

Research Article

Further Properties of β -Pascu Convex Functions of Order α

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We obtain several further properties of β -Pascu convex functions of order α which were recently introduced and studied by Ali et al. in (2006).

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1. Introduction

Let $\mathbb{N} := \{1, 2, \dots\}$, for $m, p \in \mathbb{N}$, $m \geq p + 1$, let $\mathcal{A}(p, m)$ be the class of all p -valent analytic functions $f(z) = z^p + \sum_{n=m}^{\infty} a_n z^n$ defined on the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ and let $\mathcal{A} := \mathcal{A}(1, 2)$.

Let $\mathcal{T}(p, m)$ be the subclass of $\mathcal{A}(p, m)$ consisting of functions of the form

$$f(z) = z^p - \sum_{n=m}^{\infty} a_n z^n, \quad a_n \geq 0 \text{ for } n \geq m, \quad (1.1)$$

and let $\mathcal{T} := \mathcal{T}(1, 2)$.

A function $f \in A(p, m)$ is β -Pascu convex function of order α if

$$\frac{1}{p} \operatorname{Re} \left\{ \frac{(1 - \beta)zf'(z) + (\beta/p)z(zf'(z))'}{(1 - \beta)f(z) + (\beta/p)zf'(z)} \right\} > \alpha \quad (\beta \geq 0, 0 \leq \alpha < 1). \quad (1.2)$$

We denote by $\mathcal{TPC}(p, m, \alpha, \beta)$ the subclass of $\mathcal{T}(p, m)$ consisting of β -Pascu convex function of order α . Clearly, $\mathcal{TS}^*(\alpha) := \mathcal{TPC}(1, 2, \alpha, 0)$ is the class of starlike functions with negative coefficients of order α and $\mathcal{TC}(\alpha) := \mathcal{TPC}(1, 2, \alpha, 1)$ is the class of convex functions with negative coefficients of order α (studied by Silverman [1]).

For the class $\mathcal{TPC}(p, m, \alpha, \beta)$, the following characterization was given by Ali et al. [2].

LEMMA 1.1. Let the function f be defined by (1.1). Then f is in the class $\mathcal{TPC}(p, m, \alpha, \beta)$ if and only if

$$\sum_{n=m}^{\infty} (n - p\alpha)[(1 - \beta)p + \beta n] a_n \leq p^2(1 - \alpha). \quad (1.3)$$

The result is sharp.

LEMMA 1.2. Let $f(z)$ be given by (1.1). If $f \in \mathcal{TPC}(p, m, \alpha, \beta)$, then

$$a_n \leq \frac{p^2(1 - \alpha)}{(n - p\alpha)[(1 - \beta)p + \beta n]} \quad (1.4)$$

with equality only for functions of the form

$$f_n(z) = z^p - \frac{p^2(1 - \alpha)}{(n - p\alpha)[(1 - \beta)p + \beta n]} z^n. \quad (1.5)$$

Many interesting properties such as coefficient estimate and distortion theorems for the class $\mathcal{TPC}(p, m, \alpha, \beta)$ were given by Ali et al. [2]. In the present sequel to these earlier works, we will derive several interesting properties and characteristic of the δ -neighborhood associated with the class $\mathcal{TPC}(p, m, \alpha, \beta)$.

2. Integral properties of the class $\mathcal{TPC}(p, m, \alpha, \beta)$

We recall the following definition of integral operator before we give integral properties of the class $\mathcal{TPC}(p, m, \alpha, \beta)$.

Let $\mathcal{I}_c : \mathcal{T}(p, m) \rightarrow \mathcal{T}(p, m)$ be integral operator defined by $g = \mathcal{I}_c(f)$, where $c \in (-p, \infty)$, $f \in \mathcal{T}(p, m)$ and

$$g(z) = \frac{c+p}{z^c} \int_0^z t^{c-1} f(t) dt. \quad (2.1)$$

We note that if $f \in \mathcal{T}(p, m)$ is a function of the form (1.1), then

$$g(z) = \mathcal{I}_c(f)(z) = z^p - \sum_{n=m}^{\infty} \frac{c+p}{c+n} a_n z^n. \quad (2.2)$$

THEOREM 2.1. Let $p, m \in \mathbb{N}$, $m \geq p+1$, $\alpha \in [0, 1)$, $\beta \in [0, \infty)$, and $c \in (-p, \infty)$. If $f \in \mathcal{TPC}(p, m, \alpha, \beta)$ and $g = \mathcal{I}_c(f)$, then $g \in \mathcal{TPC}(p, m, \lambda, \beta)$, where

$$\lambda = \lambda(p, m, \alpha, c) = 1 - \frac{(1 - \alpha)(c + p)(m - p)}{(m - p\alpha)(c + m) - (1 - \alpha)(c + p)p} \quad (2.3)$$

and $\alpha < \lambda$. The result is sharp.

Proof. From Lemma 1.1 and (2.2), we have $g \in \mathcal{TPC}(p, m, \lambda, \beta)$ if and only if

$$\sum_{n=m}^{\infty} \frac{(n - p\lambda)[(1 - \beta)p + \beta n](c + p)}{p^2(1 - \lambda)(c + n)} a_n \leq 1. \quad (2.4)$$

We find the largest λ such that (2.4) holds. We note that the inequalities

$$\frac{(n - p\lambda)[(1 - \beta)p + \beta n](c + p)}{p^2(1 - \lambda)(c + n)} \leq \frac{(n - p\alpha)[(1 - \beta)p + \beta n]}{p^2(1 - \alpha)} \quad (2.5)$$

imply (2.4), because $f \in \mathcal{TPC}(p, m, \alpha, \beta)$ and satisfy (1.3). But inequalities (2.5) are equivalent to

$$\frac{(n - p\lambda)(c + p)}{(1 - \lambda)(c + n)} \leq \frac{(n - p\alpha)}{(1 - \alpha)}. \quad (2.6)$$

Since $(n - p\alpha) > p(1 - \alpha)$ and $c + n > c + p$, we obtain $\lambda \leq \lambda(p, n, \alpha, c)$, where

$$\lambda(p, n, \alpha, c) = \frac{(n - p\alpha)(c + n) - (1 - \alpha)(c + p)n}{(n - p\alpha)(c + n) - (1 - \alpha)(c + p)p}. \quad (2.7)$$

Now we show that $\lambda(p, n, \alpha, c)$ is an increasing function of n , $n \geq m$. Indeed,

$$\lambda(p, n, \alpha, c) = 1 - (1 - \alpha)(c + p)E(p, n, \alpha, c), \quad (2.8)$$

where

$$E(p, n, \alpha, c) = \frac{(n - p)}{(n - p\alpha)(c + n) - (1 - \alpha)(c + p)p}, \quad (2.9)$$

and $\lambda(p, n, \alpha, c)$ increases when n increases if and only if $E(p, n, \alpha, c)$ is a strictly decreasing function of n .

Let $h(x) = E(p, x, \alpha, c)$, $x \in [m, \infty) \subset [p + 1, \infty)$, we have

$$h'(x) = -\frac{(x - p)^2}{[(x - p\alpha)(c + x) - (1 - \alpha)(c + p)p]^2} < 0. \quad (2.10)$$

We obtained

$$\lambda = \lambda(p, m, \alpha, c) \leq \lambda(p, n, \alpha, c), \quad n \geq m. \quad (2.11)$$

The result is sharp because

$$\mathcal{I}_c(f_\alpha) = f_\lambda, \quad (2.12)$$

where

$$\begin{aligned} f_\alpha(z) &= z^p - \frac{p^2(1 - \alpha)}{(m - p\alpha)[(1 - \beta)p + \beta m]} z^m, \\ f_\lambda(z) &= z^p - \frac{p^2(1 - \lambda)}{(m - p\lambda)[(1 - \beta)p + \beta m]} z^m \end{aligned} \quad (2.13)$$

are extremal functions of $\mathcal{TPC}(p, m, \alpha, \beta)$ and $\mathcal{TPC}(p, m, \lambda, \beta)$, respectively, and $\lambda = \lambda(p, m, \alpha, c)$.

Indeed, we have

$$\mathcal{G}_c(f_\alpha(z)) = z^p - \frac{p^2(1-\alpha)(c+p)}{(m-p\alpha)[(1-\beta)p+\beta m](c+m)} z^m. \quad (2.14)$$

We deduce

$$\frac{p^2(1-\lambda)}{(m-p\lambda)} = \frac{p^2(1-\alpha)(c+p)}{(m-p\alpha)(c+m)}, \quad (2.15)$$

and this implies (2.14).

From $\lambda = 1 - (1-\alpha)(c+p)(m-p)/((m-p\alpha)(c+m) - (1-\alpha)(c+p)p)$ we obtain $\lambda < 1$ and also $\lambda > \alpha$. Indeed,

$$\begin{aligned} \lambda - \alpha &= (1-\alpha) \left\{ 1 - \frac{(c+p)(m-p)}{(m-p\alpha)(c+m) - (1-\alpha)(c+p)p} \right\} \\ &= (1-\alpha) \frac{(m-p\alpha)(m-p)}{(m-p\alpha)(c+m) - (1-\alpha)(c+p)p} > 0. \end{aligned} \quad (2.16)$$

□

3. Integral means inequalities for the class $\mathcal{TPC}(p, m, \alpha, \beta)$

An analytic function g is said to be subordinate to an analytic function f (written $g < f$) if $g(z) = f(w(z))$, $z \in \mathbb{U}$, for some analytic function w with $|w(z)| \leq |z|$. In 1925, Littlewood [3] proved the following subordination result which will be required in our present investigation.

LEMMA 3.1. *If f and g are analytic in \mathbb{U} with $g < f$, then*

$$\int_0^{2\pi} |g(re^{i\theta})|^\delta d\theta \leq \int_0^{2\pi} |f(re^{i\theta})|^\delta d\theta, \quad (3.1)$$

where $\delta > 0$, $z = re^{i\theta}$, and $0 < r < 1$.

Applying Lemmas 1.1 and 3.1, we prove the following.

THEOREM 3.2. *Let $\delta > 0$. If $f \in \mathcal{TPC}(p, m, \alpha, \beta)$ and $f_m(z) = z^p - (p^2(1-\alpha)/(m-p\alpha)[(1-\beta)p+\beta m])z^m$, then for $z = re^{i\theta}$ and $0 < r < 1$,*

$$\int_0^{2\pi} |f(re^{i\theta})|^\delta d\theta \leq \int_0^{2\pi} |f_m(re^{i\theta})|^\delta d\theta. \quad (3.2)$$

Proof. Let

$$\begin{aligned} f(z) &= z^p - \sum_{n=m}^{\infty} a_n z^n, \quad a_n \geq 0, \quad n \geq m, \\ f_m(z) &= z^p - \frac{p^2(1-\alpha)}{(m-p\alpha)[(1-\beta)p+\beta m]} z^m, \end{aligned} \quad (3.3)$$

then we must show that

$$\int_0^{2\pi} \left| 1 - \sum_{n=m}^{\infty} a_n z^{n-p} \right|^\delta d\theta \leq \int_0^{2\pi} \left| 1 - \frac{p^2(1-\alpha)}{(m-p\alpha)[(1-\beta)p+\beta m]} z^{m-p} \right|^\delta d\theta. \quad (3.4)$$

By Lemma 3.1, it suffices to show that

$$1 - \sum_{n=m}^{\infty} a_n z^{n-p} < 1 - \frac{p^2(1-\alpha)}{(m-p\alpha)[(1-\beta)p+\beta m]} z^{m-p}. \quad (3.5)$$

Set

$$1 - \sum_{n=m}^{\infty} a_n z^{n-p} = 1 - \frac{p^2(1-\alpha)}{(m-p\alpha)[(1-\beta)p+\beta m]} w(z)^{m-p}. \quad (3.6)$$

From (3.6) and (1.3), we obtain

$$\begin{aligned} |w(z)|^{m-p} &= \left| \frac{(m-p\alpha)[(1-\beta)p+\beta m]}{p^2(1-\alpha)} \right| \left| \sum_{n=m}^{\infty} a_n z^{n-p} \right| \\ &\leq |z^{m-p}| \sum_{n=m}^{\infty} \frac{(n-p\alpha)[(1-\beta)p+\beta n]}{p^2(1-\alpha)} a_n \leq |z^{m-p}| \leq |z|. \end{aligned} \quad (3.7)$$

This completes the proof of the theorem. \square

The proof for the first derivative is similar.

THEOREM 3.3. *Let $\delta > 0$. If $f \in \mathcal{TPC}(p, m, \alpha, \beta)$ and $f_m(z) = z^p - (p^2(1-\alpha)/(m-p\alpha)[(1-\beta)p+\beta m])z^m$, then for $z = re^{i\theta}$ and $0 < r < 1$,*

$$\int_0^{2\pi} |f'(re^{i\theta})|^\delta d\theta \leq \int_0^{2\pi} |f'_m(re^{i\theta})|^\delta d\theta. \quad (3.8)$$

Proof. It suffices to show that

$$1 - \sum_{n=m}^{\infty} \frac{n}{p} a_n z^{n-p} < 1 - \frac{mp(1-\alpha)}{(m-p\alpha)[(1-\beta)p+\beta m]} z^{m-p}. \quad (3.9)$$

This follows because

$$\begin{aligned} |w(z)|^{m-p} &= \left| \sum_{n=m}^{\infty} \frac{(n-p\alpha)[(1-\beta)p+\beta n]}{p^2(1-\alpha)} a_n z^{n-p} \right| \\ &\leq |z|^{n-p} \sum_{n=m}^{\infty} \frac{(n-p\alpha)[(1-\beta)p+\beta n]}{p^2(1-\alpha)} a_n \leq |z|^{n-p} \leq |z|. \end{aligned} \quad (3.10)$$

\square

4. Neighborhoods of the class $\mathcal{TPC}(p, m, \alpha, \beta)$

For $f \in \mathcal{T}(p, m)$ and $\gamma \geq 0$, Frasin [4] defined

$$M_\gamma^q(f) = \left\{ g \in \mathcal{T}(p, m) : g(z) = z^p - \sum_{n=m}^{\infty} b_n z^n, \sum_{n=m}^{\infty} n^{q+1} |a_n - b_n| \leq \gamma \right\}, \quad (4.1)$$

which was called q - γ -neighborhood of f . So, for $e(z) = z$, we see that

$$M_\gamma^q(e) = \left\{ g \in \mathcal{T}(p, m) : g(z) = z^p - \sum_{n=m}^{\infty} b_n z^n, \sum_{n=m}^{\infty} n^{q+1} |b_n| \leq \gamma \right\}, \quad (4.2)$$

where q is a fixed positive integer. Note that $M_\gamma^0(f) \equiv N_\gamma(f)$ and $M_\gamma^1(f) \equiv M_\gamma(f)$. $N_\gamma(f)$ is called a γ -neighborhood of f by Ruscheweyh [5] and $M_\gamma(f)$ was defined by Silverman [6].

Now, we consider q - γ -neighborhood for function in the class $\mathcal{TPC}(p, m, \alpha, \beta)$.

THEOREM 4.1. *Let*

$$\gamma = \frac{m^{q+1} p^2 (1 - \alpha)}{(m - p\alpha)[(1 - \beta)p + \beta m]}, \quad (4.3)$$

then $\mathcal{TPC}(p, m, \alpha, \beta) \subset M_\gamma^q(e)$.

Proof. If $f \in \mathcal{TPC}(p, m, \alpha, \beta)$, then

$$\sum_{n=m}^{\infty} n^{q+1} a_n \leq \frac{m^{q+1} p^2 (1 - \alpha)}{(m - p\alpha)[(1 - \beta)p + \beta m]} = \gamma. \quad (4.4)$$

This gives that $\mathcal{TPC}(p, m, \alpha, \beta) \subset M_\gamma^q(e)$. □

Putting $p = 1$, $m = 2$ and $\beta = 0$ in Theorem 4.1, we have the following.

COROLLARY 4.2. $\mathcal{TS}^*(\alpha) \subset M_\gamma^q(e)$, where $\gamma = 2^{q+1}(1 - \alpha)/(2 - \alpha)$.

Putting $p = 1$, $m = 2$, and $\beta = 1$ in Theorem 4.1, we have the following.

COROLLARY 4.3. $\mathcal{TC}(\alpha) \subset M_\gamma^q(e)$, where $\gamma = 2^q(1 - \alpha)/(2 - \alpha)$.

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