DUAL SERIES EQUATIONS INVOLVING GENERALIZED LAGUERRE POLYNOMIALS

B. M. SINGH, J. ROKNE, AND R. S. DHALIWAL

Received 1 April 2005

An exact solution is obtained for the dual series equations involving generalized Laguerre polynomials.

1. Introduction

We consider the following dual series equations:

$$\sum_{n=0}^{\infty} \frac{A_n L_n^{(\alpha)}\left[(x+b)^h\right]}{\Gamma(\alpha+n+1)} = f(x), \quad 0 < x < a, \tag{1.1}$$

$$\sum_{n=0}^{\infty} \frac{A_n L_n^{(\sigma)}\left[(x+b)^h\right]}{\Gamma(\alpha+n+\beta)} = g(x), \quad a < x < \infty, \tag{1.2}$$

where $\alpha+\beta+1 > \beta > 1-m$, $\sigma+1 > \alpha+\beta > 0$, m is a positive integer, and $0 < h < \infty$, $0 \le b < \infty$, and h and b are finite constants. $L_n^{(\alpha)}[(x+b)^h]$ is a Laguerre polynomial, A_n are unknown coefficients, and $f(x)$ and $g(x)$ are prescribed functions.

Srivastava [5, 6] has solved the following dual series equations:

$$\sum_{n=0}^{\infty} \frac{A_n L_n^{(\alpha)}(x)}{\Gamma(\alpha+n+1)} = f(x), \quad 0 < x < a, \tag{1.3}$$

$$\sum_{n=0}^{\infty} \frac{A_n L_n^{(\sigma)}(x)}{\Gamma(\alpha+n+\beta)} = g(x), \quad a < x < \infty. \tag{1.4}$$

The triple series equations (1.3) and (1.4) are a special case of the dual series equations (1.1) and (1.2) when

$$h = 1, \qquad b = 0. \tag{1.5}$$

Recently, Lowndes and Srivastava [3] have solved the triple series equations involving Laguerre polynomials. References for the solutions of dual and triple series equations

International Journal of Mathematics and Mathematical Sciences 2005:7 (2005) 1135–1139
DOI: 10.1155/IJMMS.2005.1135

involving Laguerre polynomials are given in [3]. Connected to this work, references and solutions for dual series equations are given by Sneddon [4].

The dual series equations (1.1) and (1.2) are new in the literature and have importance due to the closed-form solution. The results of this note are shown to be in agreement with those of Srivastava [5]. The analysis is purely formal and no justification had been given for the change of the order of integrations and summation.

2. Some useful results

In this section, we will discuss some results which are useful in solving dual series equations (1.1) and (1.2). The orthogonality relation for Laguerre polynomials is given by [2, page 292, equation (2)] and [2, page 293, equation (2)], from which we have

$$\int_0^\infty x^\alpha e^{-x} L_n^{(\alpha)}(x) L_m^{(\alpha)}(x)\,dx = \frac{\Gamma(\alpha+n+1)}{\Gamma(n+1)}\delta_{nm}, \quad \alpha > -1, \tag{2.1}$$

where δ_{nm} is the Kronecker delta.

We can easily find, with the help of integrals [2, page 293, equation (5)] and [2, page 405, equation (20)], that

$$\begin{aligned}\int_0^\xi & x^\alpha (\xi - x)^{\beta+m-2} L_n^{(\alpha)}(x)\,dx \\ &= \frac{\Gamma(\alpha+n+1)\Gamma(\beta+m-1)}{\Gamma(\alpha+\beta+m+n)} \xi^{\alpha+\beta+m-1} L_n^{(\alpha+\beta+m-1)}(\xi), \quad \alpha > -1,\ \beta+m > 1,\end{aligned} \tag{2.2}$$

$$\int_\xi^\infty e^{-x}(x-\xi)^{\sigma-\alpha-\beta} L_n^{(\sigma)}(x)\,dx = \Gamma(\sigma-\alpha-\beta+1) e^{-\xi} L_n^{\alpha+\beta-1}(\xi), \quad \sigma+1 > \alpha+\beta > 0. \tag{2.3}$$

From [1, page 190, equation (27)], we find that

$$\frac{d^m}{dx^m}\left[x^{\alpha+m} L_n^{(\alpha+m)}(x)\right] = \frac{\Gamma(\alpha+m+n+1)}{\Gamma(\alpha+n+1)} x^\alpha L_n^\alpha(x). \tag{2.4}$$

3. Solution of dual series equations (1.1) and (1.2)

We assume that

$$\begin{aligned} x+b = X^{1/h}, \qquad & f\left(X^{1/h}-b\right) = f_1(X), \\ g\left(X^{1/h}-b\right) = g_1(X), \qquad & b^h = c, \qquad (a+b)^h = d, \end{aligned} \tag{3.1}$$

then the dual series equations (1.1) and (1.2) can be written in the following form:

$$\sum_{n=0}^\infty \frac{A_n L_n^{(\alpha)}(X)}{\Gamma(\alpha+n+1)} = f_1(X), \quad c < X < d, \tag{3.2}$$

$$\sum_{n=0}^\infty \frac{A_n L_n^{(\sigma)}(X)}{\Gamma(\alpha+\beta+n)} = g_1(X), \quad d < X < \infty. \tag{3.3}$$

We assume that

$$\sum_{n=0}^{\infty} \frac{A_n L_n^{(\alpha)}(X)}{\Gamma(\alpha+n+1)} = f_2(X), \quad 0 < X < c. \tag{3.4}$$

Combining the series equations (3.2) and (3.4), we can write the dual series equations (3.2) and (3.3) in the form

$$\sum_{n=0}^{\infty} \frac{A_n L_n^{(\alpha)}(X)}{\Gamma(\alpha+n+1)} = F(X), \quad 0 < X < d, \tag{3.5}$$

$$\sum_{n=0}^{\infty} \frac{A_n L_n^{(\sigma)}(X)}{\Gamma(\alpha+\beta+n)} = g_1(X), \quad d < X < \infty, \tag{3.6}$$

where

$$F(X) = \begin{cases} f_2(X), & 0 < X < c, \\ f_1(X), & c < X < d. \end{cases} \tag{3.7}$$

Multiplying (3.5) by $X^{\alpha}(\xi - X)^{\beta+m-2}$, where m is a positive integer, integrating with respect to X over $(0,\xi)$, and interchanging the order of integrations, we find on using (2.2) that

$$\sum_{n=0}^{\infty} \frac{A_n L_n^{(\alpha+\beta+m-1)}(\xi)}{\Gamma(\alpha+\beta+m+n)} = \frac{\xi^{-\alpha-\beta-m+1}}{\Gamma(\beta+m-1)} \int_0^{\xi} X^{\alpha}(\xi - X)^{\beta+m-2} F(X)\,dX, \quad 0 < \xi < d, \tag{3.8}$$

where

$$\alpha > -1, \qquad \beta + m > 1. \tag{3.9}$$

If we now multiply (3.8) by $\xi^{\alpha+\beta+m-1}$, differentiate both sides m times with respect to ξ, and use formula (2.4), we find that

$$\sum_{n=0}^{\infty} \frac{A_n L_n^{(\alpha+\beta-1)}(\xi)}{\Gamma(\alpha+\beta+n)} = \frac{\xi^{-\alpha-\beta+1}}{\Gamma(\beta+m-1)} \frac{d^m}{d\xi^m} \int_0^{\xi} X^{\alpha}(\xi - X)^{\beta+m-2} F(X)\,dX, \quad 0 < \xi < d, \tag{3.10}$$

where

$$\alpha > -1, \qquad \beta + m > 1. \tag{3.11}$$

Multiplying (3.6) by $e^{-X}(X-\xi)^{\sigma-\alpha-\beta}$, integrating with respect to x over (ξ,∞), and interchanging the order of integrations, we find by using formula (2.3) that

$$\sum_{n=0}^{\infty}\frac{A_n L_n^{(\alpha+\beta-1)}(\xi)}{\Gamma(\alpha+\beta+n)}=\frac{e^{\xi}}{\Gamma(\sigma-\alpha-\beta+1)}\int_{\xi}^{\infty}e^{-X}(X-\xi)^{\sigma-\alpha-\beta}g_1(X)dX,\quad d<\xi<\infty,\tag{3.12}$$

where

$$\sigma+1>\alpha+\beta>0.\tag{3.13}$$

The left-hand sides of (3.10) and (3.12) are now identical. Making use of the orthogonality relation (2.1), we find from (3.10) and (3.12) that

$$A_n=\Gamma(n+1)\left[\int_0^d\frac{e^{-\xi}L_n^{(\alpha+\beta-1)}(\xi)F_1(\xi)d\xi}{\Gamma(\beta+m-1)}+\int_d^{\infty}\frac{\xi^{\alpha+\beta-1}L_n^{(\alpha+\beta-1)}(\xi)G(\xi)d\xi}{\Gamma(\sigma-\alpha-\beta+1)}\right],\tag{3.14}$$

where

$$F_1(\xi)=\frac{d^m}{d\xi^m}\int_0^{\xi}X^{\alpha}(\xi-X)^{\beta+m-2}F(X)dX,\tag{3.15}$$

$$G(X)=\int_{\xi}^{\infty}e^{-X}(X-\xi)^{\sigma-\alpha-\beta}g_1(X)dX,\tag{3.16}$$

provided that $\alpha+\beta+1>1-m$ and $\sigma+1>\alpha+\beta>0$.

With the help of (3.7), (3.15) can be written in the form:

$$F_1(\xi)=\frac{d^m}{d\xi^m}\left[\int_0^c X^{\alpha}(\xi-X)^{\beta+m-2}f_2(X)dX+\int_X^{\xi}X^{\alpha}(\xi-X)^{\beta+m-2}f_1(X)dX\right],\quad c<\xi.\tag{3.17}$$

When we put

$$b=0,\qquad h=1,\qquad f_2(X)=0\tag{3.18}$$

in the solution of the dual series equations (1.1) and (1.2), we then obtain the solution of the dual series equations (1.3) and (1.4) and the results are in complete agreement with those of [5].

References

[1] A. Erdélyi, W. Magnus, F. Oberhettinger, and F. G. Tricomi, *Higher Transcendental Functions. Vols. I, II*, McGraw-Hill, New York, 1953.

[2] ———, *Tables of Integral Transforms. Vol. II*, McGraw-Hill, New York, 1954.

[3] J. S. Lowndes and H. M. Srivastava, *Some triple series and triple integral equations*, J. Math. Anal. Appl. **150** (1990), no. 1, 181–187.

[4] I. N. Sneddon, *Mixed Boundary Value Problems in Potential Theory*, North-Holland, Amsterdam, 1966.

[5] H. M. Srivastava, *A note on certain dual series equations involving Laguerre polynomials*, Pacific J. Math. **30** (1969), 525–527.

[6] ———, *Dual series relations involving generalized Laguerre polynomials*, J. Math. Anal. Appl. **31** (1970), 587–594.

B. M. Singh: Department of Mathematics and Statistics, The University of Calgary, Calgary, AB, Canada T2N 1N4

J. Rokne: Department of Computer Science, The University of Calgary, Calgary, AB, Canada T2N 1N4
E-mail address: rokne@cpsc.ucalgary.ca

R. S. Dhaliwal: Department of Mathematics and Statistics, The University of Calgary, Calgary, AB, Canada T2N 1N4
E-mail addresses: dhaliwal@math.ucalgary.ca; dhali.r@shaw.ca

Journal of Applied Mathematics and Decision Sciences

Special Issue on Intelligent Computational Methods for Financial Engineering

Call for Papers

As a multidisciplinary field, financial engineering is becoming increasingly important in today's economic and financial world, especially in areas such as portfolio management, asset valuation and prediction, fraud detection, and credit risk management. For example, in a credit risk context, the recently approved Basel II guidelines advise financial institutions to build comprehensible credit risk models in order to optimize their capital allocation policy. Computational methods are being intensively studied and applied to improve the quality of the financial decisions that need to be made. Until now, computational methods and models are central to the analysis of economic and financial decisions.

However, more and more researchers have found that the financial environment is not ruled by mathematical distributions or statistical models. In such situations, some attempts have also been made to develop financial engineering models using intelligent computing approaches. For example, an artificial neural network (ANN) is a nonparametric estimation technique which does not make any distributional assumptions regarding the underlying asset. Instead, ANN approach develops a model using sets of unknown parameters and lets the optimization routine seek the best fitting parameters to obtain the desired results. The main aim of this special issue is not to merely illustrate the superior performance of a new intelligent computational method, but also to demonstrate how it can be used effectively in a financial engineering environment to improve and facilitate financial decision making. In this sense, the submissions should especially address how the results of estimated computational models (e.g., ANN, support vector machines, evolutionary algorithm, and fuzzy models) can be used to develop intelligent, easy-to-use, and/or comprehensible computational systems (e.g., decision support systems, agent-based system, and web-based systems)

This special issue will include (but not be limited to) the following topics:

- **Computational methods**: artificial intelligence, neural networks, evolutionary algorithms, fuzzy inference, hybrid learning, ensemble learning, cooperative learning, multiagent learning
- **Application fields**: asset valuation and prediction, asset allocation and portfolio selection, bankruptcy prediction, fraud detection, credit risk management
- **Implementation aspects**: decision support systems, expert systems, information systems, intelligent agents, web service, monitoring, deployment, implementation

Authors should follow the Journal of Applied Mathematics and Decision Sciences manuscript format described at the journal site http://www.hindawi.com/journals/jamds/. Prospective authors should submit an electronic copy of their complete manuscript through the journal Manuscript Tracking System at http://mts.hindawi.com/, according to the following timetable:

Manuscript Due	December 1, 2008
First Round of Reviews	March 1, 2009
Publication Date	June 1, 2009

Guest Editors

Lean Yu, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, China; Department of Management Sciences, City University of Hong Kong, Tat Chee Avenue, Kowloon, Hong Kong; yulean@amss.ac.cn

Shouyang Wang, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, China; sywang@amss.ac.cn

K. K. Lai, Department of Management Sciences, City University of Hong Kong, Tat Chee Avenue, Kowloon, Hong Kong; mskklai@cityu.edu.hk