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Research Article

On the Composition of Distributions $x^{-s}\ln|x|$ and $|x|^{\mu}$

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Let F be a distribution and let f be a locally summable function. The distribution $F(f)$ is defined as the neutrix limit of the sequence $\{F_n(f)\}$, where $F_n(x) = F(x) * \delta_n(x)$ and $\{\delta_n(x)\}$ is a certain sequence of infinitely differentiable functions converging to the Dirac delta-function $\delta(x)$. The composition of the distributions $x^{-s}\text{In}|x|$ and $|x|^{\mu}$ is evaluated for $s = 1, 2, \ldots, \mu > 0$ and $\mu s \neq 1, 2, \ldots$.

In the following, we let $\mathcal{D}$ be the space of infinitely differentiable functions with compact support, let $\mathcal{D}(a,b)$ be the space of infinitely differentiable functions with support contained in the interval (a,b), and let $\mathcal{D}'$ be the space of distributions defined on $\mathcal{D}$.

We define the locally summable functions x_+^{λ}, x_-^{λ}, $x_+^{\lambda}\ln x_+$, $x_-^{\lambda}\ln x_-$, $|x|^{\lambda}$, and $|x|^{\lambda}\ln|x|$ for $\lambda > -1$ (see [1]) by

$$
\begin{gathered}
x_+^{\lambda} = \begin{cases} x^{\lambda}, & x > 0, \\ 0, & x < 0, \end{cases} \qquad x_-^{\lambda} = \begin{cases} |x|^{\lambda}, & x < 0, \\ 0, & x > 0, \end{cases} \\
x_+^{\lambda}\ln x_+ = \begin{cases} x^{\lambda}\ln x, & x > 0, \\ 0, & x < 0, \end{cases} \qquad x_-^{\lambda}\ln x_- = \begin{cases} |x|^{\lambda}\ln|x|, & x < 0, \\ 0, & x > 0, \end{cases} \\
|x|^{\lambda} = x_+^{\lambda} + x_-^{\lambda}, \qquad |x|^{\lambda}\ln|x| = x_+^{\lambda}\ln x_+ + x_-^{\lambda}\ln x_-.
\end{gathered} \tag{1}
$$

The distributions x_+^{λ} and x_-^{λ} are then defined inductively for $\lambda < -1$ and $\lambda \neq -2, -3, \ldots$ by

$$
(x_+^{\lambda})' = \lambda x_+^{\lambda-1}, \qquad (x_-^{\lambda})' = -\lambda x_-^{\lambda-1}. \tag{2}
$$

It follows that if r is a positive integer and $-r-1<\lambda<-r$, then

$$\begin{aligned}\langle x_+^\lambda,\varphi(x)\rangle &= \int_0^\infty x^\lambda\left[\varphi(x)-\sum_{k=0}^{r-1}\frac{\varphi^{(k)}(0)}{k!}x^k\right]dx,\\ \langle x_-^\lambda,\varphi(x)\rangle &= \int_{-\infty}^0 |x|^\lambda\left[\varphi(x)-\sum_{k=0}^{r-1}\frac{\varphi^{(k)}(0)}{k!}x^k\right]dx\end{aligned}\tag{3}$$

for arbitrary φ in $\mathcal{D}$. In particular, if φ has its support contained in the interval $[-1,1]$, then

$$\begin{aligned}\langle x_+^\lambda,\varphi(x)\rangle &= \int_0^1 x^\lambda\left[\varphi(x)-\sum_{k=0}^{r-1}\frac{\varphi^{(k)}(0)}{k!}x^k\right]dx+\sum_{k=0}^{r-1}\frac{\varphi^{(k)}(0)}{k!(\lambda+k+1)},\\ \langle x_-^\lambda,\varphi(x)\rangle &= \int_{-1}^0 |x|^\lambda\left[\varphi(x)-\sum_{k=0}^{r-1}\frac{\varphi^{(k)}(0)}{k!}x^k\right]dx+\sum_{k=0}^{r-1}\frac{(-1)^k\varphi^{(k)}(0)}{k!(\lambda+k+1)},\\ \langle |x|^\lambda,\varphi(x)\rangle &= \int_{-1}^1 |x|^\lambda\left[\varphi(x)-\sum_{k=0}^{r-1}\frac{\varphi^{(k)}(0)}{k!}x^k\right]dx+\sum_{k=0}^{r-1}\frac{[1+(-1)^k]\varphi^{(k)}(0)}{k!(\lambda+k+1)},\end{aligned}\tag{4}$$

$$\langle |x|^\lambda\ln|x|,\varphi(x)\rangle = \int_{-1}^1 |x|^\lambda\ln|x|\left[\varphi(x)-\sum_{k=1}^{r-1}\frac{\varphi^{(k)}(0)}{k!}x^k\right]dx+\sum_{k=0}^{r-1}\frac{[1+(-1)^k]\varphi^{(k)}(0)}{k!(\lambda+k+1)^2}\tag{5}$$

if $-r-1<\lambda<-r$.

We define the distribution $x^{-1}\ln|x|$ by

$$x^{-1}\ln|x| = \frac{1}{2}(\ln^2|x|)',\tag{6}$$

and we define the distribution $x^{-r-1}\ln|x|$ inductively by

$$x^{-r-1}\ln|x| = \frac{x^{-r-1}-(x^{-r}\ln|x|)'}{r}\tag{7}$$

for $r=1,2,\ldots$. It follows by induction that

$$x^{-r-1}\ln|x| = \phi(r)x^{-r-1}+\frac{(-1)^r(x^{-1}\ln|x|)^{(r)}}{r!} = \phi(r)x^{-r-1}+\frac{(-1)^r(\ln^2|x|)^{(r+1)}}{2r!},\tag{8}$$

where

$$\phi(r)=\begin{cases}\sum_{i=1}^{r} i^{-1}, & r=1,2,\ldots,\\ 0, & r=0.\end{cases}\tag{9}$$

In the following, we let N be the neutrix, see [2], having domain N' the positive integers and range N'' the real numbers, with negligible functions which are finite linear sums of the functions

$$n^{\lambda}\ln^{r-1}n,\ \ln^{r}n,\quad \lambda>0,\ r=1,2,\dots \tag{10}$$

as well as all functions which converge to zero in the usual sense as n tends to infinity.

Now let $\rho(x)$ be an infinitely differentiable function having the following properties:

(i) $\rho(x)=0$ for $|x|\geq 1$,
(ii) $\rho(x)\geq 0$,
(iii) $\rho(x)=\rho(-x)$,
(iv) $\int_{-1}^{1}\rho(x)dx=1$.

Putting $\delta_n(x)=n\rho(nx)$ for $n=1,2,\dots$, it follows that $\{\delta_n(x)\}$ is a regular sequence of infinitely differentiable functions converging to the Dirac delta-function $\delta(x)$.

Next, for an arbitrary distribution f in $\mathcal{D}'$, we define

$$f_n(x)=(f*\delta_n)(x)=\langle f(t),\delta_n(x-t)\rangle \tag{11}$$

for $n=1,2,\dots$. It follows that $\{f_n(x)\}$ is a regular sequence of infinitely differentiable functions converging to the distribution $f(x)$.

The following definition was given in [3].

Definition 1. Let F be a distribution and let f be a locally summable function. Say that the distribution $F(f(x))$ exists and is equal to h on the open interval (a,b) if

$$\mathrm{N}-\lim_{n\to\infty}\int_{-\infty}^{\infty}F_n(f(x))\varphi(x)dx=\langle h(x),\varphi(x)\rangle \tag{12}$$

for all test functions φ with compact support contained in (a,b).

The following theorems were proved in [4, 5] and [6], respectively.

Theorem 2. *The distribution $(x^r)^{-s}$ exists and*

$$(x^r)^{-s}=x^{-rs} \tag{13}$$

for $r,s=1,2,\dots$.

Theorem 3. *The distribution $(|x|^{\mu})^{-s}$ exists and*

$$(|x|^{\mu})^{-s}=|x|^{-\mu s} \tag{14}$$

for $s=1,2,\dots$, $\mu>0$ and $\mu s\neq 1,2,\dots$.

Theorem 4. *If $F_s(x)$ denotes the distribution $x^{-s}\ln|x|$, then the distribution $F_s(x^r)$ exists and*

$$F_s(x^r)=rF_{rs}(x) \tag{15}$$

for $r,s=1,2,\dots$.

We need the following lemma which can be easily proved by induction.

Lemma 5.

$$\int_{-1}^{1} v^i \rho^{(r)}(v)\,dv = \begin{cases} 0, & 0 \le i < r, \\ (-1)^r r!, & i = r \end{cases} \tag{16}$$

for $r = 0, 1, 2, \ldots$.

We now prove the following theorem on the composition of distributions in the neutrix setting.

Theorem 6. *If* $F_s(x)$ *denotes the distribution* $x^{-s}\ln|x|$, *then the distribution* $F_s(|x|^\mu)$ *exists and*

$$F_s(|x|^\mu) = \mu|x|^{-\mu s}\ln|x| \tag{17}$$

for $s = 1, 2, \ldots$, $\mu > 0$ *and* $\mu s \neq 1, 2, \ldots$.

Proof. We will suppose that $r < \mu s < r + 1$ for some positive integer r. We put

$$\begin{aligned} [F_s(|x|^\mu)]_n &= F_s(|x|^\mu) * \delta_n(x) \\ &= \phi(s-1)[(|x|^\mu)^{-s}]_n - \frac{(-1)^s}{2(s-1)!}\int_{-1/n}^{1/n} \ln^2 \big|\, |x|^\mu - t \,\big| \delta_n^{(s)}(t)\,dt, \end{aligned} \tag{18}$$

and note that

$$\begin{aligned} &\int_{-1}^{1} x^k \int_{-1/n}^{1/n} \ln^2 \big|\, |x|^\mu - t \,\big| \delta_n^{(s)}(t)\,dt\,dx \\ &\quad = \begin{cases} 0, & k \text{ odd}, \\ 2\displaystyle\int_0^1 x^k \int_{-1/n}^{1/n} \ln^2 \big|\, |x|^\mu - t \,\big| \delta_n^{(s)}(t)\,dt\,dx, & k \text{ even}. \end{cases} \end{aligned} \tag{19}$$

Then

$$\begin{aligned} &\int_0^1 x^k \int_{-1/n}^{1/n} \ln^2 |x^\mu - t| \delta_n^{(s)}(t)\,dt\,dx \\ &\quad = \int_{-1/n}^{1/n} \delta_n^{(s)}(t) \int_0^{n^{-1/\mu}} x^k \ln^2 |x^\mu - t|\,dx\,dt \\ &\qquad + \int_{-1/n}^{1/n} \delta_n^{(s)}(t) \int_{n^{-1/\mu}}^{1} x^k \ln^2 |x^\mu - t|\,dx\,dt \\ &\quad = \frac{n^{(\mu s - k - 1)/\mu}}{\mu} \int_{-1}^{1} \rho^{(s)}(v) \int_0^1 u^{-(\mu-k-1)/\mu} \ln^2 \left|\frac{u - v}{n}\right| du\,dv \\ &\qquad + \frac{n^{(\mu s - k - 1)/\mu}}{\mu} \int_{-1}^{1} \rho^{(s)}(v) \int_1^n u^{-(\mu-k-1)/\mu} \ln^2 \left|\frac{u - v}{n}\right| du\,dv \\ &\quad = I_1 + I_2, \end{aligned} \tag{20}$$

on using the substitutions $u = nx^\mu$ and $v = nt$.

It is easily seen that

$$\mathrm{N}-\lim_{n\to\infty} I_1 = 0 \tag{21}$$

for $k = 0, 1, \ldots, r-1$.

Now,

$$\begin{aligned}
I_2 &= \frac{n^{(\mu s-k-1)/\mu}}{\mu}\int_{-1}^{1}\rho^{(s)}(v)\int_{1}^{n}u^{-(\mu-k-1)/\mu}\Big[\ln\left|\frac{1-v}{u}\right| + \ln u - \ln n\Big]^2 du\,dv \\
&= \frac{n^{(\mu s-k-1)/\mu}}{\mu}\int_{-1}^{1}\rho^{(s)}(v)\int_{1}^{n}u^{-(\mu-k-1)/\mu}\ln^2\left|\frac{1-v}{u}\right| du\,dv \\
&\quad + \frac{2n^{(\mu s-k-1)/\mu}}{\mu}\int_{-1}^{1}\rho^{(s)}(v)\int_{1}^{n}u^{-(\mu-k-1)/\mu}\ln u \ln\left|\frac{1-v}{u}\right| du\,dv \\
&\quad - \frac{2n^{(\mu s-k-1)/\mu}}{\mu}\ln n\int_{-1}^{1}\rho^{(s)}(v)\int_{1}^{n}u^{-(\mu-k-1)/\mu}\ln\left|\frac{1-v}{u}\right| du\,dv \\
&= J_1 + J_2 + J_3,
\end{aligned} \tag{22}$$

since $\int_{-1}^{1}\rho^{(s)}(v)\,dv = 0$ for $s = 1, 2, \ldots$, by Lemma 5.

It is easily seen that

$$\mathrm{N}-\lim_{n\to\infty} J_3 = 0. \tag{23}$$

Next, we have

$$\begin{aligned}
J_1 &= \frac{n^{(\mu s-k-1)/\mu}}{\mu}\int_{-1}^{1}\rho^{(s)}(v)\int_{1}^{n}u^{-(\mu-k-1)/\mu}\left(\sum_{i=1}^{\infty}\frac{v^i}{iu^i}\right)^2 du\,dv \\
&= \frac{2n^{(\mu s-k-1)/\mu}}{\mu}\sum_{i=1}^{\infty}\frac{\phi(i)}{i+1}\int_{-1}^{1}v^{i+1}\rho^{(s)}(v)\int_{1}^{n}u^{(k+1)/\mu-i-2}du\,dv \\
&= \frac{2n^{(\mu s-k-1)/\mu}}{\mu}\sum_{i=1}^{\infty}\frac{\phi(i)}{i+1}\,\frac{\mu\big(n^{(k+1)/\mu-i-1}-1\big)}{k-\mu(i+1)+1}\int_{-1}^{1}v^{i+1}\rho^{(s)}(v)\,dv,
\end{aligned} \tag{24}$$

and it follows that

$$\mathrm{N}-\lim_{n\to\infty} J_1 = \frac{2\phi(s-1)}{s(\mu s-k-1)}\int_{-1}^{1}v^s\rho^{(s)}(v)\,dv = \frac{2(-1)^s\phi(s-1)(s-1)!}{\mu s-k-1}, \tag{25}$$

on using Lemma 5, for $k = 0, 1, \ldots, r-1$.

Finally,

$$\begin{aligned}
J_2 &= \frac{2n^{(\mu s-k-1)/\mu}}{\mu}\sum_{i=1}^{\infty}\frac{1}{i}\int_{-1}^{1}v^i\rho^{(s)}(v)\int_{1}^{n}u^{(k+1)/\mu-i-1}\ln u\,du\,dv \\
&= 2\sum_{i=1}^{\infty}\frac{1}{i}\left[\frac{n^{s-i}\ln n}{k-\mu i+1} - \frac{\mu\big(n^{s-i}-n^{(\mu s-k-1)/\mu}\big)}{(k-\mu i+1)^2}\right]\int_{-1}^{1}v^i\rho^{(s)}(v)\,dv,
\end{aligned} \tag{26}$$

and it follows that

$$\mathrm{N}-\lim_{n\to\infty} J_2 = \frac{2\mu(-1)^{s-1}(s-1)!}{(\mu s-k-1)^2}, \tag{27}$$

on using Lemma 5, for $k = 0,1,\ldots,r-1$.

Hence,

$$\begin{aligned} &\mathrm{N}-\lim_{n\to\infty}\int_0^1 x^k \int_{-1/n}^{1/n} \ln^2|x^\mu - t|\,\delta_n^{(s)}(t)\,dt\,dx \\ &\quad = 2(-1)^s(s-1)!\left[\frac{\phi(s-1)}{\mu s-k-1} - \frac{\mu}{(\mu s-k-1)^2}\right] \end{aligned} \tag{28}$$

for $k = 0,1,\ldots,r-1$, on using (19) to (23). Then using (19) and (25), we see that

$$\begin{aligned} &\mathrm{N}-\lim_{n\to\infty}\int_{-1}^1 x^k \int_{-1/n}^{1/n} \ln^2||x|^\mu - t|\,\delta_n^{(s)}(t)\,dt\,dx \\ &\quad = 2(-1)^s(s-1)!\left[\frac{\phi(s-1)}{\mu s-k-1} - \frac{\mu}{(\mu s-k-1)^2}\right]((-1)^k+1) \end{aligned} \tag{29}$$

for $k = 0,1,\ldots,r-1$.

When $k = r$, (18) still holds, but now we have

$$I_1 = \frac{n^{(\mu s-r-1)/\mu}}{\mu}\int_{-1}^1 \rho^{(s)}(v)\int_0^1 u^{-(\mu-r-1)/\mu}\ln^2\left|\frac{u-v}{n}\right|du\,dv, \tag{30}$$

and it follows that for any continuous function ψ

$$\lim_{n\to\infty}\int_0^{n^{-1/\mu}} x^r \int_{-1/n}^{1/n}\ln^2||x|^\mu - t|\,\delta_n^{(s)}(t)\,dt\,\psi(x)\,dx = 0. \tag{31}$$

Similarly,

$$\lim_{n\to\infty}\int_{-n^{-1/\mu}}^{0} x^r \int_{-1/n}^{1/n}\ln^2||x|^\mu - t|\,\delta_n^{(s)}(t)\,dt\,\psi(x)\,dx = 0. \tag{32}$$

Next, when $|x|^\mu \ge 1/n$, we have

$$\begin{aligned} \int_{-1/n}^{1/n}\ln^2||x|^\mu - t|\,\delta_n^{(s)}(t)\,dt &= n^s\int_{-1}^1 \ln^2||x|^\mu - v/n|\,\rho^{(s)}(v)dv \\ &= n^s\int_{-1}^1\left[\ln|x|^\mu - \sum_{i=1}^\infty \frac{v^i}{in^i|x|^{\mu i}}\right]^2\rho^{(s)}(v)dv \\ &= \sum_{i=s}^\infty \frac{-2\ln|x|^\mu + 2\phi(i-1)}{in^{i-s}|x|^{\mu i}}\int_{-1}^1 v^i\rho^{(s)}(v)dv. \end{aligned} \tag{33}$$

It follows that

$$\left|\int_{-1/n}^{1/n}\ln^2||x|^\mu - t|\,\delta_n^{(s)}(t)\,dt\right| \le \sum_{i=s}^\infty \frac{(4\mu\ln|x| + 4\phi(i-1))K_s}{in^{i-s}|x|^{\mu i}} \tag{34}$$

for $s = 1, 2, \ldots$, where

$$K_s = \int_{-1}^{1} \left| \rho^{(s)}(v) \right| dv. \tag{35}$$

If now $n^{-1/\mu} < \eta < 1$, then

$$\begin{aligned}
&\int_{n^{-1/\mu}}^{\eta} x^r \left| \int_{-1/n}^{1/n} \ln^2 \left| |x|^\mu - t \right| \delta_n^{(s)}(t) \right| dt\, dx \\
&\quad \leq \sum_{i=s}^{\infty} \frac{4K_s\mu}{in^{i-s}} \int_{n^{-1/\mu}}^{\eta} x^{r-\mu i} \ln x\, dx + \sum_{i=s}^{\infty} \frac{4K_s\phi(s-1)}{in^{i-s}} \int_{n^{-1/\mu}}^{\eta} x^{r-\mu i}\, dx \\
&\quad = \sum_{i=s}^{\infty} \frac{4K_s\mu}{in^{i-s}} \left[\frac{\eta^{r+1-\mu i}\ln\eta - n^{i-(r+1)/\mu}\ln n^{-1/\mu}}{r+1-\mu i} - \frac{\eta^{r+1-\mu i} - n^{i-(r+1)/\mu}}{(r+1-\mu i)^2} \right] \\
&\qquad + \sum_{i=s}^{\infty} \frac{4K_s\phi(s-1)}{in^{i-s}} \frac{\eta^{r+1-\mu i} - n^{i-(r+1)/\mu}}{r+1-\mu i}.
\end{aligned} \tag{36}$$

It follows that

$$\lim_{n\to\infty} \int_{n^{-1/\mu}}^{\eta} x^r \left| \int_{-1/n}^{1/n} \ln^2 \left| |x|^\mu - t \right| \delta_n^{(s)}(t) \right| dt\, dx = O(\eta \ln \eta) \tag{37}$$

for $s = 1, 2, \ldots$.

Thus, if ψ is a continuous function, then

$$\lim_{n\to\infty} \left| \int_{n^{-1/\mu}}^{\eta} x^r \psi(x) \int_{-1/n}^{1/n} \ln^2 \left| |x|^\mu - t \right| \delta_n^{(s)}(t)\, dt\, dx \right| = O(\eta \ln \eta) \tag{38}$$

for $s = 1, 2, \ldots$.

Similarly,

$$\lim_{n\to\infty} \left| \int_{-\eta}^{-n^{-1/\mu}} x^r \psi(x) \int_{-1/n}^{1/n} \ln^2 \left| |x|^\mu - t \right| \delta_n^{(s)}(t) \right| dt\, dx = O(\eta \ln \eta) \tag{39}$$

for $s = 1, 2, \ldots$.

Now let $\varphi(x)$ be an arbitrary function in $\mathcal{D}$ with support contained in the interval $[-1, 1]$. By Taylor's theorem, we have

$$\varphi(x) = \sum_{k=0}^{r-1} \frac{x^k}{k!} \varphi^{(k)}(0) + \frac{x^r}{r!} \varphi^{(r)}(\xi x), \tag{40}$$

where $0<\xi<1$. Then

$$\begin{aligned}\left\langle \int_{-1/n}^{1/n} \ln^2 \left| |x|^{\mu} - t \right| \delta_n^{(s)}(t)\,dt, \varphi(x) \right\rangle &= \int_{-1}^{1} \varphi(x) \int_{-1/n}^{1/n} \ln^2 \left| |x|^{\mu} - t \right| \delta_n^{(s)}(t)\,dt\,dx \\ &= \sum_{k=0}^{r-1} \frac{\varphi^{(k)}(0)}{k!} \int_{-1}^{1} x^k \int_{-1/n}^{1/n} \ln^2 \left| |x|^{\mu} - t \right| \delta_n^{(s)}(t)dt\,dx \\ &+ \int_{-n^{-1/\mu}}^{n^{-1/\mu}} \frac{x^r}{r!}\varphi^{(r)}(\xi x) \int_{-1/n}^{1/n} \ln^2 \left| |x|^{\mu} - t \right| \delta_n^{(s)}(t)dt\,dx \\ &+ \int_{n^{-1/\mu}}^{\eta} \frac{x^r}{r!}\varphi^{(r)}(\xi x) \int_{-1/n}^{1/n} \ln^2 \left| |x|^{\mu} - t \right| \delta_n^{(s)}(t)dt\,dx \\ &+ \int_{\eta}^{1} \frac{x^r}{r!}\varphi^{(r)}(\xi x) \int_{-1/n}^{1/n} \ln^2 \left| |x|^{\mu} - t \right| \delta_n^{(s)}(t)dt\,dx \\ &+ \int_{-\eta}^{-n^{-1/\mu}} \frac{x^r}{r!}\varphi^{(r)}(\xi x) \int_{-1/n}^{1/n} \ln^2 \left| |x|^{\mu} - t \right| \delta_n^{(s)}(t)dt\,dx \\ &+ \int_{-1}^{-\eta} \frac{x^r}{r!}\varphi^{(r)}(\xi x) \int_{-1/n}^{1/n} \ln^2 \left| |x|^{\mu} - t \right| \delta_n^{(s)}(t)dt\,dx. \end{aligned} \tag{41}$$

Using equations (27) to (32) and noting that on the intervals $[-1,-\eta]$ and $[\eta,1]$,

$$\lim_{n\to\infty} \frac{(-1)^s}{2(s-1)!} \int_{-1/n}^{1/n} \ln^2 \left| |x|^{\mu} - t \right| \delta_n^{(s)}(t)dt = \phi(s-1)|x|^{-\mu s} - \mu |x|^{-\mu s} \ln|x|. \tag{42}$$

Since $|x|^{\mu}$ and $F_s(x)$ are continuous on these intervals, it follows that

$$\begin{aligned} \mathrm{N}-\lim_{n\to\infty} & \frac{(-1)^s}{2(s-1)!} \left\langle \int_{-1/n}^{1/n} \ln^2 \left| |x|^{\mu} - t \right| \delta_n^{(s)}(t)dt, \varphi(x) \right\rangle \\ &= \sum_{k=0}^{r-1} \left[\frac{\phi(s-1)}{\mu s - k - 1} - \frac{\mu}{(\mu s - k - 1)^2} \right] \frac{\varphi^{(k)}(0)}{k!} \left((-1)^k + 1\right) \\ &+ O(\eta|\ln\eta|) + \int_{\eta}^{1} \frac{x^{r-\mu s}}{r!} \varphi^{(r)}(\xi x)\left(\phi(s-1) - \mu \ln x\right) dx \\ &+ \int_{-1}^{-\eta} \frac{|x|^{r-\mu s}}{r!} \varphi^{(r)}(\xi x)\left(\phi(s-1) - \mu \ln|x|\right) dx. \end{aligned} \tag{43}$$

Since η can be made arbitrarily small, it follows that

$$\begin{aligned}
\mathrm{N}-\lim_{n\to\infty} \frac{(-1)^s}{2(s-1)!}\left\langle \int_{-1/n}^{1/n} \ln^2 \big|\, |x|^\mu - t\,\big|\, \delta_n^{(s)}(t)\,dt, \varphi(x) \right\rangle
&= \sum_{k=0}^{r-1}\left[\frac{\phi(s-1)}{\mu s-k-1} - \frac{\mu}{(\mu s-k-1)^2}\right]\frac{\varphi^{(k)}(0)}{k!}\left((-1)^k+1\right) \\
&\quad + \phi(s-1)\int_{-1}^{1} |x|^{-\mu s}\left[\varphi(x) - \sum_{k=0}^{r-1}\frac{x^k}{k!}\varphi^{(k)}(0)\right]dx \\
&\quad - \mu\int_{-1}^{1} |x|^{-\mu s}\ln|x|\left[\varphi(x) - \sum_{k=0}^{r-1}\frac{x^k}{k!}\varphi^{(k)}(0)\right]dx \\
&= \phi(s-1)\left\langle |x|^{-\mu s}, \varphi(x)\right\rangle - \mu\left\langle |x|^{-\mu s}\ln|x|, \varphi(x)\right\rangle
\end{aligned} \tag{44}$$

on using (5). This proves (15) on the interval $[-1,1]$. However, (15) clearly holds on any interval not containing the origin, and the proof is complete. □

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As a multidisciplinary field, financial engineering is becoming increasingly important in today's economic and financial world, especially in areas such as portfolio management, asset valuation and prediction, fraud detection, and credit risk management. For example, in a credit risk context, the recently approved Basel II guidelines advise financial institutions to build comprehensible credit risk models in order to optimize their capital allocation policy. Computational methods are being intensively studied and applied to improve the quality of the financial decisions that need to be made. Until now, computational methods and models are central to the analysis of economic and financial decisions.

However, more and more researchers have found that the financial environment is not ruled by mathematical distributions or statistical models. In such situations, some attempts have also been made to develop financial engineering models using intelligent computing approaches. For example, an artificial neural network (ANN) is a nonparametric estimation technique which does not make any distributional assumptions regarding the underlying asset. Instead, ANN approach develops a model using sets of unknown parameters and lets the optimization routine seek the best fitting parameters to obtain the desired results. The main aim of this special issue is not to merely illustrate the superior performance of a new intelligent computational method, but also to demonstrate how it can be used effectively in a financial engineering environment to improve and facilitate financial decision making. In this sense, the submissions should especially address how the results of estimated computational models (e.g., ANN, support vector machines, evolutionary algorithm, and fuzzy models) can be used to develop intelligent, easy-to-use, and/or comprehensible computational systems (e.g., decision support systems, agent-based system, and web-based systems)

This special issue will include (but not be limited to) the following topics:

- **Computational methods**: artificial intelligence, neural networks, evolutionary algorithms, fuzzy inference, hybrid learning, ensemble learning, cooperative learning, multiagent learning
- **Application fields**: asset valuation and prediction, asset allocation and portfolio selection, bankruptcy prediction, fraud detection, credit risk management
- **Implementation aspects**: decision support systems, expert systems, information systems, intelligent agents, web service, monitoring, deployment, implementation

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