

Research Article

Expression of a Tensor Commutation Matrix in Terms of the Generalized Gell-Mann Matrices

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We have expressed the tensor commutation matrix $n \otimes n$ as linear combination of the tensor products of the generalized Gell-Mann matrices. The tensor commutation matrices $3 \otimes 2$ and $2 \otimes 3$ have been expressed in terms of the classical Gell-Mann matrices and the Pauli matrices.

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1. Introduction

When we had worked on Raelina Andriambololona idea on the use of tensor product in Dirac equation [1, 2], we had met the unitary matrix

$$U_{2 \otimes 2} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad (1.1)$$

This matrix is frequently found in quantum information theory [3–5] where one writes, by using the Pauli matrices [3–5],

$$U_{2 \otimes 2} = \frac{1}{2} I_2 \otimes I_2 + \frac{1}{2} \sum_{i=1}^3 \sigma_i \otimes \sigma_i \quad (1.2)$$

with I_2 the 2×2 unit matrix. We call this matrix a tensor commutation matrix $2 \otimes 2$. The tensor commutation matrix $3 \otimes 3$ is expressed by using the Gell-Mann matrices under

the following form [6]:

$$U_{3\otimes 3} = \frac{1}{3}I_3 \otimes I_3 + \frac{1}{2} \sum_{i=1}^8 \lambda_i \otimes \lambda_i. \quad (1.3)$$

We have to talk a bit about different types of matrices because in the generalization of the above formulas, we will consider a commutation matrix as a matrix of fourth-order tensor and in expressing the commutation matrices $U_{3\otimes 2}$, $U_{2\otimes 3}$, at the last section, a commutation matrix will be considered as matrix of second-order tensor.

$\mathcal{M}_{m \times n}(\mathbb{C})$ denotes the set of $m \times n$ matrices whose elements are complex numbers.

2. Tensor product of matrices

2.1. Matrices. If the elements of a matrix are considered as the components of a second-order tensor, we adopt the habitual notation for a matrix, without parentheses inside, whereas if the elements of the matrix are, for instance, considered as the components of sixth-order tensor, three times covariant and three times contravariant, then we represent the matrix of the following way, for example:

$$M = \left(\left(\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 1 \\ 3 & 2 \end{pmatrix} \right) \quad \left(\begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix} \quad \begin{pmatrix} 7 & 8 \\ 9 & 0 \end{pmatrix} \right) \right. \\ \left(\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \right) \quad \left(\begin{pmatrix} 3 & 4 \\ 5 & 6 \end{pmatrix} \quad \begin{pmatrix} 9 & 8 \\ 7 & 6 \end{pmatrix} \right) \\ \left(\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 \\ 3 & 2 \end{pmatrix} \right) \quad \left(\begin{pmatrix} 5 & 4 \\ 3 & 2 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix} \right) \\ \left(\begin{pmatrix} 4 & 5 \\ 1 & 6 \end{pmatrix} \quad \begin{pmatrix} 1 & 7 \\ 8 & 9 \end{pmatrix} \right) \quad \left(\begin{pmatrix} 3 & 4 \\ 5 & 6 \end{pmatrix} \quad \begin{pmatrix} 7 & 8 \\ 9 & 0 \end{pmatrix} \right) \\ \left(\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \quad \begin{pmatrix} 9 & 8 \\ 7 & 6 \end{pmatrix} \right) \quad \left(\begin{pmatrix} 9 & 8 \\ 7 & 6 \end{pmatrix} \quad \begin{pmatrix} 5 & 4 \\ 3 & 2 \end{pmatrix} \right) \\ \left(\begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix} \quad \begin{pmatrix} 5 & 4 \\ 3 & 2 \end{pmatrix} \right) \quad \left(\begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix} \quad \begin{pmatrix} 3 & 4 \\ 5 & 6 \end{pmatrix} \right) \Bigg),$$

$$M = (M_{j_1 j_2 j_3}^{i_1 i_2 i_3}) \quad \begin{array}{l} i_1 i_2 i_3 = 111, 112, 121, 122, 211, 212, 221, 222, 311, 312, 321, 322 \text{ row indices,} \\ j_1 j_2 j_3 = 111, 112, 121, 122, 211, 212, 221, 222 \text{ column indices.} \end{array} \quad (2.1)$$

The first indices i_1 and j_1 are the indices of the outside parenthesis which we call the first-order parenthesis; the second indices i_2 and j_2 are the indices of the next parentheses which we call the second-order parentheses; the third indices i_3 and j_3 are the indices of the most interior parentheses, of this example, which we call third-order parentheses. So, for instance, $M_{121}^{321} = 5$.

If we delete the third-order parenthesis, then the elements of the matrix M are considered as the components of a fourth-order tensor, twice contravariant and twice covariant.

A matrix is a diagonal matrix if deleting the interior parentheses, we have a habitual diagonal matrix.

A matrix is a symmetric (resp., antisymmetric) matrix if deleting the interior parentheses, we have a habitual symmetric (resp., antisymmetric) matrix.

We identify one matrix to another matrix if after deleting the interior parentheses, they are the same matrices.

2.2. Tensor product of matrices

Definition 2.1. Consider $A = (A_j^i) \in \mathcal{M}_{m \times n}(\mathbb{C})$, $B = (B_j^i) \in \mathcal{M}_{p \times r}(\mathbb{C})$. The matrix defined by

$$A \otimes B = \begin{pmatrix} A_1^1 B & \dots & A_j^1 B & \dots & A_n^1 B \\ \vdots & & \vdots & & \vdots \\ A_1^i B & \dots & A_j^i B & \dots & A_n^i B \\ \vdots & & \vdots & & \vdots \\ A_1^m B & \dots & A_j^m B & \dots & A_n^m B \end{pmatrix} \quad (2.2)$$

is called the tensor product of the matrix A by the matrix B ,

$$\begin{aligned} A \otimes B &\in \mathcal{M}_{mp \times nr}(\mathbb{C}), \\ A \otimes B &= (C_{j_1 j_2}^{i_1 i_2}) = (A_{j_1}^{i_1} B_{j_2}^{i_2}), \end{aligned} \quad (2.3)$$

(cf., e.g., [3]) where, $i_1 i_2$ are row indices and $j_1 j_2$ are column indices.

3. Generalized Gell-Mann matrices

Let us fix $n \in \mathbb{N}$, $n \geq 2$ for all continuations. The generalized Gell-Mann matrices or $n \times n$ -Gell-Mann matrices are the traceless Hermitian $n \times n$ matrices $\Lambda_1, \Lambda_2, \dots, \Lambda_{n^2-1}$ which satisfy the relation $Tr(\Lambda_i \Lambda_j) = 2\delta_{ij}$, for all $i, j \in \{1, 2, \dots, n^2 - 1\}$, where $\delta_{ij} = \delta^{ij} = \delta_j^i$ is the Kronecker symbol [7].

However, for the demonstration of Theorem 4.3, denote, for $1 \leq i < j \leq n$, the $C_n^2 = (n!/2!(n-2)!) n \times n$ -Gell-Mann matrices which are symmetric with all elements 0 except the i th row j th column and the j th row i th column which are equal to 1, by $\Lambda^{(ij)}$; the $C_n^2 = (n!/2!(n-2)!) n \times n$ -Gell-Mann matrices which are antisymmetric with all elements are 0 except the i th row j th column which is equal to $-i$ and the j th row i th column which

is equal to i , by $\Lambda^{[ij]}$ and by $\Lambda^{(d)}$, $1 \leq d \leq n-1$, the following $(n-1) \times n$ -Gell-Mann matrices are diagonal:

$$\Lambda^{(1)} = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & -1 & & \\ & & 0 & \vdots \\ \vdots & & \ddots & \\ 0 & \dots & & 0 \end{pmatrix},$$

$$\Lambda^{(2)} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & & \\ & & -2 & \vdots \\ \vdots & & 0 & \\ 0 & \dots & & 0 \end{pmatrix}, \dots, \Lambda^{(n-1)} = \frac{1}{\sqrt{C_n^2}} \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & & \\ & & 1 & \vdots \\ \vdots & & \ddots & \\ 0 & \dots & 1 & -(n-1) \end{pmatrix}. \quad (3.1)$$

For $n = 2$, we have the Pauli matrices.

4. Tensor commutation matrices

Definition 4.1. For $p, q \in \mathbb{N}$, $p \geq 2$, $q \geq 2$, call the tensor commutation matrix $p \otimes q$ the permutation matrix $U_{p \otimes q} \in \mathcal{M}_{pq \times pq}(\mathbb{C})$ formed by 0 and 1, verifying the property

$$U_{p \otimes q} \cdot (a \otimes b) = b \otimes a \quad (4.1)$$

for all $a \in \mathcal{M}_{p \times 1}(\mathbb{C})$, $b \in \mathcal{M}_{q \times 1}(\mathbb{C})$.

Considering $U_{p \otimes q}$ as a matrix of a second-order tensor, one can construct it by using the following rule [6].

Rule 4.2. Let us start in putting 1 at first row and first column, after that let us pass into second column in going down at the rate of p rows and put 1 at this place, then pass into third column in going down at the rate of p rows and put 1, and so on until there are only for us $p-1$ rows for going down (then we have obtained number of 1 : q). Then pass into the next column which is the $(q+1)$ th column, put 1 at the second row of this column and repeat the process until we have only $p-2$ rows for going down (then we have obtained number of 1 : $2q$). After that pass into the next column which is the $(2q+1)$ th column, put 1 at the third row of this column and repeat the process until we have only $p-3$ rows for going down (then we have obtained number of 1 : $3q$). Continuing in this way, we will have that the element at $p \times q$ th row and $p \times q$ th column is 1. The other elements are 0.

THEOREM 4.3. *One has*

$$U_{n \otimes n} = \frac{1}{n} I_n \otimes I_n + \frac{1}{2} \sum_{i=1}^{n^2-1} \Lambda_i \otimes \Lambda_i. \quad (4.2)$$

Proof. One has

$$\begin{aligned} I_n \otimes I_n &= (\delta_{j_1 j_2}^{i_1 i_2}) = (\delta_{j_1}^{i_1} \delta_{j_2}^{i_2}), \\ U_{n \otimes n} &= (\delta_{j_2}^{i_1} \delta_{j_1}^{i_2}), \end{aligned} \quad (4.3)$$

where, $i_1 i_2$ are row indices and $j_1 j_2$ are column indices [3].

Consider at first the C_n^2 symmetric $n \times n$ Gell-Mann matrices which can be written as

$$\begin{aligned} \Lambda^{(ij)} &= (\Lambda^{(ij)}_k^l)_{1 \leq l \leq n, 1 \leq k \leq n} \\ &= (\delta^{il} \delta_k^j)_{1 \leq l \leq n, 1 \leq k \leq n} + (\delta^{jl} \delta_k^i)_{1 \leq l \leq n, 1 \leq k \leq n} \\ &= (\delta^{il} \delta_k^j + \delta^{jl} \delta_k^i)_{1 \leq l \leq n, 1 \leq k \leq n}. \end{aligned} \quad (4.4)$$

Then

$$\Lambda^{(ij)} \otimes \Lambda^{(ij)} = \left((\Lambda^{(ij)} \otimes \Lambda^{(ij)})_{k_1 k_2}^{l_1 l_2} \right) = (\delta^{il_1} \delta_{k_1}^j + \delta^{jl_1} \delta_{k_1}^i) (\delta^{il_2} \delta_{k_2}^j + \delta^{jl_2} \delta_{k_2}^i), \quad (4.5)$$

where $l_1 l_2$ are row indices and $k_1 k_2$ are column indices.

That is,

$$(\Lambda^{(ij)} \otimes \Lambda^{(ij)})_{k_1 k_2}^{l_1 l_2} = \delta^{il_1} \delta_{k_1}^j \delta^{il_2} \delta_{k_2}^j + \delta^{il_1} \delta_{k_1}^j \delta^{jl_2} \delta_{k_2}^i + \delta^{jl_1} \delta_{k_1}^i \delta^{il_2} \delta_{k_2}^j + \delta^{jl_1} \delta_{k_1}^i \delta^{jl_2} \delta_{k_2}^i. \quad (4.6)$$

The C_n^2 antisymmetric $n \times n$ Gell-Mann matrices can be written as

$$\Lambda^{[ij]} = (\Lambda^{[ij]}_k^l)_{1 \leq l \leq n, 1 \leq k \leq n} = (-i \delta^{il} \delta_k^j + i \delta^{jl} \delta_k^i)_{1 \leq l \leq n, 1 \leq k \leq n}. \quad (4.7)$$

Then

$$\begin{aligned} \Lambda^{[ij]} \otimes \Lambda^{[ij]} &= \left((\Lambda^{[ij]} \otimes \Lambda^{[ij]})_{k_1 k_2}^{l_1 l_2} \right), \\ (\Lambda^{[ij]} \otimes \Lambda^{[ij]})_{k_1 k_2}^{l_1 l_2} &= -\delta^{il_1} \delta_{k_1}^j \delta^{il_2} \delta_{k_2}^j + \delta^{il_1} \delta_{k_1}^j \delta^{jl_2} \delta_{k_2}^i + \delta^{jl_1} \delta_{k_1}^i \delta^{il_2} \delta_{k_2}^j - \delta^{jl_1} \delta_{k_1}^i \delta^{jl_2} \delta_{k_2}^i, \\ \sum_{1 \leq i < j \leq n} (\Lambda^{(ij)} \otimes \Lambda^{(ij)})_{k_1 k_2}^{l_1 l_2} &+ \sum_{1 \leq i < j \leq n} (\Lambda^{[ij]} \otimes \Lambda^{[ij]})_{k_1 k_2}^{l_1 l_2} \\ &= 2 \sum_{1 \leq i < j \leq n} (\delta^{il_1} \delta_{k_1}^j \delta^{jl_2} \delta_{k_2}^i + \delta^{jl_1} \delta_{k_1}^i \delta^{il_2} \delta_{k_2}^j) = 2 \sum_{i \neq j} \delta^{il_1} \delta_{k_1}^j \delta^{jl_2} \delta_{k_2}^i \end{aligned} \quad (4.8)$$

is the $l_1 l_2$ th row, $k_1 k_2$ th column of the matrix

$$\sum_{1 \leq i < j \leq n} \Lambda^{(ij)} \otimes \Lambda^{(ij)} + \sum_{1 \leq i < j \leq n} \Lambda^{[ij]} \otimes \Lambda^{[ij]}. \quad (4.9)$$

Now, consider the diagonal $n \times n$ Gell-Mann matrices. Let $d \in \mathbb{N}$, $1 \leq d \leq n-1$,

$$\Lambda^{(d)} = \frac{1}{\sqrt{C_{d+1}^2}} \left(\delta_k^l \sum_{p=1}^d \delta_k^p - d \delta_k^l \delta_k^{d+1} \right) \quad (4.10)$$

and the $l_1 l_2$ th row, $k_1 k_2$ th of the matrix $\Lambda^{(d)} \otimes \Lambda^{(d)}$ is

$$\begin{aligned} (\Lambda^{(d)} \otimes \Lambda^{(d)})_{k_1 k_2}^{l_1 l_2} &= \frac{1}{C_{d+1}^2} \delta_{k_1}^{l_1} \delta_{k_2}^{l_2} \left(\sum_{q=1}^d \sum_{p=1}^d \delta_{k_1}^q \delta_{k_2}^p \right) - \frac{1}{C_{d+1}^2} \delta_{k_1}^{l_1} \delta_{k_2}^{l_2} \left(d \delta_{k_2}^{d+1} \sum_{p=1}^d \delta_{k_1}^p \right) \\ &\quad - \frac{1}{C_{d+1}^2} \delta_{k_1}^{l_1} \delta_{k_2}^{l_2} \left(d \delta_{k_1}^{d+1} \sum_{p=1}^d \delta_{k_2}^p \right) + \frac{1}{C_{d+1}^2} \delta_{k_1}^{l_1} \delta_{k_2}^{l_2} (d^2 \delta_{k_1}^{d+1} \delta_{k_2}^{d+1}), \end{aligned} \quad (4.11)$$

$\Lambda^{(d)} \otimes \Lambda^{(d)}$ is a diagonal matrix, so all that we have to do is to calculate the elements on the diagonal where $l_1 = k_1$ and $l_2 = k_2$. Then,

$$\begin{aligned} \sum_{d=1}^{n-1} (\Lambda^{(d)} \otimes \Lambda^{(d)})_{k_1 k_2}^{l_1 l_2} &= \sum_{d=1}^{n-1} \frac{1}{C_{d+1}^2} \left(\sum_{q=1}^d \delta_{k_1}^q \right) \left(\sum_{p=1}^d \delta_{k_2}^p \right) - \sum_{d=1}^{n-1} \frac{1}{C_{d+1}^2} d \delta_{k_2}^{d+1} \sum_{p=1}^d \delta_{k_1}^p \\ &\quad - \sum_{d=1}^{n-1} \frac{1}{C_{d+1}^2} d \delta_{k_1}^{d+1} \sum_{p=1}^d \delta_{k_2}^p + \sum_{d=1}^{n-1} \frac{1}{C_{d+1}^2} d^2 \delta_{k_1}^{d+1} \delta_{k_2}^{d+1} \end{aligned} \quad (4.12)$$

is the $l_1 l_2$ th row, $k_1 k_2$ th column of the diagonal matrix $\sum_{d=1}^{n-1} \Lambda^{(d)} \otimes \Lambda^{(d)}$ with $l_1 = k_1$ and $l_2 = k_2$.

Let us distinguish two cases.

Case 1. $k_1 \neq 1$ or $k_2 \neq 1$.

Case 1.1. $k_1 \neq k_2$.

If $k_1 < k_2$,

$$\sum_{d=1}^{n-1} (\Lambda^{(d)} \otimes \Lambda^{(d)})_{k_1 k_2}^{l_1 l_2} = \sum_{d=k_2}^{n-1} \frac{1}{C_{d+1}^2} - \frac{k_2 - 1}{C_{k_2}^2} = 2 \left[\sum_{d=k_2}^{n-1} \left(\frac{1}{d} - \frac{1}{d+1} \right) - \frac{1}{k_2} \right] = -\frac{2}{n}. \quad (4.13)$$

Similarly, if $k_1 > k_2$,

$$\sum_{d=1}^{n-1} (\Lambda^{(d)} \otimes \Lambda^{(d)})_{k_1 k_2}^{l_1 l_2} = -\frac{2}{n}. \quad (4.14)$$

Case 1.2. $k_1 = k_2 \neq 1$:

$$\sum_{d=1}^{n-1} (\Lambda^{(d)} \otimes \Lambda^{(d)})_{k_1 k_2}^{l_1 l_2} = \sum_{d=k_2}^{n-1} \frac{1}{C_{d+1}^2} + \frac{(k_2 - 1)^2}{C_{k_2}^2} = \frac{2}{k_2} - \frac{2}{n} + \frac{(k_2 - 1)^2}{C_{k_2}^2} = 2 - \frac{2}{n}. \quad (4.15)$$

Case 2. $k_1 = k_2 = 1$:

$$\sum_{d=1}^{n-1} (\Lambda^{(d)} \otimes \Lambda^{(d)})_{k_1 k_2}^{l_1 l_2} = \sum_{d=1}^{n-1} \frac{1}{C_{d+1}^2} = 2 - \frac{2}{n}. \quad (4.16)$$

We can condense these cases in one formula as

$$\sum_{d=1}^{n-1} (\Lambda^{(d)} \otimes \Lambda^{(d)})_{k_1 k_2}^{l_1 l_2} = -\frac{2}{n} \delta_{k_1}^{l_1} \delta_{k_2}^{l_2} + 2 \sum_{i=1}^n \delta^{il_1} \delta_{k_1}^i \delta^{il_2} \delta_{k_2}^i, \quad (4.17)$$

which yields the diagonal of the diagonal matrix $\sum_{d=1}^{n-1} \Lambda^{(d)} \otimes \Lambda^{(d)}$.

For all the $n \times n$ Gell-Mann matrices, we have

$$\begin{aligned} \sum_{1 \leq i < j \leq n} (\Lambda^{(ij)} \otimes \Lambda^{(ij)})_{k_1 k_2}^{l_1 l_2} + \sum_{1 \leq i < j \leq n} (\Lambda^{[ij]} \otimes \Lambda^{[ij]})_{k_1 k_2}^{l_1 l_2} + \sum_{d=1}^{n-1} (\Lambda^{(d)} \otimes \Lambda^{(d)})_{k_1 k_2}^{l_1 l_2} \\ = -\frac{2}{n} \delta_{k_1}^{l_1} \delta_{k_2}^{l_2} + 2 \sum_{i=1}^n \delta^{il_1} \delta_{k_1}^i \delta^{il_2} \delta_{k_2}^i + 2 \sum_{i \neq j} \delta^{il_1} \delta_{k_1}^j \delta^{il_2} \delta_{k_2}^i \\ = -\frac{2}{n} \delta_{k_1}^{l_1} \delta_{k_2}^{l_2} + 2 \sum_{j=1}^n \sum_{i=1}^n \delta^{il_1} \delta_{k_1}^j \delta^{il_2} \delta_{k_2}^i \\ = -\frac{2}{n} \delta_{k_1}^{l_1} \delta_{k_2}^{l_2} + 2 \delta_{k_2}^{l_1} \delta_{k_1}^{l_2} \end{aligned} \quad (4.18)$$

for all $l_1, l_2, k_1, k_2 \in \{1, 2, \dots, n\}$.

Hence, by using (4.3),

$$\sum_{i=1}^{n^2-1} \Lambda_i \otimes \Lambda_i = -\frac{2}{n} I_n \otimes I_n + 2 U_{n \otimes n} \quad (4.19)$$

and the theorem is proved. \square

5. Expression of $U_{3 \otimes 2}$ and $U_{2 \otimes 3}$

In this section, we derive formulas for $U_{3 \otimes 2}$ and $U_{2 \otimes 3}$, naturally in terms of the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (5.1)$$

and the Gell-Mann matrices

$$\begin{aligned}\lambda_1 &= \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, & \lambda_2 &= \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, & \lambda_3 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\ \lambda_4 &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, & \lambda_5 &= \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, & \lambda_6 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \\ \lambda_7 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, & \lambda_8 &= \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.\end{aligned}\quad (5.2)$$

For $r \in \mathbb{N}^*$, define $E_{ij}^{(r)}$ as the elementary $r \times r$ matrix whose elements are zeros except the i th row and j th column which is equal to 1. We construct $U_{3 \otimes 2}$ by using Rule 4.2, and then we have

$$U_{3 \otimes 2} = E_{11}^{(6)} + E_{23}^{(6)} + E_{35}^{(6)} + E_{42}^{(6)} + E_{54}^{(6)} + E_{66}^{(6)}. \quad (5.3)$$

Take

$$E_{11}^{(6)} = E_{11}^{(3)} \otimes E_{11}^{(2)}. \quad (5.4)$$

Let

$$E_{11}^{(3)} = \alpha_0 I_3 + \alpha_3 \lambda_3 + \alpha_8 \lambda_8 \quad (5.5)$$

with $\alpha_0, \alpha_3, \alpha_8 \in \mathbb{C}$, then

$$\begin{aligned}\alpha_0 &= \frac{1}{3}, & \alpha_3 &= \frac{1}{2}, & \alpha_8 &= \frac{\sqrt{3}}{6}, \\ E_{11}^{(3)} &= \frac{1}{3} I_3 + \frac{1}{2} \lambda_3 + \frac{\sqrt{3}}{6} \lambda_8.\end{aligned}\quad (5.6)$$

Let

$$E_{11}^{(2)} = \beta_0 I_2 + \beta_3 \sigma_3 \quad (5.7)$$

with $\beta_0, \beta_3 \in \mathbb{C}$, then

$$\begin{aligned}\beta_0 &= \frac{1}{2}, & \beta_3 &= \frac{1}{2}, \\ E_{11}^{(2)} &= \frac{1}{2} I_2 + \frac{1}{2} \sigma_3.\end{aligned}\quad (5.8)$$

So we have

$$E_{11}^{(6)} = \left(\frac{1}{3} I_3 + \frac{1}{2} \lambda_3 + \frac{\sqrt{3}}{6} \lambda_8 \right) \otimes \left(\frac{1}{2} I_2 + \frac{1}{2} \sigma_3 \right). \quad (5.9)$$

In a similar way, we have

$$\begin{aligned}
E_{23}^{(6)} &= \left(\frac{1}{2}\lambda_1 + \frac{i}{2}\lambda_2 \right) \otimes \left(\frac{1}{2}\sigma_1 - \frac{i}{2}\sigma_2 \right), \\
E_{35}^{(6)} &= \left(\frac{1}{2}\lambda_6 + \frac{i}{2}\lambda_7 \right) \otimes \left(\frac{1}{2}I_2 + \frac{1}{2}\sigma_3 \right), \\
E_{42}^{(6)} &= \left(\frac{1}{2}\lambda_1 - \frac{i}{2}\lambda_2 \right) \otimes \left(\frac{1}{2}I_2 - \frac{1}{2}\sigma_3 \right), \\
E_{54}^{(6)} &= \left(\frac{1}{2}\lambda_6 - \frac{i}{2}\lambda_7 \right) \otimes \left(\frac{1}{2}\sigma_1 + \frac{i}{2}\sigma_2 \right), \\
E_{66}^{(6)} &= \left(\frac{1}{3}I_3 - \frac{\sqrt{3}}{3}\lambda_8 \right) \otimes \left(\frac{1}{2}I_2 - \frac{1}{2}\sigma_3 \right).
\end{aligned} \tag{5.10}$$

Hence

$$\begin{aligned}
U_{3\otimes 2} &= \left(\frac{1}{3}I_3 + \frac{1}{2}\lambda_3 + \frac{\sqrt{3}}{6}\lambda_8 \right) \otimes \left(\frac{1}{2}I_2 + \frac{1}{2}\sigma_3 \right) + \left(\frac{1}{2}\lambda_1 + \frac{i}{2}\lambda_2 \right) \otimes \left(\frac{1}{2}\sigma_1 - \frac{i}{2}\sigma_2 \right) \\
&\quad + \left(\frac{1}{2}\lambda_6 + \frac{i}{2}\lambda_7 \right) \otimes \left(\frac{1}{2}I_2 + \frac{1}{2}\sigma_3 \right) + \left(\frac{1}{2}\lambda_1 - \frac{i}{2}\lambda_2 \right) \otimes \left(\frac{1}{2}I_2 - \frac{1}{2}\sigma_3 \right) \\
&\quad + \left(\frac{1}{2}\lambda_6 - \frac{i}{2}\lambda_7 \right) \otimes \left(\frac{1}{2}\sigma_1 + \frac{i}{2}\sigma_2 \right) + \left(\frac{1}{3}I_3 - \frac{\sqrt{3}}{3}\lambda_8 \right) \otimes \left(\frac{1}{2}I_2 - \frac{1}{2}\sigma_3 \right).
\end{aligned} \tag{5.11}$$

In an analogous way,

$$\begin{aligned}
U_{2\otimes 3} &= \left(\frac{1}{2}I_2 + \frac{1}{2}\sigma_3 \right) \otimes \left(\frac{1}{3}I_3 + \frac{1}{2}\lambda_3 + \frac{\sqrt{3}}{6}\lambda_8 \right) + \left(\frac{1}{2}\sigma_1 + \frac{i}{2}\sigma_2 \right) \otimes \left(\frac{1}{2}\lambda_1 - \frac{i}{2}\lambda_2 \right) \\
&\quad + \left(\frac{1}{2}I_2 + \frac{1}{2}\sigma_3 \right) \otimes \left(\frac{1}{2}\lambda_6 - \frac{i}{2}\lambda_7 \right) + \left(\frac{1}{2}I_2 - \frac{1}{2}\sigma_3 \right) \otimes \left(\frac{1}{2}\lambda_1 + \frac{i}{2}\lambda_2 \right) \\
&\quad + \left(\frac{1}{2}\sigma_1 - \frac{i}{2}\sigma_2 \right) \otimes \left(\frac{1}{2}\lambda_6 + \frac{i}{2}\lambda_7 \right) + \left(\frac{1}{2}I_2 - \frac{1}{2}\sigma_3 \right) \otimes \left(\frac{1}{3}I_3 - \frac{\sqrt{3}}{3}\lambda_8 \right).
\end{aligned} \tag{5.12}$$

One can develop these formulas in employing the distributivity of the tensor product.

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