

# A NOTE INVOLVING $p$ -VALENTLY BAZILEVIĆ FUNCTIONS

HÜSEYIN IRMAK, KRZYSZTOF PIEJKO, AND JAN STANKIEWICZ

Received 11 July 2004 and in revised form 6 December 2004

A theorem involving  $p$ -valently Bazilević functions is considered and then its certain consequences are given.

## 1. Introduction and definitions

Let  $\mathcal{A}_n(p)$  be the class of normalized functions of the form

$$f(z) = z^p + \sum_{k=n+p}^{\infty} a_k z^k \quad (n, p \in \mathbb{N} = \{1, 2, 3, \dots\}), \quad (1.1)$$

which are analytic and  $p$ -valent in the unit disc  $\mathcal{U} = \{z \in \mathbb{C} : |z| < 1\}$ . A function  $f \in \mathcal{A}_n(p)$  is said to be in the class  $\mathcal{S}_n(p, \alpha)$  if it satisfies the inequality

$$\Re e \left\{ \frac{zf'(z)}{f(z)} \right\} > \alpha \quad (0 \leq \alpha < p, p \in \mathbb{N}, z \in \mathcal{U}). \quad (1.2)$$

Also a function  $f \in \mathcal{A}_n(p)$  is said to be a  $p$ -valently Bazilević function of type  $\beta$  ( $\beta \geq 0$ ) and order  $\gamma$  ( $0 \leq \gamma < p; p \in \mathbb{N}$ ) if there exists a function  $g$  belonging to the class  $\mathcal{S}_n(p) := \mathcal{S}_n(p, 0)$  such that

$$\Re e \left\{ \frac{zf'(z)}{[f(z)]^{1-\beta} [g(z)]^\beta} \right\} > \gamma \quad (z \in \mathcal{U}). \quad (1.3)$$

We denote the class of all such functions by  $\mathcal{B}_n(p, \beta, \gamma)$ . In particular, when  $\beta = 1$ , a function  $f \in \mathcal{K}_n(p, \gamma) := \mathcal{B}_n(p, 1, \gamma)$  is said to be  $p$ -valently close-to-convex of order  $\gamma$  in  $\mathcal{U}$ . Moreover,  $\mathcal{B}_n(p, 0, \gamma) =: \mathcal{S}_n(p, \gamma)$  when  $\beta = 0$ .

## 2. Main results and their consequences

We begin with the following lemma due to Jack [2].

LEMMA 2.1. Let  $\omega(z)$  be nonconstant and regular in  $\mathcal{U}$  with  $\omega(0) = 0$ . If  $|\omega(z)|$  attains its maximum value on the circle  $|z| = r$  ( $0 < r < 1$ ) at the point  $z_0$ , then  $z_0\omega'(z_0) = c\omega(z_0)$ , where  $c \geq 1$ .

With the aid of the above lemma, we prove the following result.

THEOREM 2.2. Let  $f \in \mathcal{A}_n(p)$ ,  $w \in \mathbb{C} \setminus \{0\}$ ,  $\beta \geq 0$ ,  $0 \leq \alpha < p$ ,  $p \in \mathbb{N}$ ,  $z \in \mathcal{U}$ , and also let the function  $\mathcal{H}$  be defined by

$$\mathcal{H}(z) = \left( \frac{zf'(z)}{zf'(z) - p[f(z)]^{1-\beta}[g(z)]^\beta} \right) \cdot \left( 1 + \frac{zf''(z)}{f'(z)} - (1-\beta) \frac{zf'(z)}{f(z)} - \beta \frac{zg'(z)}{g(z)} \right), \quad (2.1)$$

where  $g \in \mathcal{S}_n(p)$ . If  $\mathcal{H}(z)$  satisfies one of the following conditions:

$$\Re\{\mathcal{H}(z)\} \begin{cases} < |w|^{-2}\Re\{w\} & \text{when } \Re\{w\} > 0, \\ \neq 0 & \text{when } \Re\{w\} = 0, \\ > |w|^{-2}\Re\{w\} & \text{when } \Re\{w\} < 0, \end{cases} \quad (2.2)$$

or

$$\Im\{\mathcal{H}(z)\} \begin{cases} < |w|^{-2}\Im\{\bar{w}\} & \text{when } \Im\{\bar{w}\} > 0, \\ \neq 0 & \text{when } \Im\{\bar{w}\} = 0, \\ > |w|^{-2}\Im\{\bar{w}\} & \text{when } \Im\{\bar{w}\} < 0, \end{cases} \quad (2.3)$$

then

$$\left| \left( \frac{zf'(z)}{[f(z)]^{1-\beta}[g(z)]^\beta} - p \right)^w \right| < p - \alpha, \quad (2.4)$$

where the value of complex power in (2.4) is taken to be as its principal value.

*Proof.* We define the function  $\Omega$  by

$$\left( \frac{zf'(z)}{[f(z)]^{1-\beta}[g(z)]^\beta} - p \right)^w = (p - \alpha)\Omega(z), \quad (2.5)$$

where  $\beta \geq 0$ ,  $w \in \mathbb{C} \setminus \{0\}$ ,  $0 \leq \alpha < p$ ,  $p \in \mathbb{N}$ ,  $z \in \mathcal{U}$ ,  $f \in \mathcal{A}_n(p)$ , and  $g \in \mathcal{S}_n(p)$ .

We see clearly that the function  $\Omega$  is regular in  $\mathcal{U}$  and  $\Omega(0) = 0$ . Making use of the logarithmic differentiation of both sides of (2.5) with respect to the known complex variable  $z$ , and if we make use of equality (2.5) once again, then we find that

$$wz \left( \frac{zf'(z)}{[f(z)]^{1-\beta}[g(z)]^\beta} - p \right)^{-1} \left( \frac{zf'(z)}{[f(z)]^{1-\beta}[g(z)]^\beta} - p \right)' = \frac{z\Omega'(z)}{\Omega(z)}, \quad (2.6)$$

which yields

$$\mathcal{H}(z) := \frac{\bar{w}}{|w|^2} \frac{z\Omega'(z)}{\Omega(z)} \quad (w \in \mathbb{C} \setminus \{0\}; z \in \mathcal{U}). \quad (2.7)$$

Assume that there exists a point  $z_0 \in \mathcal{U}$  such that

$$\max_{|z| \leq |z_0|} |\Omega(z)| = |\Omega(z_0)| = 1 \quad (z \in \mathcal{U}). \quad (2.8)$$

Applying Lemma 2.1, we can then write

$$z_0\Omega'(z_0) = c\Omega(z_0) \quad (c \geq 1). \quad (2.9)$$

Then (2.7) yields

$$\Re e\{\mathcal{H}(z_0)\} = \Re e\left\{\frac{\bar{w}}{|w|^2} \frac{z_0\Omega'(z_0)}{\Omega(z_0)}\right\} = \Re e\{c \bar{w} |w|^{-2}\}, \quad (2.10)$$

so that

$$\Re e\{\mathcal{H}(z_0)\} = \frac{c}{|w|^2} \Re e\{\bar{w}\} \begin{cases} \geq |w|^{-2} \Re e\{w\} & \text{if } \Re e\{w\} > 0, \\ = 0 & \text{if } \Re e\{w\} = 0, \\ \leq |w|^{-2} \Re e\{w\} & \text{if } \Re e\{w\} < 0, \end{cases} \quad (2.11)$$

$$\Im m\{\mathcal{H}(z_0)\} = \frac{c}{|w|^2} \Im m\{\bar{w}\} \begin{cases} \geq |w|^{-2} \Im m\{\bar{w}\} & \text{if } \Im m\{\bar{w}\} > 0, \\ = 0 & \text{if } \Im m\{\bar{w}\} = 0, \\ \leq |w|^{-2} \Im m\{\bar{w}\} & \text{if } \Im m\{\bar{w}\} < 0. \end{cases} \quad (2.12)$$

But the inequalities in (2.11) and (2.12) contradict, respectively, the inequalities in (2.2) and (2.3). Hence, we conclude that  $|\Omega(z)| < 1$  for all  $z \in \mathcal{U}$ . Consequently, it follows from (2.5) that

$$\left| \left( \frac{zf'(z)}{[f(z)]^{1-\beta} [g(z)]^\beta} - p \right)^w \right| = (p - \alpha) |\Omega(z)| < p - \alpha. \quad (2.13)$$

Therefore, the desired proof is completed.  $\square$

This theorem has many interesting and important consequences in analytic function theory and geometric function theory. We give some of these with their corresponding geometric properties.

First, if we choose the value of the parameter  $w$  as a real number with  $w := \delta \in \mathbb{R} \setminus \{0\}$  in the theorem, then we obtain the following corollary.

**COROLLARY 2.3.** *Let  $f \in \mathcal{A}_n(p)$ ,  $\delta \in \mathbb{R} \setminus \{0\}$ ,  $\beta \geq 0$ ,  $0 \leq \alpha < p$ ,  $p \in \mathbb{N}$ ,  $z \in \mathcal{U}$ , and let the function  $\mathcal{H}$  be defined by (2.1). Also, if  $\mathcal{H}$  satisfies the following conditions:*

$$\Re e\{\mathcal{H}(z)\} \begin{cases} < |\delta|^{-2} & \text{when } \delta > 0, \\ > |\delta|^{-2} & \text{when } \delta < 0, \end{cases} \quad (2.14)$$

then

$$\Re e \left\{ \frac{zf'(z)}{[f(z)]^{1-\beta} [g(z)]^\beta} \right\} > p - (p - \alpha)^{1/\delta}. \quad (2.15)$$

Putting  $w = 1$  in the theorem, we get the following corollary.

**COROLLARY 2.4.** *Let  $f \in \mathcal{A}_n(p)$ ,  $g \in \mathcal{S}_n(p)$ ,  $\beta \geq 0$ ,  $0 \leq \alpha < p$ ,  $p \in \mathbb{N}$ ,  $z \in \mathcal{U}$ , and let the function  $\mathcal{H}$  be defined by (2.1). If  $\mathcal{H}(z)$  satisfies one of the following conditions:*

$$\Re e \{ \mathcal{H}(z) \} < 1 \quad \text{or} \quad \Im m \{ \mathcal{H}(z) \} \neq 0, \quad (2.16)$$

*then  $f \in \mathcal{B}_n(p, \beta, \alpha)$ , that is,  $f$  is a  $p$ -valently Bazilević function of type  $\beta$  and order  $\alpha$  in  $\mathcal{U}$ .*

Setting  $w = 1$  and  $\beta = 0$  in the theorem, we have the following corollary.

**COROLLARY 2.5.** *Let  $f \in \mathcal{A}_n(p)$ ,  $0 \leq \alpha < p$ ,  $p \in \mathbb{N}$ ,  $z \in \mathcal{U}$ , and let the function  $\mathcal{G}$  be defined by*

$$\mathcal{G}(z) = \left( \frac{zf'(z)}{zf'(z) - pf(z)} \right) \left( 1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} \right). \quad (2.17)$$

*If  $\mathcal{G}(z)$  satisfies one of the following conditions:*

$$\Re e \{ \mathcal{G}(z) \} < 1 \quad \text{or} \quad \Im m \{ \mathcal{G}(z) \} \neq 0, \quad (2.18)$$

*then  $f \in \mathcal{S}_n(p, \alpha)$ , that is,  $f$  is  $p$ -valently starlike of order  $\alpha$  in  $\mathcal{U}$ .*

By taking  $w = 1$  and  $\beta = 1$  in the theorem, we obtain the following corollary.

**COROLLARY 2.6.** *Let  $f \in \mathcal{A}_n(p)$ ,  $g \in \mathcal{S}_n(p)$ ,  $0 \leq \alpha < p$ ,  $p \in \mathbb{N}$ ,  $z \in \mathcal{U}$ , and let the function  $\mathcal{F}$  be defined by*

$$\mathcal{F}(z) = \left( \frac{zf'(z)}{zf'(z) - pg(z)} \right) \left( 1 + \frac{zf''(z)}{f'(z)} - \frac{zg'(z)}{g(z)} \right). \quad (2.19)$$

*If  $\mathcal{F}(z)$  satisfies one of the following conditions:*

$$\Re e \{ \mathcal{F}(z) \} < 1 \quad \text{or} \quad \Im m \{ \mathcal{F}(z) \} \neq 0, \quad (2.20)$$

*then  $f \in \mathcal{K}_n(p, \alpha)$ , that is,  $f$  is  $p$ -valently close-to-convex of order  $\alpha$  in  $\mathcal{U}$ .*

Lastly, if we take  $p = 1$  in Corollaries 2.4, 2.5, and 2.6, then we easily obtain the three important results involving Bazilević functions of type  $\beta$  ( $\beta \geq 0$ ) and order  $\alpha$  ( $0 \leq \alpha < 1$ ) in  $\mathcal{U}$ , starlike functions of order  $\alpha$  ( $0 \leq \alpha < 1$ ) in  $\mathcal{U}$ , and close-to-convex functions of order  $\alpha$  ( $0 \leq \alpha < 1$ ) in  $\mathcal{U}$ , respectively, (see, e.g., [1, 3, 4, 5]).

## Acknowledgments

This work has been carried out by the help of four-month financial support (June–September, 2004) from the TÜBİTAK (The Scientific and Technical Research Council of Turkey) which is given to the first author during the scientific research at the University of Rzeszów and Rzeszów University of Technology in Poland. This present investigation was also supported by NATO and Başkent University (Ankara, Turkey). The first author would also like to acknowledge Professor Mehmet Haberal, Rector of Baskent University, who generously supports scientific researches in all aspects. I would like to extend my thanks to Professor J. Stankiewicz and Professor J. Dziok for their kind invitation to Poland and their invaluable support for this research.

## References

- [1] M. P. Ch'én and S. Owa, *Notes on certain  $p$ -valently Bazilević functions*, Panamer. Math. J. **3** (1993), no. 4, 51–59.
- [2] I. S. Jack, *Functions starlike and convex of order  $\alpha$* , J. London Math. Soc. (2) **3** (1971), 469–474.
- [3] S. S. Miller, *Distortions properties of alpha-starlike functions*, Proc. Amer. Math. Soc. **38** (1973), 311–318.
- [4] S. S. Miller, P. T. Mocanu, and M. O. Reade, *All  $\alpha$ -convex functions are univalent and starlike*, Proc. Amer. Math. Soc. **37** (1973), 553–554.
- [5] P. T. Mocanu, *Une propriété de convexité généralisée dans la théorie de la représentation conforme*, Mathematica (Cluj) **11 (34)** (1969), 127–133 (French).

Hüseyin Irmak: Department of Mathematics Education, Faculty of Education, Başkent University, Bağlıca Campus, 06530 Etimesgut, Ankara, Turkey

*E-mail address:* hisimya@baskent.edu.tr

Krzysztof Piejko: Department of Mathematics, Faculty of Management and Marketing, Rzeszów University of Technology, 2 Wincentego Pola Street, 35-959 Rzeszów, Poland

*E-mail address:* piejko@prz.rzeszow.pl

Jan Stankiewicz: Department of Mathematics, Faculty of Management and Marketing, Rzeszów University of Technology, 2 Wincentego Pola Street, 35-959 Rzeszów, Poland; Institute of Mathematics, University of Rzeszów, 16A Rejtana Street, 35-310 Rzeszów, Poland

*E-mail address:* jstan@prz.rzeszow.pl

## Special Issue on Intelligent Computational Methods for Financial Engineering

### Call for Papers

As a multidisciplinary field, financial engineering is becoming increasingly important in today's economic and financial world, especially in areas such as portfolio management, asset valuation and prediction, fraud detection, and credit risk management. For example, in a credit risk context, the recently approved Basel II guidelines advise financial institutions to build comprehensible credit risk models in order to optimize their capital allocation policy. Computational methods are being intensively studied and applied to improve the quality of the financial decisions that need to be made. Until now, computational methods and models are central to the analysis of economic and financial decisions.

However, more and more researchers have found that the financial environment is not ruled by mathematical distributions or statistical models. In such situations, some attempts have also been made to develop financial engineering models using intelligent computing approaches. For example, an artificial neural network (ANN) is a nonparametric estimation technique which does not make any distributional assumptions regarding the underlying asset. Instead, ANN approach develops a model using sets of unknown parameters and lets the optimization routine seek the best fitting parameters to obtain the desired results. The main aim of this special issue is not to merely illustrate the superior performance of a new intelligent computational method, but also to demonstrate how it can be used effectively in a financial engineering environment to improve and facilitate financial decision making. In this sense, the submissions should especially address how the results of estimated computational models (e.g., ANN, support vector machines, evolutionary algorithm, and fuzzy models) can be used to develop intelligent, easy-to-use, and/or comprehensible computational systems (e.g., decision support systems, agent-based system, and web-based systems)

This special issue will include (but not be limited to) the following topics:

- **Computational methods:** artificial intelligence, neural networks, evolutionary algorithms, fuzzy inference, hybrid learning, ensemble learning, cooperative learning, multiagent learning

- **Application fields:** asset valuation and prediction, asset allocation and portfolio selection, bankruptcy prediction, fraud detection, credit risk management
- **Implementation aspects:** decision support systems, expert systems, information systems, intelligent agents, web service, monitoring, deployment, implementation

Authors should follow the Journal of Applied Mathematics and Decision Sciences manuscript format described at the journal site <http://www.hindawi.com/journals/jamds/>. Prospective authors should submit an electronic copy of their complete manuscript through the journal Manuscript Tracking System at <http://mts.hindawi.com/>, according to the following timetable:

Manuscript Due	December 1, 2008
First Round of Reviews	March 1, 2009
Publication Date	June 1, 2009

### Guest Editors

**Lean Yu**, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, China; Department of Management Sciences, City University of Hong Kong, Tat Chee Avenue, Kowloon, Hong Kong; [yulean@amss.ac.cn](mailto:yulean@amss.ac.cn)

**Shouyang Wang**, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, China; [sywang@amss.ac.cn](mailto:sywang@amss.ac.cn)

**K. K. Lai**, Department of Management Sciences, City University of Hong Kong, Tat Chee Avenue, Kowloon, Hong Kong; [mskklai@cityu.edu.hk](mailto:mskklai@cityu.edu.hk)