{s\beta p}} \sum_{m=2^r+1}^{2^{r+1}} \sum_{i=1}^{m} \sum_{k=1}^{2^s} \frac{E\|X_{ik}\|^p}{m^{(\alpha+1)p}} \\
&\quad \approx \left(\frac{4}{\varepsilon}\right)^p \Gamma_2^* \sum_{r=u}^{\infty} \sum_{s=v}^{\infty} \sum_{m=2^r+1}^{2^{r+1}} \sum_{i=1}^{m} \sum_{k=1}^{2^s} \frac{r^p E\|X_{ik}\|^p}{m^{(\alpha+1)p} 2^{\beta sp}} \\
&\quad \le \left(\frac{4}{\varepsilon}\right)^p \Gamma_2^* \sum_{r=u}^{\infty} \sum_{s=v}^{\infty} \sum_{k=1}^{2^s} \frac{r^p}{2^{\beta sp}} \left[\sum_{i=1}^{2^r} \sum_{m=2^r+1}^{2^{r+1}} + \sum_{i=2^r+1}^{2^{r+1}} \sum_{m=i}^{2^{r+1}}\right] \frac{E\|X_{ik}\|^p}{m^{(\alpha+1)p}} \\
&\quad \le \left(\frac{4}{\varepsilon}\right)^p \Gamma_2^* \sum_{r=u}^{\infty} \sum_{s=v}^{\infty} \sum_{k=1}^{2^s} \frac{r^p}{2^{\beta sp}} \left[\sum_{i=1}^{2^r} \frac{1}{2^{r[(\alpha+1)p-1]}} + \sum_{i=2^r+1}^{2^{r+1}} \frac{1}{i^{(\alpha+1)p-1}}\right] E\|X_{ik}\|^p \\
&\quad \le \left(\frac{4}{\varepsilon}\right)^p \Gamma_2^* \sum_{s=v}^{\infty} \sum_{k=1}^{2^s} \frac{1}{2^{\beta sp}} \left\{\sum_{r=u}^{\infty} \sum_{i=1}^{2^r} \frac{r^p}{2^{r[(\alpha+1)p-1]}} + \sum_{r=u}^{\infty} \sum_{i=2^r+1}^{2^{r+1}} \frac{r^p}{i^{(\alpha+1)p-1}}\right\} E\|X_{ik}\|^p.
\end{aligned} \tag{4.17}$$

Since the first term and the second term in the bracket of (4.17) can be expressed, respectively, as

$$\sum_{r=u}^{\infty}\sum_{i=1}^{2^r}\frac{r^p}{2^{r[(\alpha+1)p-1]}} \le \sum_{i=1}^{\infty}\sum_{u<r;\, i<2^r}\frac{r^2}{2^{r[(\alpha+1)p-1]}} \le 2^{(\alpha+1)p}\left(\sum_{i=1}^{2^u}\frac{1}{i^{(\alpha+1)p-1}}+\sum_{i=2^u+1}^{\infty}\frac{1}{i^{(\alpha+1)p-1}}\right),$$
$$\sum_{r=u}^{\infty}\sum_{i=2^r+1}^{2^{r+1}}\frac{r^p}{i^{(\alpha+1)p-1}} \le \sum_{i=2^u+1}^{\infty}\sum_{u<r;\, i<2^{r+1}}\frac{r^2}{2^{r[(\alpha+1)p-1]}} \le 2^{(\alpha+1)p}\sum_{i=2^u+1}^{\infty}\frac{1}{i^{(\alpha+1)p-1}}, \tag{4.18}$$

hence,

$$\begin{aligned}(4.17) &\le 2^{(\alpha+1)p}\left(\frac{4}{\varepsilon}\right)^p\Gamma_2^*\sum_{s=v}^{\infty}\left[\sum_{k=1}^{2^v}+\sum_{k=2^v+1}^{2^s}\right]\left[\sum_{i=1}^{2^u}+2\sum_{i=2^u+1}^{\infty}\right]\frac{E||X_{ik}||^p}{i^{\alpha p}2^{\beta sp}}\\
&\le 2^{\alpha p}\left(\frac{8}{\varepsilon}\right)^p\Gamma_2^*\left\{\frac{1}{2^{\beta vp}}\sum_{k=1}^{2^v}\sum_{i=1}^{2^u}\frac{E||X_{ik}||^p}{i^{\alpha p}}+\frac{1}{2^{\beta vp}}\sum_{k=1}^{2^v}\sum_{i=2^u+1}^{\infty}\frac{E||X_{ik}||^p}{i^{\alpha p}}\right.\\
&\qquad\left.+\sum_{k=2^v+1}^{\infty}\sum_{i=1}^{2^u}\frac{E||X_{ik}||^p}{i^{\alpha p}k^{\beta p}}+2\sum_{k=2^v+1}^{\infty}\sum_{i=2^u+1}^{\infty}\frac{E||X_{ik}||^p}{i^{\alpha p}k^{\beta p}}\right\}.\end{aligned} \tag{4.19}$$

Similarly, for some $\Gamma_3 > 0$,

$$\begin{aligned}\sum_{r=u}^{\infty}\sum_{s=v}^{\infty}P\left[A_{rs}^{(2)}>\frac{\varepsilon}{4}\right] &\le 2^{\beta p}\left(\frac{8}{\varepsilon}\right)^p\Gamma_3\left\{\frac{1}{2^{\alpha up}}\sum_{k=1}^{2^v}\sum_{i=1}^{2^u}\frac{E||X_{ik}||^p}{k^{\beta p}}+\frac{1}{2^{\alpha up}}\sum_{i=1}^{2^u}\sum_{k=2^v+1}^{\infty}\frac{E||X_{ik}||^p}{k^{\beta p}}\right.\\
&\qquad\left.+\sum_{i=2^u+1}^{\infty}\sum_{k=1}^{2^v}\frac{E||X_{ik}||^p}{i^{\alpha p}k^{\beta p}}+2\sum_{i=2^u+1}^{\infty}\sum_{k=2^v+1}^{\infty}\frac{E||X_{ik}||^p}{i^{\alpha p}k^{\beta p}}\right\}.\end{aligned} \tag{4.20}$$

Next, since

$$\begin{aligned}&\max_{2^r<m\le 2^{r+1}}\max_{2^s<n\le 2^{s+1}}||\xi_{mn}-\xi_{m,2^s}-\xi_{2^r,n}+\xi_{2^r,2^s}||\\
&\quad\le\max_{1\le m\le 2^r}\max_{1\le n\le 2^s}\left\|\sum_{t_1=2^r+1}^{2^r+m}\sum_{t_2=2^s+1}^{2^s+n}\left(\xi_{t_1,t_2}-\xi_{(t_1-1),t_2}-\xi_{t_1,(t_2-1)}+\xi_{(t_1-1),(t_2-1)}\right)\right\|,\\
&\quad\xi_{t_1,t_2}-\xi_{(t_1-1),t_2}-\xi_{t_1,(t_2-1)}+\xi_{(t_1-1),(t_2-1)}=\sum_{i=1}^{t_1}\sum_{k=1}^{t_2}b_{ik}(t_1,t_2)X_{ik},\end{aligned} \tag{4.21}$$

where

$$\begin{aligned}b_{ik}(t_1,t_2)&=\left[\frac{1}{t_1^{\alpha}}\left(1-\frac{i-1}{t_1}\right)-\frac{1}{(t_1-1)^{\alpha}}\left(1-\frac{i-1}{t_1-1}\right)\right]\\
&\quad\cdot\left[\frac{1}{t_2^{\beta}}\left(1-\frac{k-1}{t_2}\right)-\frac{1}{(t_2-1)^{\beta}}\left(1-\frac{k-1}{t_2-1}\right)\right],\end{aligned} \tag{4.22}$$

so apparently,

$$|b_{ik}(t_1,t_2)| \le \frac{(i-1)(k-1)}{t_1^{\alpha+1}t_2^{\beta+1}} \cdot \frac{1}{(t_1-1)(t_2-1)} \le \frac{1}{t_1^{\alpha+1}t_2^{\beta+1}}. \tag{4.23}$$

Hence, for some $\Gamma_4 > 0$,

$$\begin{aligned}
&\sum_{r=u}^{\infty}\sum_{s=v}^{\infty} P\left[A_{rs}^{(3)} > \frac{\varepsilon}{4}\right] \\
&\le \sum_{r=u}^{\infty}\sum_{s=v}^{\infty} P\left[\max_{1\le m\le 2^r}\max_{1\le n\le 2^s}\left\|\sum_{t_1=2^r+1}^{2^r+m}\sum_{t_2=2^s+1}^{2^s+n}\left(\xi_{t_1,t_2}-\xi_{(t_1-1),t_2}-\xi_{t_1,(t_2-1)}+\xi_{(t_1-1),(t_2-1)}\right)\right\| > \frac{\varepsilon}{4}\right] \\
&\le \left(\frac{4}{\varepsilon}\right)^p \Gamma_4 \sum_{r=u}^{\infty}\sum_{s=v}^{\infty}(\log_2 2\cdot 2^r)^p(\log_2 2\cdot 2^s)^p \sum_{m=2^r+1}^{2^{r+1}}\sum_{n=2^s+1}^{2^{s+1}}\sum_{i=1}^{m}\sum_{k=1}^{n}\frac{E\|X_{ik}\|^p}{m^{(\alpha+1)p}n^{(\beta+1)p}} \\
&\le \left(\frac{4}{\varepsilon}\right)^p \Gamma_4 \sum_{r=u}^{\infty}\sum_{s=v}^{\infty}(r+1)^p(s+1)^p \sum_{m=2^r+1}^{2^{r+1}}\sum_{n=2^s+1}^{2^{s+1}}\sum_{i=1}^{m}\sum_{k=1}^{n}\frac{E\|X_{ik}\|^p}{m^{(\alpha+1)p}n^{(\beta+1)p}} \\
&\approx \left(\frac{4}{\varepsilon}\right)^p \Gamma_4 \sum_{r=u}^{\infty}\sum_{s=v}^{\infty} r^p s^p\left(\sum_{i=1}^{2^r}\sum_{m=2^r+1}^{2^{r+1}}+\sum_{i=2^r+1}^{2^{r+1}}\sum_{m=i}^{2^{r+1}}\right)\left(\sum_{k=1}^{2^s}\sum_{n=2^s+1}^{2^{s+1}}+\sum_{k=2^s+1}^{2^{s+1}}\sum_{n=i}^{2^{s+1}}\right)\frac{E\|X_{ik}\|^p}{m^{(\alpha+1)p}n^{(\beta+1)p}} \\
&\le 2^{(\alpha+\beta)p}\left(\frac{16}{\varepsilon}\right)^p\Gamma_4\left\{\frac{1}{2^{\alpha up}2^{\beta vp}}\sum_{i=1}^{2^u}\sum_{k=1}^{2^v}E\|X_{ik}\|^p + \frac{2}{2^{\alpha up}}\sum_{i=1}^{2^u}\sum_{k=2^v+1}^{\infty}\frac{E\|X_{ik}\|^p}{k^{\beta p}}\right. \\
&\qquad\left. +\frac{2}{2^{\beta vp}}\sum_{i=2^u+1}^{\infty}\sum_{k=1}^{2^v}\frac{E\|X_{ik}\|^p}{i^{\alpha p}} + 4\sum_{i=2^u+1}^{\infty}\sum_{k=2^v+1}^{\infty}\frac{E\|X_{ik}\|^p}{i^{\alpha p}k^{\beta p}}\right\}.
\end{aligned} \tag{4.24}$$

Combining the results in (4.9), (4.10), (4.19), (4.20), and (4.24), we can conclude that $P[\lim_{m,n\to\infty}\|\xi_{mn}\| = 0] = 1$. Since $P[\sup_{m\ge s,n\ge t}\|\xi_{mn}\| > \varepsilon] \to 0$ as $s,t\to\infty$ by applying Lemmas 2.5 and 2.6 on the right-hand side of the previous inequalities, then the proof is completed by a convention that $\sum_{i=1}^{2^r}\sum_{k=1}^{2^r} = 0 = \sum_{i=1}^{\infty}\sum_{k=1}^{2^r} = \sum_{i=1}^{2^r}\sum_{k=1}^{\infty}$ and $\sum_{i=2^r}^{\infty}\sum_{k=2^r}^{\infty} = \sum_{i=1}^{\infty}\sum_{k=1}^{\infty}$ if $r=-1$. □

Proof of Theorem 3.2. Similar to the previous proof, we start out from assuming that $2^r \le m \le 2^{r+1}$ and $2^s \le n \le 2^{s+1}$ with nonnegative integers u and v, and letting

$$\tau_{mn} = \frac{1}{m^\alpha n^\beta}\sum_{i=1}^{m}\sum_{k=1}^{n}\left(1-\frac{i-1}{m}\right)X_{ik}. \tag{4.25}$$

First, we can write

$$\sum_{r=u}^{\infty}\sum_{s=v}^{\infty}P\left[\max_{2^r\le m\le 2^{r+1}}\max_{2^s\le n\le 2^{s+1}}||\tau_{mn}||>\varepsilon\right]\le\sum_{r=u}^{\infty}\sum_{s=v}^{\infty}P\left[||\tau_{2^r,2^s}||>\frac{\varepsilon}{4}\right]+\sum_{j=1}^{3}D_{rs}^{(j)}, \tag{4.26}$$

where

$$\begin{aligned}
D_{rs}^{(1)}&=\sum_{r=u}^{\infty}\sum_{s=v}^{\infty}P\left[\max_{2^r<m\le 2^{r+1}}||\tau_{m,2^s}-\tau_{2^r,2^s}||>\frac{\varepsilon}{4}\right],\\
D_{rs}^{(2)}&=\sum_{r=u}^{\infty}\sum_{s=v}^{\infty}P\left[\max_{2^s<n\le 2^{s+1}}||\tau_{2^r,n}-\tau_{2^r,2^s}||>\frac{\varepsilon}{4}\right],\\
D_{rs}^{(3)}&=\sum_{r=u}^{\infty}\sum_{s=v}^{\infty}P\left[\max_{2^r<m\le 2^{r+1}}\max_{2^s<n\le 2^{s+1}}||\tau_{mn}-\tau_{m,2^s}-\tau_{2^r,n}+\tau_{2^r,2^s}||>\frac{\varepsilon}{4}\right].
\end{aligned} \tag{4.27}$$

Comparing (4.9) to (4.24), we can easily obtain that for some $\Gamma_5>0$,

$$\sum_{r=u}^{\infty}\sum_{s=v}^{\infty}P\left[||\tau_{2^r,2^s}||>\frac{\varepsilon}{4}\right]\le\left(\frac{4}{\varepsilon}\right)^p\Gamma_5\sum_{i=2^u+1}^{\infty}\sum_{k=2^v+1}^{\infty}\frac{E||X_{ik}||^p}{i^{\alpha p}k^{\beta p}} \tag{4.28}$$

and for some $\Gamma_6,\Gamma_6^*>0$,

$$\begin{aligned}
D_{rs}^{(1)}&\le\left(\frac{4}{\varepsilon}\right)^p\Gamma_6\sum_{r=u}^{\infty}\sum_{s=v}^{\infty}[\log_2 2\cdot 2^r]^p\sum_{m=2^r+1}^{2^{r+1}}\sum_{i=1}^{m}\sum_{k=1}^{2^s}\frac{E||X_{ik}||^p}{m^{(\alpha+1)p}2^{\beta sp}}\\
&\le 2^{\alpha p}\left(\frac{8}{\varepsilon}\right)^p\Gamma_6^*\left\{\frac{1}{2^{\beta vp}}\sum_{i=1}^{2^u}\sum_{k=1}^{2^v}\frac{E||X_{ik}||^p}{i^{\alpha p}}+\frac{1}{2^{\beta vp}}\sum_{i=2^u+1}^{\infty}\sum_{k=1}^{2^v}\frac{E||X_{ik}||^p}{i^{\alpha p}}\right.\\
&\qquad\left.+\sum_{i=1}^{2^u}\sum_{k=2^v+1}^{\infty}\frac{E||X_{ik}||^p}{i^{\alpha p}k^{\beta p}}+2\sum_{i=2^u+1}^{\infty}\sum_{k=2^v+1}^{\infty}\frac{E||X_{ik}||^p}{i^{\alpha p}k^{\beta p}}\right\}.
\end{aligned} \tag{4.29}$$

Next, we need to examine that

$$\begin{aligned}
\max_{2^s<n\le 2^{s+1}}||\tau_{2^r,n}-\tau_{2^r,2^s}||&=\max_{2^s<n\le 2^{s+1}}\left\|\sum_{t=2^s+1}^{2^s+n}(\tau_{2^r,t}-\tau_{2^r,t-1})\right\|,\\
\tau_{2^r,t}-\tau_{2^r,t-1}&=\sum_{i=1}^{2^r}\sum_{k=1}^{t}c_{ik}(r,t)X_{ik},
\end{aligned} \tag{4.30}$$

where

$$c_{ik}(r,t)=\begin{cases}\dfrac{1}{2^{\alpha r}}\left(1-\dfrac{i-1}{2^r}\right)\left[\dfrac{1}{t^\beta}-\dfrac{1}{(t-1)^\beta}\right], & k=1,2,\dots,t-1,\\[2ex] \dfrac{1}{2^{\alpha r}}\left(1-\dfrac{i-1}{2^r}\right)\cdot\dfrac{1}{t^\beta}, & k=t.\end{cases} \tag{4.31}$$

Hence, we use basic calculus to obtain

$$|c_{ik}(r,t)| \leq \frac{\beta}{2^{\alpha r} t^{\beta}(t-1)^{\beta}}, \quad k=1,\ldots,t-1, \qquad |c_{ik}(r,t)| \leq \frac{1}{2^{\alpha r} t^{\beta}}, \quad k=t. \tag{4.32}$$

Following the arguments in the proof of Theorem 3.1, we can get that for some $\Gamma_7, \Gamma_7^* > 0$,

$$\begin{aligned}
D_{rs}^{(2)} &\leq \sum_{r=u}^{\infty}\sum_{s=v}^{\infty} P\left[\max_{1\leq n\leq 2^s}\left\|\sum_{t=2^s+1}^{2^s+n}\sum_{i=1}^{2^r}\sum_{k=1}^{t} c_{ik}(r,t)X_{ik}\right\| > \frac{\varepsilon}{4}\right] \\
&\leq \left(\frac{4}{\varepsilon}\right)^p \Gamma_7 \sum_{r=u}^{\infty}\sum_{s=v}^{\infty} \frac{[\log_2 2^{s+1}]^p}{2^{\alpha r p}} \sum_{i=1}^{2^r}\sum_{n=2^s+1}^{2^{s+1}}\sum_{k=1}^{n} |c_{ik}(r,t)|^p E\|X_{ik}\|^p \\
&\approx \left(\frac{4}{\varepsilon}\right)^p \Gamma_7 \sum_{r=u}^{\infty}\sum_{s=v}^{\infty} \frac{[s+1]^p}{2^{\alpha r p}} \sum_{i=1}^{2^r}\sum_{n=2^s+1}^{2^{s+1}} \left\{\sum_{k=1}^{n-1} \frac{E\|X_{ik}\|^p}{n^{2\beta p}} + \frac{E\|X_{in}\|^p}{n^{\beta p}}\right\} \\
&\leq 2^{\beta p}\left(\frac{8}{\varepsilon}\right)^p \Gamma_7^* \left\{\frac{1}{2^{\alpha u p}}\sum_{i=1}^{2^u}\sum_{k=1}^{2^v}\frac{E\|X_{ik}\|^p}{k^{\beta p}} + \frac{1}{2^{\alpha u p}}\sum_{i=1}^{2^u}\sum_{k=2^v+1}^{\infty}\frac{E\|X_{ik}\|^p}{k^{\beta p}}\right. \\
&\qquad + \sum_{i=2^u+1}^{\infty}\sum_{k=1}^{2^v}\frac{E\|X_{ik}\|^p}{i^{\alpha p}k^{\beta p}} + \sum_{i=2^u+1}^{\infty}\sum_{k=2^v+1}^{\infty}\frac{E\|X_{ik}\|^p}{i^{\alpha p}k^{\beta p}} \\
&\qquad \left. + \sum_{i=2^u+1}^{\infty}\sum_{k=2^v+1}^{\infty}\frac{E\|X_{ik}\|^p}{i^{\alpha p}k^{\beta p}}\left[1+\log_2^p(k+1)\right]\right\}.
\end{aligned} \tag{4.33}$$

The second term in (4.33) is obtained by the following basic calculation. Next, similar to the procedure of getting (4.24), we have

$$\begin{aligned}
&\max_{2^r<m\leq 2^{r+1}}\max_{2^s<n\leq 2^{s+1}} \|\tau_{mn} - \tau_{m,2^s} - \tau_{2^r,n} + \tau_{2^r,2^s}\| \\
&\quad \leq \max_{1\leq m\leq 2^r}\max_{1\leq n\leq 2^s}\left\|\sum_{t_1=2^r+1}^{2^r+m}\sum_{t_2=2^s+1}^{2^s+n}\left(\tau_{t_1,t_2} - \tau_{t_1-1,t_2} - \tau_{t_1,t_2-1} + \tau_{t_1-1,t_2-1}\right)\right\|,
\end{aligned} \tag{4.34}$$

where

$$\tau_{t_1,t_2} - \tau_{t_1-1,t_2} - \tau_{t_1,t_2-1} + \tau_{t_1-1,t_2-1} = \sum_{i=1}^{t_1}\sum_{k=1}^{t_2} d_{ik}(t_1,t_2)X_{ik},$$

$$d_{ik}(t_1,t_2) = \begin{cases} \left[\dfrac{1}{t_1^{\alpha}}\left(1-\dfrac{i-1}{t_1}\right) - \dfrac{1}{(t_1-1)^{\alpha}}\left(1-\dfrac{i-1}{t_1-1}\right)\right]\left[\dfrac{1}{t_2^{\beta}} - \dfrac{1}{(t_2-1)^{\beta}}\right], & k\leq n-1, \\ \left[\dfrac{1}{t_1^{\alpha}}\left(1-\dfrac{i-1}{t_1}\right) - \dfrac{1}{(t_1-1)^{\alpha}}\left(1-\dfrac{i-1}{t_1-1}\right)\right]\dfrac{1}{t_2^{\beta}}, & k=n. \end{cases} \tag{4.35}$$

Similarly

$$|d_{ik}(t_1,t_2)| \le \frac{\beta}{t_1^{\alpha+1}t_2^{2\beta}}, \quad k \le n, \qquad |d_{ik}(t_1,t_2)| \le \frac{1}{t_1^{\alpha+1}t_2^{\beta}}, \quad k = n. \tag{4.36}$$

Then, by the analogous approaches of the proof of Theorem 3.1, for some $H > 0$, we have

$$\begin{aligned} D_{rs}^{(3)} \le \left(\frac{4}{\varepsilon}\right)^p H \cdot \Bigg\{ &(\text{lower order terms}) + \sum_{i=2^u+1}^{\infty} \sum_{k=2^v+1}^{\infty} \frac{E\|X_{ik}\|^p}{i^{\alpha p}k^{\beta p}} \\ &+ \sum_{i=2^u+1}^{\infty} \sum_{k=2^v+1}^{\infty} \frac{E\|X_{ik}\|^p}{i^{\alpha p}k^{\beta p}} \cdot \left[1+\log_2^p(k+1)\right] \Bigg\}. \end{aligned} \tag{4.37}$$

Consequently, following the analogous arguments in the previous proof, we can conclude the desired result. □

Proof of Theorem 3.3. By $\sum_{i=1}^{\infty}\sum_{k=1}^{\infty}(\tau_{ik}^p/i^{\alpha p}k^{\beta p}) = \infty$, there exists an array $\{\varepsilon_{ik}\}$ of nonincreasing positive numbers converging to 0 as $\max\{i,k\} \to \infty$ such that $\sum_{i=1}^{\infty}\sum_{k=1}^{\infty}(\tau_{ik}^p\varepsilon_{ik}^p/i^{\alpha p}k^{\beta p}) = \infty$. Next, define an array of independent random variables $\{W_{ik}\}$ with the following properties [8]:

$$P\left[W_{ik} = \frac{i^\alpha k^\beta}{\varepsilon_{ik}}\right] = \frac{\tau_{ik}^p\varepsilon_{ik}^p}{i^{\alpha p}k^{\beta p}}, \qquad P[W_{ik} = 0] = 1 - \frac{\tau_{ik}^p\varepsilon_{ik}^p}{i^{\alpha p}k^{\beta p}}. \tag{4.38}$$

Then, it is easy to have that

$$E|W_{ik}|^p = \tau_{ik}^p, \qquad 0 \le E(W_{ik}) = \frac{\tau_{ik}^p\varepsilon_{ik}^{p-1}}{i^{\alpha(p-1)}k^{\beta(p-1)}} \le \varepsilon_{ik}^{p-1}. \tag{4.39}$$

Now let $X_{ik} = \tau_{ik} \cdot ((W_{ik} - E(W_{ik}))/\delta_{ik}) = X_{ik}^+ - X_{ik}^-$, where $\delta_{ik}^p = E|W_{ik} - E(W_{ik})|^p$, $X_{ik}^+ = \tau_{ik}W_{ik}/\delta_{ik}$, and $X_{ik}^- = \tau_{ik}E(W_{ik})/\delta_{ik}$. Since W_{ik} and $E(W_{ik})$ are all positive, and by the dominated convergence theorem for δ_{ik}, we have

$$\frac{\tau_{ik}}{\delta_{ik}} \ge 1, \qquad \lim_{i+k\to\infty}\frac{\tau_{ik}}{\delta_{ik}} = 1. \tag{4.40}$$

Consequently, we can obtain that $X_{ik}^+ \geq \max\{0, W_{ik}\}$, $i, k = 1, 2, 3, \ldots$, and $|X_{ik}^-| \leq 2\varepsilon_{ik}^{p-1}$, when $i + k$ is sufficiently large. Then,

$$\begin{aligned}
&\left|\frac{1}{m^\alpha n^\beta}\sum_{i=1}^{m}\sum_{k=1}^{n}\left(1-\frac{i-1}{m}\right)\left(1-\frac{k-1}{n}\right)X_{ik}\right| \\
&\quad\geq \left|\left|\frac{1}{m^\alpha n^\beta}\sum_{i=1}^{m}\sum_{k=1}^{n}\left(1-\frac{i-1}{m}\right)\left(1-\frac{k-1}{n}\right)X_{ik}^+\right|\right. \\
&\qquad\left.-\left|\frac{1}{m^\alpha n^\beta}\sum_{i=1}^{m}\sum_{k=1}^{n}\left(1-\frac{i-1}{m}\right)\left(1-\frac{k-1}{n}\right)X_{ik}^-\right|\right| \\
&\quad\geq \frac{1}{m^\alpha n^\beta}\sum_{i=1}^{m}\sum_{k=1}^{n}\left(1-\frac{i-1}{m}\right)\left(1-\frac{k-1}{n}\right)X_{ik}^+ - \frac{2\varepsilon_{11}^{p-1}}{m^\alpha n^\beta}\left(\frac{m+1}{2}\right)\left(\frac{n+1}{2}\right) \quad (X_{ik}^+ \geq 0) \\
&\quad\geq \frac{C}{m^\alpha n^\beta}\left\{\sum_{i=[m/2]}^{m}\left(1-\frac{i-1}{m}\right)\sum_{k=[n/2]}^{n}\left(1-\frac{k-1}{n}\right)\right\}X_{[m/2],[n/2]}^+ - \frac{C_1 mn}{m^\alpha n^\beta}, \\
&\qquad\qquad C, C_1 > 0,\ (X_{ik}^+ \text{ are nondecreasing when } i+k \text{ is large enough}) \\
&\quad\geq \frac{C}{m^\alpha n^\beta}\cdot\frac{m}{4}\cdot\frac{n}{4}\cdot X_{[m/2],[n/2]}^+ - \frac{C_1 mn}{m^\alpha n^\beta} \\
&\quad\geq \frac{C}{m^\alpha n^\beta}\cdot\frac{m}{4}\cdot\frac{n}{4}\cdot\frac{\tau_{[m/2],[n/2]}}{\delta_{[m/2],[n/2]}}W_{[m/2],[n/2]} - \frac{C_1 mn}{m^\alpha n^\beta} \\
&\quad\geq \frac{C}{m^\alpha n^\beta}\cdot\frac{m}{4}\cdot\frac{n}{4}\cdot\frac{[m/2]^\alpha[n/2]^\beta}{\varepsilon_{[m/2],[n/2]}} - \frac{C_1 mn}{m^\alpha n^\beta} \\
&\quad\approx \frac{C\cdot m\cdot n}{64\cdot\varepsilon_{[m/2],[n/2]}} - \frac{C_1 mn}{m^\alpha n^\beta} \approx \frac{C_2}{\varepsilon_{[m/2],[n/2]}}\cdot\frac{m\cdot n}{m^\alpha n^\beta}, \quad C_2 > 0,
\end{aligned} \tag{4.41}$$

where $[\cdot]$ denotes the greatest integer function. Finally, employing the Borel-Cantelli lemma and the assumption conditions can yield that

$$\limsup_{m+n\to\infty}\left|\frac{1}{m^\alpha n^\beta}\sum_{i=1}^{m}\sum_{k=1}^{n}\left(1-\frac{i-1}{m}\right)\left(1-\frac{k-1}{n}\right)X_{ik}\right| = \infty, \quad \text{a.s.} \tag{4.42}$$

This completes the desired proof. □

Proof of Theorem 3.4. Here we define an array $\{W_{ik}\}$ of independent random variables as follows [8]:

$$\begin{aligned}
P\left[W_{ik} = \frac{i^\alpha k^\beta}{\varepsilon_{ik}\log_2(k+1)}\right] &= \frac{\tau_{ik}^p\varepsilon_{ik}^p}{i^{\alpha p}k^{\beta p}}\log_2^p(k+1), \\
P[W_{ik} = 0] &= 1 - \frac{\tau_{ik}^p\varepsilon_{ik}^p}{i^{\alpha p}k^{\beta p}}\log_2^p(k+1).
\end{aligned} \tag{4.43}$$

Hence, we also have that

$$E|W_{ik}|^p = \tau_{ik}^p, \qquad 0 \le E(W_{ik}) = \frac{\tau_{ik}^p \varepsilon_{ik}^{p-1} \log_2^{p-1}(k+1)}{i^{\alpha(p-1)} k^{\beta(p-1)}} \le \varepsilon_{ik}^{p-1}. \tag{4.44}$$

Then, choosing $\{X_{ik}\}$ and following the similar steps as in the proof of Theorem 3.3, we can obtain the desired results. □

Acknowledgment

The author would like to thank the referee for his careful reading and valuable comments which make this paper more clear and readable.

References

[1] K.-L. Su, "Strong limit theorems of Cesaro type means for arrays of orthogonal random elements with multi-dimensional indices in Banach spaces," *Stochastic Analysis and Applications*, vol. 22, no. 2, pp. 237–250, 2004.

[2] P. Révész, *The Laws of Large Numbers*, vol. 4 of *Probability and Mathematical Statistics*, Academic Press, New York, NY, USA, 1968.

[3] W. F. Stout, *Almost Sure Convergence*, Academic Press, New York, NY, USA, 1974.

[4] P. Warren and J. Howell, "A strong law of large numbers for orthogonal Banach space-valued random variables," in *Probability in Banach Spaces (Proc. First Internat. Conf., Oberwolfach, 1975)*, vol. 526 of *Lecture Notes in Math.*, pp. 253–262, Springer, Berlin, Germany, 1976.

[5] F. Móricz, "Strong limit theorems for quasi-orthogonal random fields," *Journal of Multivariate Analysis*, vol. 30, no. 2, pp. 255–278, 1989.

[6] F. Móricz, "Strong limit theorems for quasi-orthogonal random fields. II," *Acta Scientiarum Mathematicarum*, vol. 55, no. 3-4, pp. 309–316, 1991.

[7] F. Móricz, K.-L. Su, and R. L. Taylor, "Strong laws of large numbers for arrays of orthogonal random elements in Banach spaces," *Acta Mathematica Hungarica*, vol. 65, no. 1, pp. 1–16, 1994.

[8] R. L. Taylor, *Stochastic Convergence of Weighted Sums of Random Elements in Linear Spaces*, vol. 672 of *Lecture Notes in Mathematics*, Springer, Berlin, Germany, 1978.

[9] J. O. Howell and R. L. Taylor, "Marcinkiewicz-Zygmund weak laws of large numbers for unconditional random elements in Banach spaces," in *Probability in Banach Spaces, III (Medford, Mass., 1980)*, vol. 860 of *Lecture Notes in Math.*, pp. 219–230, Springer, Berlin, Germany, 1981.

[10] A. N. Shiryayev, *Probability*, vol. 95 of *Graduate Texts in Mathematics*, Springer, New York, NY, USA, 1984, translated by R. P. Boas.

Kuo-Liang Su: Department of Business, National Open University, Lu Chow 247, Taipei, Taiwan
Email address: klsu@mail.nou.edu.tw

Mathematical Problems in Engineering

Special Issue on Time-Dependent Billiards

Call for Papers

This subject has been extensively studied in the past years for one-, two-, and three-dimensional space. Additionally, such dynamical systems can exhibit a very important and still unexplained phenomenon, called as the Fermi acceleration phenomenon. Basically, the phenomenon of Fermi acceleration (FA) is a process in which a classical particle can acquire unbounded energy from collisions with a heavy moving wall. This phenomenon was originally proposed by Enrico Fermi in 1949 as a possible explanation of the origin of the large energies of the cosmic particles. His original model was then modified and considered under different approaches and using many versions. Moreover, applications of FA have been of a large broad interest in many different fields of science including plasma physics, astrophysics, atomic physics, optics, and time-dependent billiard problems and they are useful for controlling chaos in Engineering and dynamical systems exhibiting chaos (both conservative and dissipative chaos).

We intend to publish in this special issue papers reporting research on time-dependent billiards. The topic includes both conservative and dissipative dynamics. Papers discussing dynamical properties, statistical and mathematical results, stability investigation of the phase space structure, the phenomenon of Fermi acceleration, conditions for having suppression of Fermi acceleration, and computational and numerical methods for exploring these structures and applications are welcome.

To be acceptable for publication in the special issue of Mathematical Problems in Engineering, papers must make significant, original, and correct contributions to one or more of the topics above mentioned. Mathematical papers regarding the topics above are also welcome.

Authors should follow the Mathematical Problems in Engineering manuscript format described at http://www.hindawi.com/journals/mpe/. Prospective authors should submit an electronic copy of their complete manuscript through the journal Manuscript Tracking System at http://mts.hindawi.com/ according to the following timetable:

Manuscript Due	December 1, 2008
First Round of Reviews	March 1, 2009
Publication Date	June 1, 2009

Guest Editors

Edson Denis Leonel, Departamento de Estatística, Matemática Aplicada e Computação, Instituto de Geociências e Ciências Exatas, Universidade Estadual Paulista, Avenida 24A, 1515 Bela Vista, 13506-700 Rio Claro, SP, Brazil ; edleonel@rc.unesp.br

Alexander Loskutov, Physics Faculty, Moscow State University, Vorob'evy Gory, Moscow 119992, Russia; loskutov@chaos.phys.msu.ru