THE INTEGRAL EQUATION METHODS FOR THE PERTURBED HELMHOLTZ EIGENVALUE PROBLEMS

ABDESSATAR KHELIFI

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It is well known that the main difficulty in solving eigenvalue problems under shape deformation relates to the continuation of multiple eigenvalues of the unperturbed configuration. These eigenvalues may evolve, under shape deformation, as separated, distinct eigenvalues. In this paper, we address the integral equation method in the evaluation of eigenfunctions and the corresponding eigenvalues of the two-dimensional Laplacian operator under boundary variations of the domain. Using surface potentials, we show that the eigenvalues are the characteristic values of meromorphic operator-valued functions.

1. Introduction

The properties of eigenvalue problems under shape deformation have been the subject of comprehensive studies [7, 13] and the area continues to carry great importance up to now [14]. A substantial portion of these investigations is related to smoothness properties of eigenfunctions with respect to boundary perturbations. Recently, Bruno and Reitich have presented in [4] explicit constructions of high-order boundary perturbation expansions for eigenelements. Their algorithm is based on certain properties of joint analytic dependence on the boundary perturbations and spatial variables of the eigenfunctions. The main difficulty in solving eigenvalue problems relates to the continuation of multiple eigenvalues of the unperturbed configuration. These eigenvalues may evolve, under shape deformation, as separated, distinct eigenvalues, and this splitting may only become apparent at high orders in their Taylor expansions.

In this paper, we use the technique of integral equation to evaluate the analyticity properties and asymptotic expansions of the eigenfunctions and the eigenvalues of the Laplacian operator under boundary variations of the domain of definition. Using surface potentials, we show that the eigenvalues are the characteristic values of meromorphic operator-valued functions which are of Fredholm type with index 0. We then proceed using the generalized Rouché's theorem [6] and the result found in [10] to construct their complete asymptotic expressions. Our approach concerning the question of analytic dependence and asymptotic expansion is based on a holomorphic formulation of the boundary integral equations and its characteristic problem version.

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Our analysis and uniform asymptotic formulas of the eigenfunctions, which are represented as sum of a single-layer potential and of a double-layer potential involving the Green's function, are considerably different from that in [9]. We would here like to find an efficient and accurate method, different from what we have presented in [2]. Powerful techniques from the theory of meromorphic operator-valued functions and careful asymptotic analysis of integral kernels are combined for solving this problem. Similar results can be obtained when considering the homogeneous Neumann boundary condition with only minor modifications of the arguments and techniques presented in this work. We just replace the Dirichlet Green's function by the Neumann Green's function in the integral representation and in a way completely similar, we establish our asymptotic formulas.

2. Problem formulation

Let $\gamma(t):[0,1]\to\mathbb{R}^2$ and $\beta(t):[0,1]\to\mathbb{R}^2$ be two analytic, 1-periodic functions satisfying $|\gamma'(t)|>C$ for all $t\in[0,1]$, where C is a positive constant. We introduce

$$\gamma_\epsilon(t)=\gamma(t)+\epsilon\beta(t),\quad \epsilon\in\mathbb{R}. \tag{2.1}$$

With this definition, $(t,\epsilon)\mapsto\gamma_\epsilon(t)$ is an analytic function on $[0,1]\times\mathbb{R}$, 1-periodic in the variable t, and satisfies

$$\left|\gamma_\epsilon'(t)\right|>\frac{C}{2} \tag{2.2}$$

for any ϵ such that $2|\epsilon||\beta'(t)|_{\mathscr{C}^0([0,1])}<C$. We consider the bounded domain Ω_ϵ in $\mathbb{R}^2$ with boundary $\partial\Omega_\epsilon$ parameterized by the function $\gamma_\epsilon(t)$:

$$\partial\Omega_\epsilon=\{\gamma_\epsilon(t),\ t\in[0,1]\}. \tag{2.3}$$

The outward unit normal to $\partial\Omega_\epsilon$ is denoted by ν_ϵ.

In this paper, we deal with the asymptotics of eigenvalues and eigenfunctions associated to the following eigenvalue problem:

$$\Delta u(\epsilon)+\lambda^2(\epsilon)u(\epsilon)=0\quad\text{in }\Omega_\epsilon,\qquad u(\epsilon)=0\quad\text{on }\partial\Omega_\epsilon. \tag{2.4}$$

It is well known that the operator $-\Delta$ on $L^2(\Omega_\epsilon)$ with domain $H^2(\Omega_\epsilon)\cap H_0^1(\Omega_\epsilon)$ is selfadjoint with compact resolvent. Consequently, its spectrum only consists of isolated, real, and positive eigenvalues with finite multiplicity, and there are corresponding eigenfunctions which build an orthonormal basis of $L^2(\Omega_\epsilon)$.

Let $\lambda_0^2>0$ denote an eigenvalue of the eigenvalue problem (2.4) for $\epsilon=0$ with geometric multiplicity $m\geq1$. There exists a small constant $r_0>0$ such that λ_0^2 is the unique eigenvalue of (2.4) for $\epsilon=0$ in the set $\{\lambda^2,\lambda\in D_{r_0}(\lambda_0)\}$, where $D_{r_0}(\lambda_0)$ is a disc of center λ_0 and radius r_0. We define the λ_0-group as the whole perturbed eigenvalues of (2.4) for $\epsilon>0$ generated by splitting from λ_0. The following analyticity result is well known [7].

Theorem 2.1. *There exists $\epsilon_0>0$ such that for $|\epsilon|<\epsilon_0$, the λ_0-group consists of m eigenvalues, $\lambda_i(\epsilon)$, $i=1,\dots,m$ (repeated according to their multiplicity). Moreover, these $\lambda_i(\epsilon)$ are*

analytic functions with respect to ϵ satisfying $\lambda_i(0) = \lambda_0$, $i = 1,\dots,m$. The normalized eigenfunctions associated to the λ_0-group of eigenvalues are analytic with respect to ϵ and their values at 0 are m linearly independent solutions of the unperturbed eigenvalue problem.

Classical regularity results and the previous theorem imply that the eigenfunctions associated to the λ_0-group of eigenvalues are separately analytic in the small parameter ϵ and the spatial variable x. Using an integral equation technique, we will also establish the joint analytic dependence of these functions with respect to (x,ϵ). As it is well known, joint analyticity does not follow from separate real analyticity properties.

3. Integral equation method

The use of integral equations is a convenient tool for a number of investigations [5]. We now develop a boundary integral formulation for solving the eigenvalue problem (2.4). We use this method to characterize the eigenvalue and the normal derivative of the eigenfunction as characteristic value and root function of some operator-valued function. This characterization is the key point in deriving our results and asymptotic formulae.

Let u be the solution to the Helmholtz equation:

$$\Delta u + \lambda^2 u = 0, \quad \text{in } \mathbb{R}^2. \tag{3.1}$$

We begin by defining the outgoing Green's function $G(x,y)$ as the solution of

$$\Delta_x G(x,y) + \lambda^2 G(x,y) = -\delta_y(x), \quad \text{in } \mathbb{R}^2, \tag{3.2}$$

with the radiation condition as $|x| \to +\infty$:

$$\left| \frac{\partial G}{\partial |x|} - i\lambda G \right| = O\left(\frac{1}{|x|}\right). \tag{3.3}$$

In fact, G is explicitly given as

$$G(x,y) = \frac{i}{4} H_0^{(1)}(\lambda |x-y|), \tag{3.4}$$

where $H_0^{(1)}(z)$ is the Hankel function of the first kind and of order zero [1]. Its singularity has the form

$$G(x,y) \sim \frac{1}{2\pi} \log|x-y| + \cdots \quad \text{as } x \longrightarrow y. \tag{3.5}$$

3.1. Preliminary results. Consider (3.1) for the function u in the exterior of Ω_ϵ, multiply by the Green's function G and integrate by parts, we get that for $x \in \mathbb{R}^2 \setminus \overline{\Omega_\epsilon}$,

$$u(x) = \int_{\partial\Omega_\epsilon} \left(\frac{\partial u}{\partial \nu_\epsilon}\bigg|_+ (y) G(x,y) - u(y) \frac{\partial G}{\partial \nu_\epsilon(y)}\bigg|_+ (x,y) \right) d\sigma(y). \tag{3.6}$$

The jump condition for $\partial u/\partial \nu_\epsilon$ on $\partial\Omega_\epsilon$ yields

$$u(x) = -\int_{\partial\Omega_\epsilon} \frac{\partial G}{\partial \nu_\epsilon(y)}\bigg|_+ (x,y) u(y) d\sigma(y) + \int_{\partial\Omega_\epsilon} G(x,y) \frac{\partial u}{\partial \nu_\epsilon}\bigg|_- (y) d\sigma(y). \tag{3.7}$$

Of course, the above equations does not hold up to the boundary of Ω_ϵ, but if we take the limit as $x \to \partial\Omega_\epsilon$, we get from, for instance, [5, 12] that

$$\begin{aligned}\frac{1}{2}u\Big|_{\partial\Omega_\epsilon}(x) = &-\int_{\partial\Omega_\epsilon}\frac{\partial G}{\partial\nu_\epsilon(y)}\Big|_{+}(x,y)u(y)d\sigma(y)\\ &+\int_{\partial\Omega_\epsilon}G(x,y)\frac{\partial u}{\partial\nu_\epsilon}\Big|_{-}(y)d\sigma(y)\quad \text{for } x\in\partial\Omega_\epsilon.\end{aligned}\tag{3.8}$$

We introduce the following operators, called single- and double-layer potentials, respectively:

$$\begin{aligned}\mathrm{Sl}(\lambda)g(x) &= \int_{\partial\Omega_\epsilon}G(x,y)g(y)d\sigma(y),\quad x\in\mathbb{R}^2\setminus\partial\Omega_\epsilon,\\ \mathrm{Dl}(\lambda)g(x) &= \int_{\partial\Omega_\epsilon}\partial_{\nu_\epsilon(y)}G(x,y)g(y)d\sigma(y),\quad x\in\mathbb{R}^2\setminus\partial\Omega_\epsilon.\end{aligned}\tag{3.9}$$

For $g\in\mathcal{C}^\infty(\partial\Omega_\epsilon)$, or even $g\in L^1(\partial\Omega_\epsilon)$, the functions $\mathrm{Sl}(\lambda)g$ and $\mathrm{Dl}(\lambda)g$ are well defined and smooth for $x\in\mathbb{R}^2\setminus\partial\Omega_\epsilon$.

Now define the following operators:

$$S(\lambda):H^{-1/2}(\partial\Omega_\epsilon)\longrightarrow H^{1/2}(\partial\Omega_\epsilon),\qquad D(\lambda):H^{1/2}(\partial\Omega_\epsilon)\longrightarrow H^{1/2}(\partial\Omega_\epsilon),\tag{3.10}$$

where

$$\begin{aligned}S(\lambda):g &\longrightarrow \int_{\partial\Omega_\epsilon}G(\cdot,y)g(y)d\sigma(y),\\ D(\lambda):g &\longrightarrow \int_{\partial\Omega_\epsilon}\frac{\partial G}{\partial\nu_\epsilon(y)}\Big|_{+}(\cdot,y)g(y)d\sigma(y).\end{aligned}\tag{3.11}$$

For such g and every $x\in\partial\Omega_\epsilon$, we denote by $g_+(x)$ and $g_-(x)$ the limits of $g(y)$ as $y\to x$, from $y\in\Omega_\epsilon$ and $y\in\mathbb{R}^2\setminus\overline{\Omega_\epsilon}$, respectively, when these limits exist. It is a well-known classical result that, for $x\in\partial\Omega_\epsilon$,

$$\begin{aligned}\big(\mathrm{Sl}(\lambda)g\big)_+(x) &= \big(\mathrm{Sl}(\lambda)g\big)_-(x) = S(\lambda)g(x),\\ \big(\mathrm{Dl}(\lambda)g\big)_\pm(x) &= \pm\frac{1}{2}g(x)+D(\lambda)g(x),\end{aligned}\tag{3.12}$$

where $S(\lambda)$ is pseudodifferential operator of order -1.

Throughout this paper, we use for simplicity the notation $H^s_\sharp(]0,1[) = H^s(\mathbb{R}/]0,1[)$, for $s\in\mathbb{R}$, where $H^s(\mathbb{R}/]0,1[)$ denotes the classical Sobolev H^s-space on the quotient $\mathbb{R}/]0,1[$.

Using a change of variables and integral equations, the following important result immediately holds from Taylor [16, page 184].

Proposition 3.1. *Let $L_\epsilon(\lambda):H^{-1/2}_\sharp(]0,1[)\to H^{1/2}_\sharp(]0,1[)$ be defined as follows:*

$$\begin{aligned}L_\epsilon(\lambda)f(t) &= \big(S(\lambda)f(\gamma_\epsilon^{-1})\big)(\gamma_\epsilon(t))\\ &= \frac{i}{4}\int_0^1 H_0^{(1)}\big(\lambda\,|\gamma_\epsilon(t)-\gamma_\epsilon(s)|\big)\,|\gamma_\epsilon'(s)|\,f(s)ds\quad \text{for } f\in H^{-1/2}_\sharp(]0,1[).\end{aligned}\tag{3.13}$$

Then the operator-valued function $L_\epsilon(\lambda)$ is Fredholm analytic with index 0 in $\mathbb{C}\setminus i\mathbb{R}^-$. Moreover, $L_\epsilon^{-1}(\lambda)$ is a meromorphic function and its poles are in $\{\Im(z)\le 0\}$. $\Im(z)$ means the imaginary part of z and $\Re(z)$ is the real part.

From now on we will focus our attention on solving the eigenvalue problem (2.4).

3.2. Joint analyticity of kernel. Based on the result found in [3] and on the argument of Millar [11], we will now prove the following proposition. The following result will be useful in Sections 4 and 5.

Proposition 3.2. *There exist a constant $\eta>0$ and a complex neighborhood $\mathcal{V}$ of 0 such that for every function $\varphi(t,\epsilon)\in H_\sharp^{-1/2}(]0,1[)$ analytic in $(t,\epsilon)\in\{|\Im(t)|\le\eta\}\times\mathcal{V}$, the function $L_\epsilon(\lambda)\varphi(t,\epsilon)\in H_\sharp^{1/2}(]0,1[)$ is analytic with respect to (t,ϵ,λ) in $\{|\Im(t)|\le\eta\}\times\mathcal{V}\times D_{r_0}(\lambda_0)$, where $D_{r_0}(\lambda_0)$ is a disc of center λ_0 and radius r_0.*

Proof. The proof of this proposition heavily relies on the work of Bruno and Reitich [3] where they expressed the kernel of $L_\epsilon(\lambda)$ in the following form:

$$\begin{aligned}&\frac{i}{4}H_0^{(1)}\left(\lambda\left|\gamma_\epsilon(t)-\gamma_\epsilon(s)\right|\right)\left|\gamma_\epsilon'(s)\right|\\&\quad=-\sum_{p=-1}^{1}\log\left(|t-s-p|\right)h\left(\lambda^2\left|\gamma_\epsilon(t)-\gamma_\epsilon(s)\right|^2\right)\left|\gamma_\epsilon'(s)\right|+k(t,s,\epsilon,\lambda),\end{aligned}\tag{3.14}$$

for $(t,s,\epsilon,\lambda)\in\mathcal{J}\cap\{\Im(t)=0\}\cap\{\Im(s)=0\}\cap\{\Im(\epsilon)=0\}$, $s\notin\{t,t+1,t-1\}$, where h is an analytic function in $\mathbb{C}$, and k is analytic in $\mathcal{J}$. Here $\mathcal{J}=\{|\Im(t)|\le\eta;|\Im(s)|\le\eta;|\epsilon|\le\delta;\lambda\in D_{r_0}(\lambda_0);-\rho\le\Re(t)\le1+\rho;-\rho\le\Re(s)\le1+\rho\}$, for the positive numbers ρ, r_0, η, and δ.

The central difficulty to prove the analytic property of the operator L_ϵ comes from the spatial regularity of its kernel. We show that logarithmic singularity of the kernel of $L_\epsilon(\lambda)$ is independent of the parameter ϵ.

We introduce

$$\Phi(t,\epsilon,\lambda)=-\int_0^1\sum_{p=-1}^{1}\log|t-s-p|h\left(\lambda^2\left|\gamma_\epsilon(t)-\gamma_\epsilon(s)\right|^2\right)\varphi(s,\epsilon)\left|\gamma_\epsilon'(s)\right|ds.\tag{3.15}$$

Using a change of variables, this function can be rewritten as follows:

$$\Phi(t,\epsilon,\lambda)=-\int_{-1}^{2}\log|t-s|h\left(\lambda^2\left|\gamma_\epsilon(t)-\gamma_\epsilon(s)\right|^2\right)\varphi(s,\epsilon)\left|\gamma_\epsilon'(s)\right|ds.\tag{3.16}$$

An integration by parts leads to

$$\begin{aligned}\Phi(t,\epsilon,\lambda)&=\log\left(|t-2|\right)\psi(t,2,\epsilon,\lambda)\\&\quad-\log\left(|t+1|\right)\psi(t,-1,\epsilon,\lambda)-\int_{-1}^{2}\frac{\psi(t,s,\epsilon,\lambda)}{|t-s|}ds,\end{aligned}\tag{3.17}$$

where

$$\psi(t,s,\epsilon,\lambda) = -\int_t^s h\left(\lambda^2 \left|\, \gamma_\epsilon(t) - \gamma_\epsilon(z) \,\right|^2\right) \varphi(z,\epsilon) \left|\, \gamma_\epsilon'(z) \,\right| dz. \tag{3.18}$$

Clearly, $\Phi(t,\epsilon,\lambda)$ can be extended to a complex analytic function in $\mathbb{C} \times \mathcal{V} \times D_{r_0}(\lambda_0)$. Furthermore, from the following identity

$$L_\epsilon(\lambda)\varphi(t,\epsilon) = \Phi(t,\epsilon,\lambda) + \int_0^1 k(t,s,\epsilon,\lambda)\varphi(s,\epsilon)ds, \tag{3.19}$$

we deduce the desired result applying (3.14). □

4. Reduction to a characteristic value problem

In this section, we first recall some definitions and notations from [6, 10]. Using boundary integral equations, we reduce the problem (2.4) to the existence and the distribution of the characteristic values of selfadjoint integral operators in the complex plane.

4.1. Notations and definitions. Let $\mathscr{E}$ and $\mathscr{F}$ be two Banach spaces and let $\mathscr{L}(\mathscr{E},\mathscr{F})$ be the algebra of all bounded-valued function acting from $\mathscr{E}$ into $\mathscr{F}$. $\mathscr{E}'$ and $\mathscr{F}'$ denote the dual spaces of $\mathscr{E}$ and $\mathscr{F}$, respectively.

Let λ_0 be a fixed complex value in $\mathbb{C}$. We denote by $\mathscr{A}(\lambda)$ an operator-valued function acting from $D_{r_0}(\lambda_0)$ into $\mathscr{L}(\mathscr{E},\mathscr{F})$, where $D_{r_0}(\lambda_0)$ is a disc of center λ_0 and radius $r_0 > 0$. λ_0 is called a *characteristic* value of $\mathscr{A}(\lambda)$ if

(i) $\mathscr{A}(\lambda)$ is holomorphic in some neighborhood of λ_0, except possibly at λ_0,

(ii) there exists a vector-valued function $\phi(\lambda)$: $D_{r_0}(\lambda_0) \to \mathscr{H}$ holomorphic at λ_0 and verifying $\phi(\lambda_0) \neq 0$, such that $\mathscr{A}(\lambda)\phi(\lambda)$ is holomorphic at λ_0 and vanishes at this point.

$\phi(\lambda)$ is called a *root* function of $\mathscr{A}(\lambda)$ associated to λ_0 and the vector $\phi_0 = \phi(\lambda_0)$ is called an *eigenvector*. The closure of the linear set of eigenvectors corresponding to λ_0 is denoted by $\operatorname{Ker}\mathscr{A}(\lambda_0)$.

Suppose that λ_0 is a characteristic value of the function $\mathscr{A}(\lambda)$ and $\phi(\lambda)$ is a root function satisfying (ii). Then there exists a number $m(\phi) \geq 1$ and a vector-valued function $\psi(\lambda)$: $D_{r_0}(\lambda_0) \to \mathscr{H}$ holomorphic such that

$$\mathscr{A}(\lambda)\phi(\lambda) = (\lambda - \lambda_0)^{m(\phi)}\psi(\lambda), \qquad \psi(\lambda_0) \neq 0. \tag{4.1}$$

The number $m(\phi)$ is called the *multiplicity* of the root function $\phi(\lambda)$. Let ϕ_0 be an eigenvector corresponding to λ_0 and let

$$\mathscr{R}(\phi_0) = \{m(\phi); \phi(\lambda) \text{ is a root function such } \phi(\lambda_0) = \phi_0\}. \tag{4.2}$$

Then by *rank* of ϕ_0 we mean $\operatorname{rank}(\phi_0) = \max\mathscr{R}(\phi_0)$. Suppose that $n = \dim\operatorname{Ker}\mathscr{A}(\lambda_0) < +\infty$ and that the ranks of all vectors in $\operatorname{Ker}\mathscr{A}(\lambda_0)$ are finite. A system of eigenvectors ϕ_0^j, $j = 1,\ldots,n$, is called a *canonical system of eigenvectors* of $\mathscr{A}(\lambda)$ associated to λ_0 if the ranks possess the following property: $\operatorname{rank}(\phi_0^j)$ is the maximum of the ranks of all eigenvectors

in some direct complement in $\dim \operatorname{Ker} \mathcal{A}(\lambda_0)$ of the linear span of the vectors $\phi_0^1, \ldots, \phi_0^{j-1}$. Let $r_j = \operatorname{rank}(\phi_0^j)$. Then $(r_j)_j$ uniquely determines the function $\mathcal{A}(\lambda)$. We call

$$N(\mathcal{A}(\lambda_0)) = \sum_{j=1}^{n} r_j \tag{4.3}$$

the *null multiplicity of the characteristic* value λ_0 of $\mathcal{A}(\lambda)$.

If λ_0 is not a characteristic value of $\mathcal{A}(\lambda)$, we put $N(\mathcal{A}(\lambda_0)) = 0$.

Suppose that $\mathcal{A}^{-1}(\lambda)$ exists and is holomorphic in some neighborhood of λ_0, except possibly at λ_0. Then the number

$$M(\mathcal{A}(\lambda_0)) = N(\mathcal{A}(\lambda_0)) - N(\mathcal{A}^{-1}(\lambda_0)) \tag{4.4}$$

is called the *multiplicity* of the characteristic value λ_0 of $\mathcal{A}(\lambda)$. Suppose that λ_1 is a pole of the operator-valued function. The Laurent expansion of $\mathcal{A}(\lambda)$ in λ_1 is given by

$$\mathcal{A}(\lambda) = \sum_{j \geq -s} (\lambda - \lambda_1)^j A_j. \tag{4.5}$$

If in the last expression the operators A_{-j}, $j = 1, \ldots, s$, are finite dimensional, then $\mathcal{A}(\lambda)$ is called *finitely meromorphic* at λ_1.

The operator-valued function $\mathcal{A}(\lambda)$ is said to be of Fredholm type at the point λ_1 if the operator A_0 in the last expansion is a Fredholm operator.

If $\mathcal{A}(\lambda)$ is holomorphic at the point λ_0 and the operator $\mathcal{A}(\lambda_0)$ is invertible, then λ_0 is called a *regular point* of $\mathcal{A}(\lambda)$.

We set

$$\begin{aligned} (y \otimes v)(w) &:= (w, v)y \quad (w \in \mathcal{F}), \\ (v \otimes y)(u) &:= (y, u)v \quad (u \in \mathcal{E}') \end{aligned} \tag{4.6}$$

for $y \in \mathcal{E}$ and $v \in \mathcal{F}'$. Note that $y \otimes v \in \mathcal{L}(\mathcal{F}, \mathcal{E})$, $v \otimes y \in \mathcal{L}(\mathcal{E}', \mathcal{F}')$, and $(y \otimes v)^* = v \otimes y$. An ordered set $\{y_0, y_1, \ldots, y_h\} \subset \mathcal{E}$ is called a chain of eigenvectors and associated vectors (CEAV) of $\mathcal{A}$ at λ if

$$Y =: \sum_{l=0}^{h} (\cdot - \lambda)^l y_l \tag{4.7}$$

is a root function of $\mathcal{A}$ at λ with $m(Y) \geq h + 1$. Conversely, if Y is a root function of $\mathcal{A}$ at λ and $m(Y) \geq h + 1$, then the function

$$Y(\mu) =: \sum_{l=0}^{\infty} (\mu - \lambda)^l y_l, \tag{4.8}$$

$\{y_0, y_1, \ldots, y_h\}$ is CEAV.

A system $\{y_l^{(j)} : 1 \le j \le h,\ 0 \le l \le m_j\}$ is called a canonical system of eigenvectors and associated vectors (CSEAV) of $\mathcal{A}$ at λ if $\{y_0^{(j)} : 1 \le j \le h\}$ is a basis of $\mathrm{Ker}(\mathcal{A}(\lambda))$, $\{y_0^{(j)}, y_1^{(j)}, \ldots, y_{mj}^{(j)}\}$ is a CEAV of $\mathcal{A}$ at λ $(j = 1, \ldots, h)$, $mj = \sup\{m(y) : y \in \mathrm{Ker}(\mathcal{A}(\lambda)) \setminus \mathrm{Span}\{y_0^{(k)} : k < j\}\}$; $(1 \le j \le h)$. Obviously $mj = m(y_0^{(j)})$.

We recall the generalization of Steinberg's theorem [15].

Theorem 4.1. *Suppose that $\mathcal{A}$ is an operator-valued function which is finitely meromorphic and of Fredholm type in the domain $D_{r_0}(\lambda_0)$. If the operator $\mathcal{A}(\lambda)$ is invertible at one point of D_{r_0}, then $\mathcal{A}(\lambda)$ has a bounded inverse for all $\lambda \in D_{r_0}$, except possibly for certain isolated points.*

We will reduce, as mentioned in the introduction, the eigenvalue problem to some characteristic problem. From Proposition 3.1, we know that if λ_0^2 is an eigenvalue of (2.4), then λ_0 is a characteristic value of $L_0(\lambda)$. Moreover, for r_0 small enough, the function $L_0^{-1}(\lambda)$ is meromorphic in $\overline{D_{r_0}}(\lambda_0)$, where $D_{r_0}(\lambda_0)$ the disc of center λ_0 and radius r_0, and λ_0 is its unique pole in $\overline{D_{r_0}}$.

We begin by establishing the following lemma which characterises the eigenvalues of (2.4) if $\epsilon = 0$.

Lemma 4.2. *Any eigenvalue of the problem (2.4) is a simple pole of $L_0^{-1}(\lambda)$.*

Proof. Let λ_0^2 be an eigenvalue of (2.4). Let u_0 be an associated eigenfunction normalized in $L^2(\Omega_0)$. Using an integration by parts, we know that $\partial_{\nu_0} u_0$ is in fact a characteristic function of the operator-valued function $\lambda \mapsto L_0(\lambda)$ corresponding to the characteristic value λ_0, that is, $L_0(\lambda_0)\partial_{\nu_0} u_0 = 0$ on $\partial\Omega_0$. We define $\phi(\lambda)$ as a root function of $L_0(\lambda)$ corresponding to $(\lambda_0; \partial_{\nu_0} u_0)$; it is holomorphic in $D_{r_0}(\lambda_0)$, $\phi(\lambda_0) = \partial_{\nu_0} u_0$ and satisfies the identity $L_0(\lambda)\phi(\lambda)|_{\lambda=\lambda_0} = 0$. The multiplicity of $\phi(\lambda)$ is the order of λ_0 as a zero of $L_0(\lambda)\phi(\lambda)$. But it is well known that the order of λ_0 as a pole of $L_0^{-1}(\lambda)$ is precisely the maximum of the ranks of the eigenvectors in $\mathrm{Ker}\, L_0(\lambda_0)$. Then it suffices to show that the rank of an arbitrary eigenvector is equal to one.

We write $L_0(\lambda)\phi(\lambda) = (\lambda^2 - \lambda_0^2)\psi(\lambda)$, where $\psi(\lambda)$ is a holomorphic function in $H^{1/2}(\partial\Omega_0)$. For $\lambda \in D_{r_0}(\lambda_0)$, we denote by $u(\lambda)$ the unique solution of $(\Delta + \lambda^2)u(\lambda) = 0$ in Ω_0 with the boundary condition $u(\lambda) = \psi(\lambda)\gamma^{-1}$ on $\partial\Omega_0$. Using a trivial integration by parts over Ω_0, we find that

$$\int_{\Omega_0} u(\lambda)\overline{u_0}\, dx = \left(\lambda^2 - \lambda_0^2\right)^{-1} \int_{\partial\Omega_0} u(\lambda)\overline{\partial_{\nu_0} u_0}\, d\sigma(x) = \int_{\partial\Omega_0} \psi(\lambda)\overline{\partial_{\nu_0} u_0}\, d\sigma(x), \tag{4.9}$$

which immediately implies that

$$\int_{\partial\Omega_0} \psi(\lambda_0)\overline{\partial_{\nu_0} u_0}\, d\sigma(x) = \int_{\Omega_0} |u_0|^2 dx = 1, \tag{4.10}$$

since $\int_{\Omega_0} u(\lambda)\overline{u_0} dx$ is holomorphic in $D_{r_0}(\lambda_0)$. Therefore, $|\psi(\lambda_0)|^2_{L^2(\partial\Omega_0)} \neq 0$ and thus the function $\psi(\lambda_0)$ is nontrivial. □

Our main results in this section are summarized in the following theorem.

Theorem 4.3. *There exists a positive constant $\epsilon_1 = \epsilon_1(r_0, \epsilon_0)$ such that $\epsilon_1 \le \epsilon_0$ and for $|\epsilon| < \epsilon_1$, the operator-valued function $\lambda \mapsto L_\epsilon(\lambda)$ has exactly m characteristic values $(\lambda_i(\epsilon))_i$ (counted according to their multiplicity) in $\overline{D_{r_0}}(\lambda_0)$. These characteristic values build the λ_0-group associated to the perturbed eigenvalue problem (2.4), and are analytic with respect to ϵ in $]-\epsilon_1, \epsilon_1[$. They satisfy $\lambda_i(0) = \lambda_0$, for $i = 1, \ldots, m$. Moreover, if $(\hat{\lambda}_i(\epsilon))_{i=1}^{\tau_\epsilon}$ denotes the set of distinct values of $(\lambda_i(\epsilon))_{i=1}^{m}$, then the following assertions hold:*

$$\begin{aligned} \sum_{i=1}^{\tau_\epsilon} M\left(L_\epsilon\left(\hat{\lambda}_i(\epsilon)\right)\right) &= m, \\ L_\epsilon^{-1}(\lambda) &= \sum_{i=1}^{\tau_\epsilon} \left(\lambda - \hat{\lambda}_i(\epsilon)\right)^{-1} \ell_i(\epsilon) + \mathcal{R}_\epsilon(\lambda), \end{aligned} \tag{4.11}$$

where $\ell_i(\epsilon) : \mathrm{Ker}(L_\epsilon(\hat{\lambda}_i(\epsilon))) \to \mathrm{Ker}(L_\epsilon(\hat{\lambda}_i(\epsilon)))$ and $\mathcal{R}_\epsilon(\lambda)$ is a holomorphic function with respect to $(\epsilon, \lambda) \in]-\epsilon_0, \epsilon_0[\times D_{r_0}(\lambda_0)$.

Proof. We first recall that m is the geometric multiplicity of λ_0^2 as an eigenvalue of the eigenvalue problem (2.4). Proposition 3.2 implies that $L_\epsilon(\lambda)$ is an analytic operator-valued function with respect to $(\epsilon, \lambda) \in \mathbb{R} \times D_{r_0}(\lambda_0)$. Then there exists a constant $\epsilon_1(r_0, \epsilon_0) > 0$ such that for any ϵ lying in $]-\epsilon_1, \epsilon_1[$, the following holds:

$$\left| \left(L_\epsilon(\lambda) - L_0(\lambda)\right) L_0^{-1}(\lambda) \right|_{\mathcal{L}(H_\sharp^{1/2}, H_\sharp^{1/2})} < 1, \quad \forall \lambda \in \partial D_{r_0}(\lambda_0). \tag{4.12}$$

From the generalized Rouché's theorem and the results of Gohberg and Sigal [6], we deduce that $L_\epsilon(\lambda)$ is invertible on ∂D_{r_0} and has τ_ϵ characteristic values $\left(\hat{\lambda}_i(\epsilon)\right)_i$ in $D_{r_0}(\lambda_0)$ which are (obviously) the poles of the function $L_\epsilon^{-1}(\lambda)$ in the disc $D_{r_0}(\lambda_0)$. Thus, with the definitions introduced earlier, the following holds:

$$\sum_{i=1}^{\tau_\epsilon} M\left(L_\epsilon\left(\hat{\lambda}_i(\epsilon)\right)\right) = M\left(L_0(\lambda_0)\right). \tag{4.13}$$

Using Theorem 2.1 and Proposition 3.2, it can now be easily seen that the set of these characteristic values build precisely the λ_0-group of eigenvalues introduced in the last section, that is, $(\hat{\lambda}_i(\epsilon))_i \equiv (\lambda_i(\epsilon))_i$. Hence, they are analytic in the variable ϵ. Notice that in general we have $M(L_0(\lambda_0)) \ge \dim \mathrm{Ker}\, L_0(\lambda_0) = m$. But Lemma 4.2 implies $M(L_0(\lambda_0)) = m$. Furthermore, we have the Laurent expansion

$$L_0^{-1}(\lambda) = (\lambda - \lambda_0)^{-1} \ell_0 + \mathcal{R}_0(\lambda), \tag{4.14}$$

where $\ell_0 = \sum_{i=1}^{\tau_0} \ell_i(0)$ and $\mathcal{R}_0(\lambda)$ is a holomorphic function. From the decomposition (4.14), we obtain

$$\begin{aligned} &\ell_0 L_0(\lambda_0) = 0 \quad \text{in } \mathcal{L}(H_\sharp^{-1/2}(]0,1[), H_\sharp^{-1/2}(]0,1[)), \\ &L_0(\lambda_0)\ell_0 = 0 \quad \text{in } \mathcal{L}(H_\sharp^{1/2}(]0,1[), H_\sharp^{1/2}(]0,1[)). \end{aligned} \tag{4.15}$$

It follows that $\ell_0 : \operatorname{Ker} L_0^*(\lambda_0) \to \operatorname{Ker} L_0(\lambda_0)$. But from the properties of the Green's function $G(x, y)$, we know that

$$\operatorname{Ker} L_0^*(\lambda_0) = \operatorname{Ker} L_0(\lambda_0). \tag{4.16}$$

Note that using similar arguments, we can prove that $(\lambda_i(\epsilon))_i$ are also simple poles of $L_\epsilon^{-1}(\lambda)$ and $M(L_\epsilon(\lambda_i(\epsilon))) = \dim \operatorname{Ker} L_\epsilon(\lambda_i(\epsilon))$, for $i = 1, \ldots, m$. Moreover, we have

$$\begin{aligned} L_\epsilon^{-1}(\lambda) &= \sum_{i=1}^{\tau_\epsilon} \left(\lambda - \hat{\lambda}_i(\epsilon)\right)^{-1} \ell_i(\epsilon) + \mathcal{R}_\epsilon(\lambda), \\ \ell_i(\epsilon) &: \operatorname{Ker}\left(L_\epsilon(\hat{\lambda}_i(\epsilon))\right) \longrightarrow \operatorname{Ker}\left(L_\epsilon(\hat{\lambda}_i(\epsilon))\right), \end{aligned} \tag{4.17}$$

where $\mathcal{R}_\epsilon(\lambda)$ is a holomorphic function which completes the proof of the theorem. □

Let λ_0 be a characteristic value of $L_0(\lambda)$. From Keldys's theorem which is simplified in [10, page 462], there exist $\{\phi_0^i : 1 \le i \le m\}$ CSEAV of L_0 at λ_0 and $\{\psi_0^i : 1 \le i \le m\}$ CSEAV of L_0^* such that the operator

$$A_0 = \frac{1}{2i\pi} \int_{|\lambda - \lambda_0| = \rho} \left(L_0(\lambda)\right)^{-1} d\lambda = \sum_{i=1}^{m} \phi_0^i \otimes \psi_0^i \tag{4.18}$$

is well defined.

Analogously, by the result of Reinhard and Möller which is due to Keldyš [8], for each characteristic value $\lambda_i(\epsilon)$ $(1 \le i \le m)$, there exist $\{\phi_{i,j}(\epsilon) : 1 \le i \le m,\ 1 \le j \le m_i\}$ CSEAV of L_ϵ at $\lambda_i(\epsilon)$ and $\{\psi_{i,j}(\epsilon) : 1 \le i \le m,\ 1 \le j \le m_i\}$ CSEAV of L_ϵ^* such that the operator

$$A_i(\epsilon) = \frac{1}{2i\pi} \int_{|\lambda - \lambda_i(\epsilon)| = \rho} \left(L_\epsilon(\lambda)\right)^{-1} d\lambda = \sum_{j=1}^{m_i} \phi_{i,j}(\epsilon) \otimes \psi_{i,j}(\epsilon) \tag{4.19}$$

is well defined. Introduce the operator

$$A(\epsilon) = \sum_{i=1}^{m} A_i(\epsilon), \quad \text{for } |\epsilon| < \epsilon_1. \tag{4.20}$$

Based on Theorem 4.3, [15] and on relation (4.20), one can see that the operator $A(\epsilon)$ is selfadjoint and holomorphic function with respect to $\epsilon \in]-\epsilon_1, \epsilon_1[$. It is quite easy to see that $A_0 = A(\epsilon = 0)$ and the following results hold.

Proposition 4.4. *For $|\epsilon| < \epsilon_1$ and for $i \in \{1, \ldots, \tau_\epsilon\}$,*

(1) *the operator $A_i(\epsilon)$ satisfies*

$$A_i(\epsilon)\big|_{(\operatorname{Ker}(L_\epsilon(\lambda_i(\epsilon))))^\perp} \equiv 0; \tag{4.21}$$

(2) *if $\lambda_i(\epsilon)$ and $\lambda_j(\epsilon)$ are two characteristic values of $L_\epsilon(\lambda)$ with $i \neq j$, then*

$$\operatorname{Ker} L_\epsilon\left(\lambda_i(\epsilon)\right) \perp \operatorname{Ker} L_\epsilon\left(\lambda_j(\epsilon)\right). \tag{4.22}$$

Proof. (1) Let $\phi \in H_\sharp^{-1/2}$ such that $A_i(\epsilon)\phi = 0$, then

$$\sum_{j=1}^{m_i} (\phi, \psi_{i,j}(\epsilon))\phi_{i,j}(\epsilon) = 0. \tag{4.23}$$

The fact that $(\phi_{ij})_{ij}$ and $(\psi_{ij})_{ij}$ are CSEAV implies that $\phi = 0$ and, therefore, yields the desired result.

(2) It suffices to prove the result only for $i = 1$ and $j = 2$. From relation (4.16), we deduce that

$$L_\epsilon(\lambda_1(\epsilon)) \operatorname{Ker} L_\epsilon(\lambda_2(\epsilon)) \subset \operatorname{Ker} L_\epsilon(\lambda_2(\epsilon)). \tag{4.24}$$

Then,

$$L_\epsilon(\lambda_1(\epsilon)) \operatorname{Ker}\left(L_\epsilon(\lambda_2(\epsilon))\right) \subset \operatorname{Ker}\left(L_\epsilon(\lambda_2(\epsilon))\right) \cap \left(\operatorname{Ker}\left(L_\epsilon(\lambda_1(\epsilon))\right)\right)^\perp. \tag{4.25}$$

Using the last relation, the relation

$$\operatorname{Ker}\left(L_\epsilon(\lambda_2(\epsilon))\right) = L_\epsilon(\lambda_1(\epsilon)) \operatorname{Ker}\left(L_\epsilon(\lambda_2(\epsilon))\right) + \left(I - L_\epsilon(\lambda_1(\epsilon))\right) \operatorname{Ker}\left(L_\epsilon(\lambda_2(\epsilon))\right) \tag{4.26}$$

becomes

$$\left(I - L_\epsilon(\lambda_1(\epsilon))\right) \operatorname{Ker}\left(L_\epsilon(\lambda_2(\epsilon))\right) = 0. \tag{4.27}$$

The operator-valued function $L_\epsilon(\lambda_1(\epsilon))$ is Fredholm of index 0 which completes the proof. □

Our strategy now is to investigate the properties of the eigenelements corresponding to the operators A_0 and A_ϵ. Let $(\mu_0^j)_{1\le j\le h}$ be the family of eigenvalues of the operator A_0 with multiplicity m_j each. Using the generalisation of Theorem 2.1, see [7, 13], we know that there exists $\epsilon_2 = \epsilon_2(\epsilon_1) > 0$ such that for $|\epsilon| < \epsilon_2$ and for $j \in \{1,\dots,h\}$, the μ_0^j-group consists of m_j eigenvalues of $A(\epsilon)$, $\mu_{j,l}(\epsilon)$, $l = 1,\dots,m_j$ (repeated according to their multiplicity).

Let $\epsilon_3 = \inf(\epsilon_1, \epsilon_2)$. For $|\epsilon| < \epsilon_3$, the following projector is well defined:

$$P_j(\epsilon) = \frac{1}{2i\pi} \int_{|\mu-\mu_{j,i}(\epsilon)|=\rho_1} \left(\mu - A(\epsilon)\right)^{-1} d\mu = \sum_{l=1}^{m_j} \sum_{s=1}^{m_{j,l}} q_{l,s}^{(j)}(\epsilon) \otimes q_{l,s}^{(j)}(\epsilon), \tag{4.28}$$

where for $1 \le j \le h$ and for $1 \le l \le m_j$, the family $(q_{l,s}^{(j)}(\epsilon))_{1\le s\le m_{jl}}$ denotes the orthogonal family of eigenfunctions corresponding to the eigenvalues $\mu_{j,l}(\epsilon)$. For $\epsilon = 0$, we have

$$P_j(0) = \frac{1}{2i\pi} \int_{|\mu-\mu_0^j|=\rho_2} \left(\mu - A(0)\right)^{-1} d\mu = \sum_{l=1}^{m_j} q_l^{(j)}(0) \otimes q_l^{(j)}(0), \tag{4.29}$$

where the family $(q_l^{(j)}(0))_{1\le l\le m_j}$ is the orthogonal family of eigenfunctions corresponding to the eigenvalue μ_0^j. Now it seems natural, from the previous results, that for all $j = 1,\dots,h$, the family $(q_l^{(j)}(0))_{1\le l\le m_j}$ is m_j-characteristic functions of $L_0(\lambda_0)$ and for all $l = 1,\dots,m_j$, the family $(q_{l,s}^{(j)}(\epsilon))_{1\le s\le m_{j,l}}$ is $m_{j,l}$-characteristic functions of $L_\epsilon(\lambda_i(\epsilon))$ and $\sum_{j=1}^h m_j = m$.

We set

$$P(\epsilon) = \sum_{j=1}^{h} P_j(\epsilon), \quad \text{for } |\epsilon| < \epsilon_3, \qquad P(0) = \sum_{j=1}^{h} P_j(0). \tag{4.30}$$

5. Analyticity and asymptotic expansion

This section is devoted to the study of the asymptotics of the characteristic elements and, therefore, the asymptotics of the eigenelements of (2.4) when the parameter ϵ goes to zero. We will give a method in order to calculate the coefficients of the expansions of the eigenelements in a neighborhood of zero when the eigenvalue λ_0^2 of $-\Delta$ is not simple. Our strategy, for deriving asymptotic expansions of the perturbations in a multiple eigenvalue λ_0 with multiplicity m that are due to boundary deformations, relies on finding the analyticity and complete asymptotic expansions of the eigenelements of $A(\epsilon)$. The following holds.

Proposition 5.1. *For* $|\epsilon| < \epsilon_3$,

(1) *the operator* $P(\epsilon)$ *is holomorphic for* $\epsilon \in]-\epsilon_3,\epsilon_3[$ *and* $P(\epsilon) = P(0) + R(\epsilon)$, *where* $R(\epsilon)$ *is holomorphic with respect to* ϵ,

(2) $P(\epsilon) = \sum_{j=1}^m q^{(j)}(\epsilon) \otimes q^{(j)}(\epsilon)$, *where* $(q^{(j)}(\epsilon))_{1\le j\le m}$ *denotes an orthonormal basis of* $\operatorname{Ker} L_\epsilon(\lambda_i(\epsilon))$. *Also,* $P(0) = \sum_{j=1}^m q^{(j)}(0) \otimes q^{(j)}(0)$, *where* $(q^{(j)}(0))_{1\le j\le m}$ *is an orthonormal basis of* $\operatorname{Ker} L_0(\lambda_0)$.

Proof. (1) This property is clear by recalling that the operator $A(\epsilon)$ is holomorphic and has the expansion $A(\epsilon) = A(0) + \widetilde{A}(\epsilon)$, where the operator $\widetilde{A}(\epsilon)$ is holomorphic with respect to ϵ and goes to 0 as $\epsilon \to 0$ and by considering, for $\epsilon \in]-\epsilon_3,\epsilon_3[$, the Neumann series

$$\big(\mu - A(\epsilon)\big)^{-1} = \big(\mu - A(0)\big)^{-1} + \sum_{p=1}^{\infty} \big(\mu - A(0)\big)^{-1}\Big[\widetilde{A}(\epsilon)\big(\mu - A(0)\big)^{-1}\Big]^p, \tag{5.1}$$

which converges uniformly with respect to μ in a neighborhood of $\mu_{j,i}$.

(2) As said at the end of Section 4, the elements of the family $(q_l^{(j)}(0))_{1\le j\le h,1\le l\le m_j}$ are characteristic functions of $L_0(\lambda_0)$ and the idea here is to organize this family as follows:

$$q_1^{(1)}(0) = q_0^{(1)}, q_2^{(1)}(0) = q_0^{(2)},\dots,q_{m_h}^{(h)}(0) = q_0^{(m)}. \tag{5.2}$$

Then, the family $(q_0^{(j)})_{1\le j\le m}$ defines an orthogonal basis in $\overline{\operatorname{Ker} L_0(\lambda_0)}$. Using similar arguments, the family $(q_{l,s}^{(j)}(\epsilon))_{1\le l\le m_j,1\le s\le m_{jl},1\le j\le h}$, given by (4.28), can be organized:

$$q_{1,1}^{(1)}(\epsilon) = q^{(1)}(\epsilon), q_{1,2}^{(1)}(0) = q^{(2)}(\epsilon),\dots,q_{m_h,m_{hm_h}}^{(h)}(\epsilon) = q^{(m)}(\epsilon), \tag{5.3}$$

where for all $j = 1,\dots,m$, $q^{(j)}(\epsilon) \to q_0^{(j)}$ as $\epsilon \to 0$. The family $(q^{(j)}(\epsilon))_{1\le j\le m}$ defines an orthogonal basis in $\bigoplus_{i=1}^{m} \operatorname{Ker}(L_\epsilon(\lambda_i(\epsilon)))$, and so the order of organization of its elements directly depends on the order of organization of the basis $(q_0^{(j)})_{1\le j\le m}$. □

Let $B_\epsilon = [a_{ls}]$ be the $(m\times m)$ matrix, where for $l = 1,\dots,m$ and for $s = 1,\dots,m$,

$$a_{ls} = \left(q_0^{(l)}, q_\epsilon^{(s)}\right). \tag{5.4}$$

Then, $B_0 = I_m$, where I_m denotes the identity matrix. From Proposition 5.1, we have

$$\sum_{j=1}^{m} \left(\cdot, q^{(j)}(\epsilon)\right) q^{(j)}(\epsilon) = \sum_{j=1}^{m} \left(\cdot, q_0^{(j)}\right) q_0^{(j)} + R(\epsilon). \tag{5.5}$$

Relation (5.5) implies that

$$\left(B_\epsilon \phi_\epsilon\right)_l = q_0^{(l)} + R(\epsilon) q_0^{(l)}, \tag{5.6}$$

where

$$q_0 = \left(q_0^{(1)},\dots,q_0^{(m)}\right)^T, \qquad \phi_\epsilon = \left(q^{(1)}(\epsilon),\dots,q^{(m)}(\epsilon)\right)^T. \tag{5.7}$$

Proposition 5.2. *There exists some constant $\epsilon_4 = \epsilon_4(\epsilon_3) > 0$, $(\epsilon_4 \le \epsilon_3)$, such that for $j \in \{1,\dots,m\}$,*

(1) *the functions $q^{(j)}(\epsilon)(t)$ are holomorphic in (t,ϵ) and satisfy the following uniform expansions: for $t \in [0,1]$,*

$$q^{(j)}(\epsilon)(t) = q_0^{(j)}(t) + \sum_{n\ge 1} q_n^{(j)}(t)\epsilon^n, \tag{5.8}$$

where the first coefficient satisfies

$$q_1^{(j)} = R_1 q_0^{(j)}, \tag{5.9}$$

and for $n \ge 2$, the coefficients $q_n^{(j)}$ are given by

$$q_n^{(j)} = R_n q_0^{(j)} - \sum_{k=1}^{n-1} \sum_{i=1}^{m} \left(q_0^{(j)}, q_k^{(i)}\right) q_{n-k}^{(i)}, \tag{5.10}$$

with $R_0 = 0$ and R_n $(n \ge 1)$ being the Taylor coefficients of $R(\epsilon)$,

(2) *the characteristic values $\lambda_j(\epsilon) = \lambda^{(j)}(\epsilon)$ satisfy*

$$\lambda^{(j)}(\epsilon) = \lambda_0 + \sum_{n\ge 1} \lambda_n^{(j)} \epsilon^n. \tag{5.11}$$

The first coefficient satisfies

$$\lambda_1^{(j)} = -1 - \frac{\left(L_0(\lambda_0) q_1^{(j)}, l_1^{(0)} q_0^{(j)}\right)}{\left\| l_1^{(0)} q_0^{(j)} \right\|^2} \tag{5.12}$$

and for $n \geq 2$,

$$\lambda_n^{(j)} = -\frac{1}{\left\|l_1^{(0)} q_0^{(j)}\right\|^2}\left[\sum_{i=2}^{n}\left(\sum_{s_1+\cdots+s_i=n}\lambda_{s_1}^{(j)}\lambda_{s_2}^{(j)}\cdots\lambda_{s_i}^{(j)}\right)\left(l_i^{(0)}q_0^{(j)}, l_1^{(0)}q_0^{(j)}\right)\right.$$
$$\left. + \sum_{l=0}^{n-1}\sum_{k=0}^{l}\left(F_{k,l-k}q_{n-l}^{(j)}, l_1^{(0)}q_0^{(j)}\right) + \sum_{k=1}^{n}\left(F_{k,n-k}q_0^{(j)}, l_1^{(0)}q_0^{(j)}\right)\right]. \tag{5.13}$$

Here for all integers k and s, the expressions $l_k^{(s)}$ and $F_{k,s}$ are two operator-valued functions with simple forms.

Proof. (1) Define the matrix: $\mathcal{D}_\epsilon = (d_{sp}^\epsilon)_{sp}$; the coefficients d_{sp}^ϵ are given by

$$d_{sp}^\epsilon = \delta_s^p + \left(R(\epsilon)q_0^{(s)}, q_0^{(p)}\right), \quad (\delta_s^p = 1 \text{ if } p = s \text{ and } \delta_s^p = 0 \text{ if } p \neq s). \tag{5.14}$$

The analyticity of the operator-valued function $R(\epsilon)$ with respect to $\epsilon \in]-\epsilon_3, \epsilon_3[$ guarantees the analyticity of $\mathcal{D}_\epsilon$. The inner product of (5.5) by $q_0^{(p)}$ gives

$$\sum_{l=1}^{m}\left(q_0^{(s)}, q^{(l)}(\epsilon)\right)\left(q^{(l)}(\epsilon), q_0^{(p)}\right) = \delta_s^p + \left(R(\epsilon)q_0^{(s)}, q_0^{(p)}\right) \tag{5.15}$$

which implies that

$$B_\epsilon^2 = \mathcal{D}_\epsilon. \tag{5.16}$$

Relation (5.5) implies

$$B_\epsilon \phi_\epsilon = \phi_0 + R(\epsilon)\phi_0. \tag{5.17}$$

We now verify that the function ϕ_ϵ is jointly analytic in (t, ϵ). Through relation (5.17), we deduce that

$$\phi_\epsilon = (B_\epsilon)^{-1}(\phi_0 + R(\epsilon)\phi_0). \tag{5.18}$$

The analyticity of $\phi_0(t)$ in t is a classical result. Then we deduce the result by using the analyticity of the matrix B_ϵ^{-1} in ϵ, which is obvious from (5.16), and the fact that the function $R(\epsilon)\phi_0(t)$ is jointly analytic in (t, ϵ). From relation (5.7), we then deduce the analyticity of the functions $q^{(j)}(\epsilon)$ follows for all $j = 1, \ldots, m$.

The analyticity of the matrix operator B_ϵ allows writing, in a neighborhood of 0, the expansion

$$B_\epsilon = B_0 + \epsilon B_1 + \epsilon^2 B_2 + \cdots. \tag{5.19}$$

By expanding

$$\phi_\epsilon = \sum_{k\geq 0}\epsilon^k q_k, \quad \text{for } |\epsilon| < \epsilon_4, \tag{5.20}$$

and (5.7), relation (5.17) implies

$$\left(\sum_{n=0}^{+\infty}\epsilon^n B_n\right)\left(\sum_{s=0}^{+\infty}\epsilon^s q_s\right)=q_0+\sum_{n=1}^{+\infty}\epsilon^n R_n q_0, \tag{5.21}$$

where we have considered $\phi_0=q_0$ and $R(\epsilon)=\sum_{n\geq 1}\epsilon^n R_n$. Then,

$$\sum_{k=0}^{n} B_k q_{n-k}=q_0+R_n q_0. \tag{5.22}$$

The jth component of the vector $B_k q_{n-k}$ is given by

$$\left(B_k q_{n-k}\right)_j=\sum_{i=1}^{m}\left(q_0^{(j)},q_k^{(i)}\right)q_{n-k}^{(i)} \tag{5.23}$$

and then relation (5.22) becomes

$$\left(B_0 q_n\right)_j+\sum_{k=1}^{n}\sum_{i=1}^{m}\left(q_0^{(j)},q_k^{(i)}\right)q_{n-k}^{(i)}=q_0^{(j)}+R_n q_0^{(j)}. \tag{5.24}$$

The relation $B_0=I_m$ implies

$$q_n^{(j)}=-\sum_{k=1}^{n}\sum_{i=1}^{m}\left(q_0^{(j)},q_k^{(i)}\right)q_{n-k}^{(i)}+q_0^{(j)}+R_n q_0^{(j)}. \tag{5.25}$$

Therefore,

$$q_n^{(j)}=-\sum_{i=1}^{m}\left(q_0^{(j)},q_n^{(i)}\right)q_0^{(i)}-\sum_{k=1}^{n-1}\sum_{i=1}^{m}\left(q_0^{(j)},q_k^{(i)}\right)q_{n-k}^{(i)}+q_0^{(j)}+R_n q_0^{(j)}. \tag{5.26}$$

Taking the inner product with $q_0^{(s)}$, $s=1,\ldots,m$, we have

$$\left(q_0^{(s)},q_n^{(j)}\right)=-\left(q_0^{(j)},q_n^{(s)}\right)-\sum_{k=1}^{n-1}\sum_{i=1}^{m}\left(q_0^{(j)},q_k^{(i)}\right)\left(q_0^{(s)},q_{n-k}^{(i)}\right)+\delta_{sj}+\left(q_0^{(s)},R_n q_0^{(j)}\right). \tag{5.27}$$

Then,

$$\left(q_0^{(j)},q_n^{(s)}\right)=-\left(q_0^{(s)},q_n^{(j)}\right)-\sum_{k=1}^{n-1}\sum_{i=1}^{m}\left(q_0^{(j)},q_k^{(i)}\right)\left(q_0^{(s)},q_{n-k}^{(i)}\right)+\delta_{sj}+\left(q_0^{(s)},R_n q_0^{(j)}\right). \tag{5.28}$$

If we replace this equality in (5.26), we find

$$\begin{aligned}q_n^{(j)}=&\sum_{i=1}^{m}\left(q_0^{(i)},q_n^{(j)}\right)q_0^{(i)}+\sum_{s=1}^{m}\left[\sum_{k=1}^{n-1}\sum_{i=1}^{m}\left(q_0^{(j)},q_k^{(i)}\right)\left(q_0^{(s)},q_{n-k}^{(i)}\right)-\delta_{sj}-\left(q_0^{(s)},R_n q_0^{(j)}\right)\right]q_0^{(s)}\\&-\sum_{k=1}^{n-1}\sum_{i=1}^{m}\left(q_0^{(j)},q_k^{(i)}\right)q_{n-k}^{(i)}+q_0^{(j)}+R_n q_0^{(j)}.\end{aligned} \tag{5.29}$$

We recall that

$$P(0)q_n^{(j)} = \sum_{i=1}^{m} \left(q_0^{(i)}, q_n^{(j)}\right) q_0^{(i)}. \tag{5.30}$$

Then relation (5.29) becomes

$$\begin{aligned}(I-P_0)q_n^{(j)} = \sum_{s=1}^{m} \Bigg[\sum_{k=1}^{n-1} \sum_{i=1}^{m} \left(q_0^{(j)}, q_k^{(i)}\right) \left(q_0^{(s)}, q_{n-k}^{(i)}\right) - \delta_{sj} - \left(q_0^{(s)}, R_n q_0^{(j)}\right) \Bigg] q_0^{(s)} \\ - \sum_{k=1}^{n-1} \sum_{i=1}^{m} \left(q_0^{(j)}, q_k^{(i)}\right) q_{n-k}^{(i)} + q_0^{(j)} + R_n q_0^{(j)}.\end{aligned} \tag{5.31}$$

From the previous relation and the properties of the operator $I - P_0$, we deduce

$$(I-P_0)\left(q_n^{(j)} + \sum_{k=1}^{n-1} \sum_{i=1}^{m} \left(q_0^{(j)}, q_k^{(i)}\right) q_{n-k}^{(i)} - R_n q_0^{(j)}\right) = 0. \tag{5.32}$$

In other words, it is obvious that $R_n q_0^{(j)} \notin \operatorname{Ker}(L_0(\lambda_0))$ for all $j = 1, \ldots, m$. Then

$$\left(q_n^{(j)} + \sum_{k=1}^{n-1} \sum_{i=1}^{m} \left(q_0^{(j)}, q_k^{(i)}\right) q_{n-k}^{(i)} - R_n q_0^{(j)}\right) \notin \operatorname{Ker}\left(L_0(\lambda_0)\right). \tag{5.33}$$

Thus, relation (5.32) means that

$$q_n^{(j)} + \sum_{k=1}^{n-1} \sum_{i=1}^{m} \left(q_0^{(j)}, q_k^{(i)}\right) q_{n-k}^{(i)} - R_n q_0^{(j)} = 0. \tag{5.34}$$

(2) In order to find out the coefficients in (5.11), our method is based on expanding the expression $L_\epsilon(\lambda^{(j)}(\epsilon))$ for ϵ near zero. To handle this, we have to expand, first, the operator-valued function $L_\epsilon(\lambda)$ around $\epsilon = 0$ and so the resulting expression around $\lambda = \lambda_0$.

We first recall the formula (see [1] and [3, page 332])

$$H_0^{(1)}(z) = J_0(z) + iY_0(z), \tag{5.35}$$

where J_0 and Y_0 are the Bessel functions

$$\begin{aligned}J_0(z) &= \sum_{n=0}^{+\infty} \frac{(-1)^n}{(n!)^2} \left(\frac{z}{2}\right)^{2n}, \\ Y_0(z) &= \frac{2}{\pi}\left(\Gamma_0 + \log\left(\frac{z}{2}\right)\right) J_0(z) - \frac{2}{\pi} \sum_{n=1}^{+\infty} \frac{(-1)^n}{(n!)^2} \left(\frac{z}{2}\right)^{2n} \left(1 + \frac{1}{2} + \cdots \frac{1}{n}\right).\end{aligned} \tag{5.36}$$

Thus, in particular, we can write

$$H_0^{(1)}(z) = h_1(z^2) + h_2(z^2)\log(z) = h_1(z^2) + (\Gamma_0 + z^2 h_3(z^2))\log(z), \tag{5.37}$$

for some entire functions h_1, h_2, and h_3, with Γ_0 being a real constant.

Combining the last result and the kernel of the operator $L_\epsilon(\lambda)$ (see Proposition 3.1), we formally obtain the following uniform expansion: for all $(\lambda,t,s) \in D_{r_0}(\lambda_0)\times[0,1]\times[0,1]$,

$$G(\lambda,t,s) = \frac{i}{4}H_0^{(1)}(\lambda\,|\,\gamma_\epsilon(t)-\gamma_\epsilon(s)\,|\,) = \sum_{n\geq 0} G_n(\lambda,t,s)\epsilon^n, \tag{5.38}$$

with the Taylor coefficients

$$\begin{aligned} G_n(\lambda,t,s) = \frac{i}{4n!}\sum_{k=0}^{n}\lambda^k\left(\frac{(\beta(t)-\beta(s),\gamma(t)-\gamma(s))}{|\,\gamma(t)-\gamma(s)\,|}\right)^k\left(\frac{(\beta'(s),\gamma'(s))}{|\,\gamma'(s)\,|}\right)^{n-k} \\ \times\frac{d^kH_0^{(1)}}{dz^k}(\lambda\,|\,\gamma(t)-\gamma(s)\,|\,). \end{aligned} \tag{5.39}$$

As an immediate consequence, the following holds:

$$L_\epsilon(\lambda) = \sum_{n=0}^{+\infty} L_n(\lambda)\epsilon^n, \quad \text{for } \epsilon \text{ small enough.} \tag{5.40}$$

By considering Proposition 3.1, it seems clear that each operator $L_n(\lambda)$ has kernel $G_n(\lambda, t,s)$. One also gets from (5.39) the following uniform expansion: for $(t,s)\in[0,1]\times[0,1]$,

$$G_n(\lambda,t,s) = \sum_{k=0}^{+\infty}(\lambda-\lambda_0)^k G_k^{(n)}(t,s), \tag{5.41}$$

where λ is in a neighborhood of λ_0, with the coefficients

$$\begin{aligned} G_k^{(n)}(t,s) = \frac{i}{4n!}\sum_{i=0}^{n}\sum_{l=0}^{k}\frac{1}{l!(k-l)!}\frac{\left[(\beta(t)-\beta(s),\gamma(t)-\gamma(s))\right]^i}{|\,\gamma(t)-\gamma(s)\,|^{i+l-k}} \\ \times\left(\frac{(\beta'(s),\gamma'(s))}{|\,\gamma'(s)\,|}\right)^{n-i}\frac{d^l(\lambda^i)}{d\lambda^l}\Big|_{\lambda=\lambda_0}\frac{d^{k+i-l}}{dz^{k+i-l}}H_0^{(1)}(\lambda_0\,|\,\gamma(t)-\gamma(s)\,|\,). \end{aligned} \tag{5.42}$$

Now, for (ϵ,λ) in a neighborhood of $(0,\lambda_0)$, we have the following expansions:

$$L_\epsilon(\lambda) = \sum_{n=0}^{+\infty}\sum_{k=0}^{+\infty}\epsilon^n(\lambda-\lambda_0)^k l_k^{(n)}, \tag{5.43}$$

where the operator $l_k^{(n)}$ has kernel $G_k^{(n)}(t,s)$. We know that $\lambda^{(j)}(\epsilon) \to \lambda_0$ as $\epsilon \to 0$. Then for ϵ small enough, we can replace $\lambda^{(j)}(\epsilon)$ given by (5.11) in relation (5.43), and so we obtain

$$L_\epsilon\left(\lambda^{(j)}(\epsilon)\right) = \sum_{n=0}^{+\infty} \left(\sum_{k=0}^{n} F_{k,n-k} \right) \epsilon^n, \tag{5.44}$$

where

$$\begin{aligned} F_{n,0} &= l_0^{(n)}, \\ F_{n,k} &= \sum_{i=1}^{k} \left(\sum_{s_1+\cdots+s_i=k} \lambda_{s_1}^{(j)} \lambda_{s_2}^{(j)} \cdots \lambda_{s_i}^{(j)} \right) l_i^{(n)}. \end{aligned} \tag{5.45}$$

Remember that

$$L_\epsilon\left(\lambda^{(j)}(\epsilon)\right) q^{(j)}(\epsilon) = 0, \quad \forall\, j = 1, \ldots, m. \tag{5.46}$$

Then by using (5.10) and (5.44) at order $n \geq 1$, we can easily write

$$\sum_{l=0}^{n} \sum_{k=0}^{l} F_{k,l-k} q_{n-l}^{(j)} = 0. \tag{5.47}$$

Thus, from (5.45) and by simple calculus, we find for $n \geq 2$,

$$\begin{aligned} \lambda_n^{(j)} = -\frac{1}{\left\| l_1^{(0)} q_0^{(j)} \right\|^2} \Bigg[& \sum_{i=2}^{n} \left(\sum_{s_1+\cdots+s_i=n} \lambda_{s_1}^{(j)} \lambda_{s_2}^{(j)} \cdots \lambda_{s_i}^{(j)} \right) \left(l_i^{(0)} q_0^{(j)}, l_1^{(0)} q_0^{(j)} \right) \\ & + \sum_{l=0}^{n-1} \sum_{k=0}^{l} \left(F_{k,l-k} q_{n-l}^{(j)}, l_1^{(0)} q_0^{(j)} \right) + \sum_{k=1}^{n} \left(F_{k,n-k} q_0^{(j)}, l_1^{(0)} q_0^{(j)} \right) \Bigg]. \end{aligned} \tag{5.48}$$

□

We now give the following lemma which seems useful to prove the fundamental result in this section.

Lemma 5.3. *Let $\mathcal{H}_0$ be a bounded neighborhood of $\overline{\Omega_0}$ in $\mathbb{R}^2$. The functions $\hat{u}_{i,j}(\epsilon)(x) = S(\hat{\lambda}_j(\epsilon)) q^{(i)}(\epsilon)(\gamma^{-1})$ are jointly analytic in the variables $(x,\epsilon) \in \mathcal{H}_0 \times]-\epsilon_4, \epsilon_4[$.*

Proof. The function $\hat{u}_{i,j}(\epsilon)(x) = S(\hat{\lambda}_j(\epsilon)) q^{(i)}(\epsilon)(\gamma^{-1})$ satisfies the Helmholtz equation in Ω_ϵ with the boundary conditions $\hat{u}_{i,j}(\epsilon)|\partial\Omega_\epsilon = 0$ and $\partial_{\nu_\epsilon} \hat{u}_{i,j}(\epsilon)(\gamma_\epsilon(t)) = q^{(j)}(\epsilon)(t)$, which are jointly analytic with respect to the variables $(t,\epsilon) \in [0,1] \times]-\epsilon_4, \epsilon_4[$. The outward unit normal ν_ϵ to $\partial\Omega_\epsilon$ is given by $\gamma_\epsilon'(t)/|\gamma_\epsilon'(t)|$, as a function of $t = \gamma^{-1}(x)$. The symbol of the operator $\Delta_x = \partial_{x_1}^2 + \partial_{x_2}^2$ is $\mathcal{S}(\xi_1, \xi_2, \epsilon) = \xi_1^2 + \xi_2^2$. Thus $\mathcal{S}(\nu_\epsilon) = 1 > 0$. Since the surface $\partial\Omega_\epsilon$ is noncharacteristic for Δ_x, the Cauchy-Kowaleski theorem implies that $\hat{u}_{i,j}(\epsilon)(x)$ is jointly analytic with respect to (x,ϵ) in $\{|x - \gamma(t)| \leq \alpha_0\} \times]-\epsilon_4, \epsilon_4[$, where α_0 is a positive constant. □

The following result holds.

Theorem 5.4. *Let $\mathcal{H}_0$ be a bounded neighborhood of $\overline{\Omega}_0$ in $\mathbb{R}^2$. Then there exists a constant $\epsilon_5 > 0$ smaller than ϵ_4 such that an orthonormal basis of eigenfunctions $(u_j(\epsilon))_j$ corresponding to the λ_0-group, $(\lambda_j(\epsilon))_j$, in $H_0^1(\Omega_\epsilon)$ can be chosen to depend holomorphically on $(x,\epsilon) \in \mathcal{H}_0 \times]-\epsilon_5, \epsilon_5[$. Moreover, these eigenfunctions satisfy the following uniform expansion: for $x \in \mathcal{H}_0$,*

$$u_j(\epsilon) = u_0^{(j)} + \sum_{n\geq 1} u_n^{(j)} \epsilon^n, \tag{5.49}$$

where the family $u_0^{(j)}$ builds a basis of eigenfunctions of (2.4) associated to λ_0^2 and normalized in $L^2(\Omega_0)$. The terms $u_n^{(j)}$ are computed from the Taylor coefficients of $q^{(j)}(\epsilon)$.

Proof. Propositions 5.1 and 5.2 imply that there exists an orthonormal basis $(q^{(i)}(\epsilon))_{1\leq i\leq mj}(t,\epsilon) \in H_\sharp^{-1/2}(]0,1[)$ of $\mathrm{Ker}(L_\epsilon(\widehat{\lambda}^{(j)}(\epsilon)))$, which is analytic in $(t,\epsilon) \in [0,1] \times]-\epsilon_4, \epsilon_4[$. We know that $S(\widehat{\lambda}_j(\epsilon))q^{(i)}(\epsilon)(\gamma^{-1})$ builds a basis of eigenfunctions of the eigenvalue problem (2.4) associated to $\widehat{\lambda}_j^2(\epsilon)$. Using the Schmidt orthogonalization process once again, we construct the desired orthonormal basis. Clearly, the functions $(\widehat{u}_{i,j}(\epsilon))_{ij}$, introduced in Lemma 5.3, build a basis of the eigenspaces corresponding to the λ_0-group, $(\lambda_j(\epsilon))_j$ in $H_0^1(\Omega_\epsilon)$. We will now give the asymptotic expansion of these functions when ϵ tends to 0. To simplify notations, we drop the subscripts i and j. Integral equations give

$$\widehat{u}(\epsilon)(x) = \int_0^1 G(\lambda(\epsilon)\,|\,x - \gamma_\epsilon(t)\,|\,)q(\epsilon)(t)\,|\,\gamma_\epsilon'(t)\,|\,dt, \quad x \in \mathcal{H}_0. \tag{5.50}$$

The perturbed eigenvalue $\lambda(\epsilon)$ lies in a small neighborhood of λ_0 for small values of ϵ. Then, there exists $\epsilon_5 > 0$ ($\epsilon_5 \leq \epsilon_4$), such that we have the following Taylor expansion:

$$G(\lambda(\epsilon)\,|\,x - \gamma_\epsilon(t)\,|\,)\,|\,\gamma_\epsilon'(t)\,| = G(\lambda_0\,|\,x - \gamma(t)\,|\,)\,|\,\gamma'(t)\,| + \sum_{k\geq 1} \epsilon^k G_k(x,t), \tag{5.51}$$

which holds uniformly in $x \in \mathcal{H}_0$ and $t \in [0,1]$. We use Proposition 5.2 to write

$$q(\epsilon)(t) = q_0(t) + \sum_{k\geq 1} \epsilon^k q_k(t), \tag{5.52}$$

uniformly in $t \in [0,1]$. Substituting the last two expansions into (5.50), we find

$$\widehat{u}(\epsilon) = \widehat{u}(0) + \sum_{k\geq 1} \epsilon^k \left[\sum_{n=1}^{k} \int_0^1 q_{k-n}(t) G_n(x,t) dt \right]. \tag{5.53}$$

Now we use the Schmidt orthogonalization process to construct from the eigenfunctions $(\widehat{u}_j(\epsilon))_j$ an orthonormal basis $(u_j(\epsilon))_j$ of the direct sum of eigenspaces associated to the λ_0-group. This method allows us to compute the asymptotic expansion of these functions. □

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Abdessatar Khelifi: Centre de Mathématiques Appliquées (CMAP), École Polytechnique, 91128 Palaiseau cedex, France

E-mail address: khelifi@cmapx.polytechnique.fr

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Celso Grebogi, Center for Applied Dynamics Research, King's College, University of Aberdeen, Aberdeen AB24 3UE, UK; grebogi@abdn.ac.uk