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Research Article

Viscosity Approximation Methods for Nonexpansive Nonself-Mappings in Hilbert Spaces

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Viscosity approximation methods for nonexpansive nonself-mappings are studied. Let C be a nonempty closed convex subset of Hilbert space H, P a metric projection of H onto C and let T be a nonexpansive nonself-mapping from C into H. For a contraction f on C and $\{t_n\} \subseteq (0,1)$, let x_n be the unique fixed point of the contraction $x \mapsto t_n f(x) + (1-t_n)(1/n)\sum_{j=1}^{n}(PT)^j x$. Consider also the iterative processes $\{y_n\}$ and $\{z_n\}$ generated by $y_{n+1} = \alpha_n f(y_n) + (1-\alpha_n)(1/(n+1))\sum_{j=0}^{n}(PT)^j y_n, n \geq 0$, and $z_{n+1} = (1/(n+1))\sum_{j=0}^{n} P(\alpha_n f(z_n) + (1-\alpha_n)(TP)^j z_n), n \geq 0$, where $y_0, z_0 \in C, \{\alpha_n\}$ is a real sequence in an interval $[0,1]$. Strong convergence of the sequences $\{x_n\}, \{y_n\}$, and $\{z_n\}$ to a fixed point of T which solves some variational inequalities is obtained under certain appropriate conditions on the real sequences $\{\alpha_n\}$ and $\{t_n\}$.

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1. Introduction

Throughout this paper, we denote the set of all nonnegative integers by $\mathbb{N}$. Let H be a real Hilbert space with norm $\|\cdot\|$ and inner product $\langle\cdot,\cdot\rangle$. Let C be a closed convex subset of H, and T a nonself-mapping from C into H. We denote the set of all fixed points of T by $F(T)$, that is, $F(T) = \{x \in C : x = Tx\}$. T is said to be *nonexpansive mapping* if

$$\|Tx - Ty\| \leq \|x - y\| \tag{1.1}$$

for all $x, y \in C$. From condition on C, there is a mapping P from H onto C which satisfies

$$\left\|x - P_C x\right\| = \min_{y \in C}\|x - y\| \tag{1.2}$$

for all $x \in C$. This mapping P is said to be *the metric projection* from H onto C. We know that the metric projection is nonexpansive. Recall that a self-mapping $f : C \to C$ is a *contraction* on C if there exists a constant $\alpha \in (0,1)$ such that

$$\|f(x) - f(y)\| \leq \alpha\|x - y\| \quad \forall x, y \in C. \tag{1.3}$$

We use Π_C to denote the collection of all contractions on C. That is,

$$\Pi_C = \{f : f : C \longrightarrow C \text{ a contraction}\}. \tag{1.4}$$

Note that each $f \in \Pi_C$ has a unique fixed point in C.

Given a real sequence $\{t_n\} \subseteq (0,1)$ and a contraction $f \in \Pi_C$, define another mapping $T_n : C \to C$ by

$$T_n x = t_n f(x) + (1 - t_n)\frac{1}{n}\sum_{j=1}^{n}(PT)^j x \quad \forall n \geq 1. \tag{1.5}$$

It is not hard to see that T_n is a contraction on C. Indeed, for $x, y \in C$, we have

$$\begin{aligned} \|T_n x - T_n y\| &= \left\| t_n(f(x) - f(y)) + (1 - t_n)\frac{1}{n}\left(\sum_{j=1}^{n}(PT)^j x - \sum_{j=1}^{n}(PT)^j y\right)\right\| \\ &\leq t_n\|f(x) - f(y)\| + (1 - t_n)\frac{1}{n}\sum_{j=1}^{n}\|(PT)^j x - (PT)^j y\| \\ &\leq t_n \alpha\|x - y\| + (1 - t_n)\|x - y\| \\ &= (1 - t_n(1 - \alpha))\|x - y\|. \end{aligned} \tag{1.6}$$

For each n, let $x_n \in C$ be the unique fixed point of T_n. Thus x_n is the unique solution of fixed point equation

$$x_n = t_n f(x_n) + (1 - t_n)\frac{1}{n}\sum_{j=1}^{n}(PT)^j x_n \quad \forall n \geq 1. \tag{1.7}$$

One of the purposes of this paper is to study the convergence of $\{x_n\}$ when $t_n \to 0$ as $n \to \infty$ in Hilbert spaces. Fix $u \in C$ and define a contraction S_n on C by

$$S_n x = t_n u + (1 - t_n)\frac{1}{n}\sum_{j=1}^{n}(PT)^j x \quad \forall n \geq 1. \tag{1.8}$$

Let $s_n \in C$ be the unique fixed point of S_n. Thus

$$s_n = t_n u + (1 - t_n)\frac{1}{n}\sum_{j=1}^{n}(PT)^j s_n \quad \forall n \geq 1. \tag{1.9}$$

Shimizu and Takahashi [1] studied the strong convergence of the sequence $\{s_n\}$ defined by (1.9) for asymptotically nonexpansive mappings in Hilbert spaces.

We also study the convergence of the following iteration schemes: for $y_0, z_0 \in C$, compute the sequences $\{y_n\}$ and $\{z_n\}$ by the iterative schemes

$$y_{n+1} = \alpha_n f(y_n) + (1-\alpha_n)\frac{1}{n+1}\sum_{j=0}^{n}(PT)^j yn, \quad n \geq 0, \tag{1.10}$$

$$z_{n+1} = \frac{1}{n+1}\sum_{j=0}^{n} P(\alpha_n f(z_n) + (1-\alpha_n)(TP)^j z_n), \quad n \geq 0, \tag{1.11}$$

where $\{\alpha_n\}$ is a real sequence in $[0,1]$, $f : C \to C$ is a contraction mapping on C, and P is the metric projection of H onto C. The first special case of (1.10) was considered by Shimizu and Takahashi [2] who introduced the following iterative process:

$$y_{n+1} = \alpha_n y + (1-\alpha_n)\frac{1}{n+1}\sum_{j=0}^{n} T^j y_n, \quad n \geq 0, \tag{1.12}$$

where y, y_0 are arbitrary (but fixed) and $\{\alpha_n\} \subseteq [0,1]$ and then they proved the following theorem.

Theorem 1.1 [2]. *Let C be a nonempty closed convex subset of a Hilbert space H, let T be a nonexpansive self-mapping of C such that $F(T)$ is nonempty, and let $P_{F(T)}$ be the metric projection from C onto $F(T)$. Let $\{\alpha_n\}$ be a real sequence which satisfies $0 \leq \alpha_n \leq 1$, $\lim_{n\to\infty}\alpha_n = 0$, and $\sum_{n=0}^{\infty}\alpha_n = \infty$. Let y and y_0 be element of C and let $\{y_n\}$ be the sequence defined by (1.12). Then $\{y_n\}$ converges strongly to $P_{F(T)}y$.*

The second special case of (1.10) and (1.11) was considered by Matsushita and Kuroiwa [3] who introduced the following iterative process:

$$\begin{aligned} y_{n+1} &= \alpha_n y + (1-\alpha_n)\frac{1}{n+1}\sum_{j=0}^{n}(PT)^j y_n, \quad n \geq 0, \\ z_{n+1} &= \frac{1}{n+1}\sum_{j=0}^{n} P(\alpha_n z + (1-\alpha_n)(TP)^j z_n), \quad n \geq 0, \end{aligned} \tag{1.13}$$

where y, z, y_0, z_0 are arbitrary (but fixed) in C and $\{\alpha_n\} \subseteq [0,1]$. More precisely, they proved the following theorem.

Theorem 1.2 [3]. *Let H be a Hilbert space, C a closed convex subset of H, P the metric projection of H onto C, and let T be a nonexpansive nonself-mapping from C into H such that $F(T)$ is nonempty, and $\{\alpha_n\}$ a sequence of real numbers such that $0 \leq \alpha_n \leq 1$, $\lim_{n\to\infty}\alpha_n = 0$, and $\sum_{n=0}^{\infty}\alpha_n = \infty$. Suppose that $\{y_n\}$ and $\{z_n\}$ are defined by (1.13), respectively. Then $\{y_n\}$ and $\{z_n\}$ converge strongly to $P_{F(T)}y$ and $P_{F(T)}z$ in $F(T)$, respectively, where $P_{F(T)}$ is the metric projection from C onto $F(T)$.*

The purpose of this paper is twofold. First, we study the convergence of the sequence $\{x_n\}$ defined by (1.7) in Hilbert spaces. Second, we prove the strong convergence of the iteration schemes $\{y_n\}$ and $\{z_n\}$ defined by (1.10) and (1.11), respectively, in Hilbert spaces. Our results extend and improve the corresponding ones announced by Shimizu and Takahashi [2], Matsushita and Kuroiwa [3], and others.

2. Preliminaries

For the sake of convenience, we restate the following concepts and results.

Lemma 2.1. *Let H be a real Hilbert space, C a closed convex subset of H, and $P_C : H \to C$ the metric (nearest point) projection. Given $x \in H$ and $y \in C$, then $y = P_C x$ if and only if there holds the inequality*

$$\langle x - y, y - z\rangle \geq 0 \quad \forall z \in C. \tag{2.1}$$

Definition 1. A mapping $T : C \to H$ is said to satisfy *nowhere normal outward* (NNO) *condition* if and only if for each $x \in C$, $Tx \in S_x^C$, where $S_x = \{y \in H : y \neq x,\ Py = x\}$ and P is the metric projection from H onto C.

The following results were proved by Matsushita and Kuroiwa [4].

Lemma 2.2 (see [4, Proposition 2, page 208]). *Let H be a Hilbert space, C a nonempty closed convex subset of H, P the metric projection of H onto C, and $T : C \to H$ a nonexpansive nonself-mapping. If $F(T)$ is nonempty, then T satisfies NNO condition.*

Lemma 2.3 (see [4, Proposition 1, page 208]). *Let H be a Hilbert space, C a nonempty closed convex subset of H, P the metric projection of H onto C, and $T : C \to H$ a nonself-mapping. Suppose that T satisfies NNO condition. Then $F(PT) = F(T)$.*

Lemma 2.4 (see [4]). *Let H be a Hilbert space, C a closed convex subset of H, and $T : C \to C$ a nonexpansive self-mapping with $F(T) \neq \emptyset$. Let $\{x_n\}$ be a sequence in C such that $\{x_{n+1} - (1/(n+1))\sum_{i=1}^{n+1} T^i x_n\}$ converges strongly to 0 as $n \to \infty$ and let $\{x_{n_j}\}$ be a subsequence of $\{x_n\}$ such that $\{x_{n_j}\}$ converges weakly to x. Then x is a fixed point of T.*

Finally, the following two lemmas are useful for the proof of our main theorems.

Lemma 2.5 (see [5]). *Let $\{\alpha_n\}$ be a sequence in $[0,1]$ that satisfies $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$. Let $\{a_n\}$ be a sequence of nonnegative real numbers such that for all $\epsilon > 0$, there exists an integer $N \geq 1$ such that for all $n \geq N$,*

$$a_{n+1} \leq (1 - \alpha_n) a_n + \alpha_n \epsilon. \tag{2.2}$$

Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.6 (see [5]). *Let H be a Hilbert space, C a nonempty closed convex subset of H, and $f : C \to C$ a contraction with coefficient $\alpha < 1$. Then*

$$\langle x - y, (I - f)x - (I - f)y\rangle \geq (1 - \alpha)\|x - y\|^2, \quad x, y \in C. \tag{2.3}$$

Remark 2.7. As in Lemma 2.6, if f is a nonexpansive mapping, then

$$\langle x - y, (I - f)x - (I - f)y\rangle \geq 0 \quad \forall x, y \in C. \tag{2.4}$$

3. Main results

THEOREM 3.1. *Let H be a Hilbert space, C a nonempty closed convex subset of H, P the metric projection of H onto C, and $T : C \to H$ a nonexpansive nonself-mapping with $F(T) \neq \emptyset$. Let $\{t_n\}$ be sequence in $(0,1)$ which satisfies $\lim_{n\to\infty} t_n = 0$. Then for a contraction mapping $f : C \to C$ with coefficient $\alpha \in (0,1)$, the sequence $\{x_n\}$ defined by (1.7) converges strongly to z, where z is the unique solution in $F(T)$ to the variational inequality*

$$\langle (I-f)z,\, x-z\rangle \geq 0, \quad x \in F(T), \tag{3.1}$$

or equivalently $z = P_{F(T)}f(z)$, where $P_{F(T)}$ is a metric projection mapping from H onto $F(T)$.

Proof. Since $F(T)$ is nonempty, it follows that T satisfies NNO condition by Lemma 2.2. We first show that $\{x_n\}$ is bounded. Let $q \in F(T)$. We note that

$$\begin{aligned} \|x_n - q\| &= \left\| t_n f(x_n) + (1-t_n)\frac{1}{n}\sum_{j=1}^{n}(PT)^j x_n - q \right\| \\ &\leq \left\| t_n\big(f(x_n) - q\big) + (1-t_n)\frac{1}{n}\sum_{j=1}^{n}\big((PT)^j x_n - (PT)^j q\big) \right\| \\ &\leq t_n\|f(x_n) - q\| + (1-t_n)\|x_n - q\| \quad \forall n \geq 1. \end{aligned} \tag{3.2}$$

So we get

$$\begin{aligned} \|x_n - q\| \leq \|f(x_n) - q\| &\leq \|f(x_n) - f(q)\| + \|f(q) - q\| \\ &\leq \alpha\|x_n - q\| + \|f(q) - q\| \quad \forall n \geq 1. \end{aligned} \tag{3.3}$$

Hence

$$\|x_n - q\| \leq \frac{1}{1-\alpha}\|f(q) - q\| \quad \forall n \geq 1. \tag{3.4}$$

This shows that $\{x_n\}$ is bounded, so are $\{f(x_n)\}$, $\{(1/n)\sum_{j=1}^{n}(PT)^j x_n\}$. Further, we note that

$$\begin{aligned} \left\| x_n - \frac{1}{n}\sum_{j=1}^{n}(PT)^j x_n \right\| &= \left\| t_n f(x_n) + (1-t_n)\frac{1}{n}\sum_{j=1}^{n}(PT)^j x_n - \frac{1}{n}\sum_{j=1}^{n}(PT)^j x_n \right\| \\ &= t_n\left\| f(x_n) - \frac{1}{n}\sum_{j=1}^{n}(PT)^j x_n \right\| \\ &\leq t_n\left(\|f(x_n)\| + \left\| \frac{1}{n}\sum_{j=1}^{n}(PT)^j x_n \right\| \right) \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \end{aligned} \tag{3.5}$$

Thus $\{x_n - (1/n)\sum_{j=1}^{n}(PT)^j x_n\}$ converges strongly to 0. Since $\{x_n\}$ is a bounded sequence, there is a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ which converges weakly to $z \in C$. By Lemmas 2.3 and 2.4, we have $z \in F(T)$. For each $n \geq 1$, since

$$x_n - z = t_n(f(x_n) - z) + (1 - t_n)\frac{1}{n}\sum_{j=1}^{n}((PT)^j x_n - z), \tag{3.6}$$

we get

$$\begin{aligned} \|x_n - z\|^2 &= (1 - t_n)\left\langle \frac{1}{n}\sum_{j=1}^{n}((PT)^j x_n - z),\, x_n - z\right\rangle + t_n\langle f(x_n) - z,\, x_n - z\rangle \\ &\leq (1 - t_n)\|x_n - z\|^2 + t_n\langle f(x_n) - z,\, x_n - z\rangle. \end{aligned} \tag{3.7}$$

Hence

$$\begin{aligned} \|x_n - z\|^2 &\leq \langle f(x_n) - z,\, x_n - z\rangle \\ &= \langle f(x_n) - f(z),\, x_n - z\rangle + \langle f(z) - z,\, x_n - z\rangle \\ &\leq \alpha\|x_n - z\|^2 + \langle f(z) - z,\, x_n - z\rangle. \end{aligned} \tag{3.8}$$

This implies that

$$\|x_n - z\|^2 \leq \frac{1}{1 - \alpha}\langle x_n - z,\ f(z) - z\rangle. \tag{3.9}$$

In particular, we have

$$\|x_{n_j} - z\|^2 \leq \frac{1}{1 - \alpha}\langle x_{n_j} - z,\ f(z) - z\rangle. \tag{3.10}$$

Since $x_{n_j} \rightharpoonup z$, it follows that

$$x_{n_j} \longrightarrow z \quad \text{as } j \longrightarrow \infty. \tag{3.11}$$

Next we show that $z \in C$ solves the variational inequality (3.1). Indeed, we note that

$$x_n = t_n f(x_n) + (1 - t_n)\frac{1}{n}\sum_{j=1}^{n}(PT)^j x_n \quad \forall n \geq 1, \tag{3.12}$$

we have

$$(I - f)x_n = -\frac{1 - t_n}{t_n}\left(x_n - \frac{1}{n}\sum_{j=1}^{n}(PT)^j x_n\right). \tag{3.13}$$

Thus for any $q \in F(T)$, we infer by Remark 2.7 that

$$\begin{aligned}
\langle (I-f)x_n, x_n - q\rangle &= -\frac{1-t_n}{t_n}\left\langle \left(I - \frac{1}{n}\sum_{j=1}^{n}(PT)^j\right)x_n, x_n - q\right\rangle \\
&= -\frac{1-t_n}{t_n}\left\langle \left(I - \frac{1}{n}\sum_{j=1}^{n}(PT)^j\right)x_n - \left(I - \frac{1}{n}\sum_{j=1}^{n}(PT)^j\right)z, x_n - q\right\rangle \\
&\leq 0 \quad \forall n \geq 1.
\end{aligned} \tag{3.14}$$

In particular

$$\langle (I-f)x_{n_j}, x_{n_j} - q\rangle \leq 0 \quad \forall j \geq 1. \tag{3.15}$$

Taking $j \to \infty$, we obtain

$$\langle (I-f)z, z - q\rangle \leq 0 \quad \forall q \in F(T), \tag{3.16}$$

or equivalent to $z = P_{F(T)}f(z)$ as required. Finally, we will show that the whole sequence $\{x_n\}$ converges strongly to z. Let another subsequence $\{x_{n_k}\}$ of $\{x_n\}$ be such that $x_{n_k} \to z' \in C$ as $k \to \infty$. Then $z' \in F(T)$, it follows from the inequality (3.16) that

$$\langle (I-f)z, z - z'\rangle \leq 0. \tag{3.17}$$

Interchange z and z' to obtain

$$\langle (I-f)z', z' - z\rangle \leq 0. \tag{3.18}$$

Adding (3.17) and (3.18) and by Lemma 2.6, we get

$$(1-\alpha)\|z - z'\|^2 \leq \langle z - z', (I-f)z - (I-f)z'\rangle \leq 0. \tag{3.19}$$

This implies that $z = z'$. Hence $\{x_n\}$ converges strongly to z. This completes the proof. □

Theorem 3.2. *Let C be a nonempty closed convex subset of a Hilbert space H, P the metric projection of H onto C, and $T : C \to H$ a nonexpansive nonself-mapping with $F(T) \neq \emptyset$. Let $\{\alpha_n\}$ be a sequence in $[0,1]$ which satisfies $\lim_{n\to\infty}\alpha_n = 0$ and $\sum_{n=1}^{\infty}\alpha_n = \infty$. Then for a contraction mapping $f : C \to C$ with coefficient $\alpha \in (0,1)$, the sequence $\{y_n\}$ defined by (1.10) converges strongly to z, where z is the unique solution in $F(T)$ of the variational inequality (3.1).*

Proof. Since $F(T)$ is nonempty, it follows that T satisfies NNO condition by Lemma 2.2. We first show that $\{y_n\}$ is bounded. Let $q \in F(T)$. We note that

$$\begin{aligned}
\|y_{n+1}-q\| &= \left\|\alpha_n f(y_n) + (1-\alpha_n)\frac{1}{n+1}\sum_{j=0}^{n}(PT)^j y_n - q\right\| \\
&\le \alpha_n\|f(y_n)-q\| + (1-\alpha_n)\frac{1}{n+1}\sum_{j=0}^{n}\|(PT)^j y_n - q\| \\
&\le \alpha_n\|f(y_n)-f(q)\| + \alpha_n\|f(q)-q\| + (1-\alpha_n)\|y_n-q\| \\
&\le \alpha_n\alpha\|y_n-q\| + \alpha_n\|f(q)-q\| + (1-\alpha_n)\|y_n-q\| \\
&= (1-\alpha_n(1-\alpha))\|y_n-q\| + \alpha_n\|f(q)-q\| \\
&\le \max\left\{\|y_n-q\|, \frac{1}{1-\alpha}\|f(q)-q\|\right\} \quad \forall n \ge 1.
\end{aligned} \tag{3.20}$$

So by induction, we get

$$\|y_n-q\| \le \max\left\{\|y_0-q\|, \frac{1}{1-\alpha}\|f(q)-q\|\right\}, \quad n \ge 0. \tag{3.21}$$

This shows that $\{y_n\}$ is bounded, so are $\{f(y_n)\}$ and $\{(1/(n+1))\sum_{j=0}^{n}(PT)^j y_n\}$. We observe that

$$\begin{aligned}
\left\|y_{n+1} - \frac{1}{n+1}\sum_{j=0}^{n}(PT)^j y_n\right\| &= \left\|\alpha_n f(y_n) + (1-\alpha_n)\frac{1}{n+1}\sum_{j=0}^{n}(PT)^j y_n - \frac{1}{n+1}\sum_{j=0}^{n}(PT)^j y_n\right\| \\
&= \alpha_n\left\|f(y_n) - \frac{1}{n+1}\sum_{j=0}^{n}(PT)^j y_n\right\| \\
&\le \alpha_n\left(\|f(y_n)\| + \left\|\frac{1}{n+1}\sum_{j=0}^{n}(PT)^j y_n\right\|\right).
\end{aligned} \tag{3.22}$$

Hence $\{y_{n+1} - (1/(n+1))\sum_{j=0}^{n}(PT)^j y_n\}$ converges strongly to 0. We next show that

$$\limsup_{n\to\infty}\langle z-y_n, z-f(z)\rangle \le 0. \tag{3.23}$$

Let $\{y_{n_j}\}$ be a subsequence of $\{y_n\}$ such that

$$\lim_{j\to\infty}\langle z-y_{n_j}, z-f(z)\rangle = \limsup_{n\to\infty}\langle z-y_n, z-f(z)\rangle, \tag{3.24}$$

and $y_{n_j} \rightharpoonup q \in C$. It follows by Lemmas 2.3 and 2.4 that $q \in F(PT) = F(T)$. By the inequality (3.1), we get

$$\limsup_{n\to\infty}\langle z-y_n, z-f(z)\rangle = \langle z-q, z-f(z)\rangle \le 0 \tag{3.25}$$

as required. Finally, we will show that $y_n \to z$. For each $n \geq 0$, we have

$$\begin{aligned}
||y_{n+1} - z||^2 &= ||y_{n+1} - z + \alpha_n(z - f(z)) - \alpha_n(z - f(z))||^2 \\
&\leq ||y_{n+1} - z + \alpha_n(z - f(z))||^2 + 2\alpha_n\langle y_{n+1} - z,\ f(z) - z\rangle \\
&= \left|\left|\alpha_n f(y_n) + (1-\alpha_n)\frac{1}{n+1}\sum_{j=0}^{n}(PT)^j y_n - (\alpha_n f(z) + (1-\alpha_n)z)\right|\right|^2 \\
&\quad + 2\alpha_n\langle y_{n+1} - z,\ f(z) - z\rangle \\
&= \left|\left|\alpha_n(f(y_n) - f(z)) + (1-\alpha_n)\frac{1}{n+1}\sum_{j=0}^{n}((PT)^j y_n - z)\right|\right|^2 \\
&\quad + 2\alpha_n\langle y_{n+1} - z,\ f(z) - z\rangle \\
&\leq \left[\alpha_n||f(y_n) - f(z)|| + (1-\alpha_n)\frac{1}{n+1}\sum_{j=0}^{n}||(PT)^j y_n - z||\right]^2 \\
&\quad + 2\alpha_n\langle y_{n+1} - z, f(z) - z\rangle \\
&\leq \left[\alpha_n\alpha||y_n - z|| + (1-\alpha_n)\frac{1}{n+1}\sum_{j=0}^{n}||y_n - z||\right]^2 \\
&\quad + 2\alpha_n\langle y_{n+1} - z, f(z) - z\rangle \\
&= (1 - \alpha_n(1-\alpha))^2||y_n - z||^2 + 2\alpha_n\langle y_{n+1} - z, f(z) - z\rangle \\
&\leq (1 - \alpha_n(1-\alpha))||y_n - z||^2 + 2\alpha_n\langle y_{n+1} - z, f(z) - z\rangle.
\end{aligned} \tag{3.26}$$

Now, let $\epsilon > 0$ be arbitrary. Then, by the fact (3.23), there exists a natural number N such that

$$\langle z - y_n,\ z - f(z)\rangle \leq \frac{\epsilon}{2} \quad \forall n \geq N. \tag{3.27}$$

From (3.26), we get

$$||y_{n+1} - z||^2 \leq (1 - \alpha_n(1-\alpha))||y_n - z||^2 + \alpha_n\epsilon. \tag{3.28}$$

By Lemma 2.5, the sequence $\{y_n\}$ converges strongly to a fixed point z of T. This completes the proof. □

By using the same arguments and techniques as those of Theorem 3.2, we have also the following main theorem.

Theorem 3.3. *Let C be a nonempty closed convex subset of a Hilbert space H, P the metric projection of H onto C, and $T : C \to H$ a nonexpansive nonself-mapping with $F(T) \neq \emptyset$. Let $\{\alpha_n\}$ be sequence in $[0,1]$ which satisfies $\lim_{n\to\infty}\alpha_n = 0$ and $\sum_{n=1}^{\infty}\alpha_n = \infty$. Then for a contraction mapping $f : C \to C$ with coefficient $\alpha \in (0,1)$, the sequence $\{z_n\}$ defined by (1.11) converges strongly to z, where z is the unique solution in $F(T)$ of the variational inequality (3.1).*

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This subject has been extensively studied in the past years for one-, two-, and three-dimensional space. Additionally, such dynamical systems can exhibit a very important and still unexplained phenomenon, called as the Fermi acceleration phenomenon. Basically, the phenomenon of Fermi acceleration (FA) is a process in which a classical particle can acquire unbounded energy from collisions with a heavy moving wall. This phenomenon was originally proposed by Enrico Fermi in 1949 as a possible explanation of the origin of the large energies of the cosmic particles. His original model was then modified and considered under different approaches and using many versions. Moreover, applications of FA have been of a large broad interest in many different fields of science including plasma physics, astrophysics, atomic physics, optics, and time-dependent billiard problems and they are useful for controlling chaos in Engineering and dynamical systems exhibiting chaos (both conservative and dissipative chaos).

We intend to publish in this special issue papers reporting research on time-dependent billiards. The topic includes both conservative and dissipative dynamics. Papers discussing dynamical properties, statistical and mathematical results, stability investigation of the phase space structure, the phenomenon of Fermi acceleration, conditions for having suppression of Fermi acceleration, and computational and numerical methods for exploring these structures and applications are welcome.

To be acceptable for publication in the special issue of Mathematical Problems in Engineering, papers must make significant, original, and correct contributions to one or more of the topics above mentioned. Mathematical papers regarding the topics above are also welcome.

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