A REFINEMENT OF NORMAL APPROXIMATION TO POISSON BINOMIAL

K. NEAMMANEE

Received 8 August 2004 and in revised form 6 December 2004

Let $X_1, X_2, \ldots, X_n$ be independent Bernoulli random variables with $P(X_j = 1) = 1 - P(X_j = 0) = p_j$ and let $S_n := X_1 + X_2 + \cdots + X_n$. S_n is called a Poisson binomial random variable and it is well known that the distribution of a Poisson binomial random variable can be approximated by the standard normal distribution. In this paper, we use Taylor's formula to improve the approximation by adding some correction terms. Our result is better than before and is of order $1/n$ in the case $p_1 = p_2 = \cdots = p_n$.

1. Introduction and main result

Let $X_1, X_2, \ldots, X_n$ be independent Bernoulli random variables with $P(X_j = 1) = p_j$ and $P(X_j = 0) = q_j$, where $0 < p_j < 1$ and $p_j + q_j = 1$ for $j = 1, 2, \ldots, n$. Let $S_n := X_1 + X_2 + \cdots + X_n$, $\mu := ES_n = p_1 + p_2 + \cdots + p_n$, and $\sigma^2 := \operatorname{Var} S_n = p_1q_1 + p_2q_2 + \cdots + p_nq_n$. In connection with Bernoulli's theorem, the following important question arises: when the number of trials is large, how can one find, at least approximately, the probability

$$P(a \le S_n \le b), \tag{1.1}$$

where $a, b = 1, 2, \ldots, n$?

De Moivre [3] was the first one who successfully attacked this difficult problem in case of $p_1 = p_2 = \cdots = p_n$ by using the standard normal distribution

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-(1/2)t^2} dt. \tag{1.2}$$

After him, in essentially the same way, but using more powerful analytical tools, Laplace [7] succeeded in establishing a simple approximation formula which is given in all books on probability. A general bound was given by Feller [5] for independent and nonidentically distributed random variables with finite third moments and by Chen and Shao [2] without assuming the existence of third moments. In the case when the random variables are identically distributed, it has long been known that the best bound is of order $1/\sqrt{n}$.

International Journal of Mathematics and Mathematical Sciences 2005:5 (2005) 717–728
DOI: 10.1155/IJMMS.2005.717

In this paper, we investigate the approximation of S_n by its asymptotic expansions. If two terms,

$$G(x) := \Phi(x) + \frac{1}{6\sqrt{2\pi}\sigma^3} \sum_{j=1}^{n} p_j q_j (p_j - q_j)(1 - x^2) e^{-x^2/2}, \tag{1.3}$$

are used, then the accuracy of the approximation is better. The use of asymptotic expansions is one of the most natural ways of refinement (see, e.g., Uspensky [12], Kolassa [6], Petrov [11], and Bhattacharya and Rao [1]). The refinement of the central limit theorem for sums of independent Bernoulli random variables has a long history.

In what follows, let

$$\Delta_n := \left| P(a \le S_n \le b) - (G(x_2) - G(x_1)) \right|, \tag{1.4}$$

where

$$x_1 = \frac{1}{\sigma}\left(a - \mu - \frac{1}{2}\right), \qquad x_2 = \frac{1}{\sigma}\left(b - \mu + \frac{1}{2}\right). \tag{1.5}$$

In the case when S_n is a binomial random variable, that is, $p_1 = p_2 = \cdots = p_n$, Uspensky [12] shows that

$$\Delta_n \le \frac{0.26 + 0.36\,|q - p|}{\sigma^2} + 2e^{-(3/2)\sigma} \tag{1.6}$$

under the condition that

$$\sigma^2 \ge 25 \tag{1.7}$$

and in 1955, Makabe [8] improved the result of Uspensky in the form of

$$\Delta_n \le \frac{0.106 + 0.054(q - p) + 0.108(q - p)^2}{\sigma^2} + 2e^{-(3/2)\sigma} \tag{1.8}$$

under the conditions that

$$p < \frac{1}{2}, \quad \sigma^2 \ge 25, \quad n \ge 100. \tag{1.9}$$

In this paper, we consider the correction terms in the case when p_i's are not necessarily equal, that is, S_n is a Poisson binomial random variable. In this case, Makabe [9] shows that

$$\Delta_n \le \frac{C}{\sigma^2} \tag{1.10}$$

for some constant $C > 0$ under the conditions that

$$\sigma^2 \ge 25, \qquad p_i < \frac{1}{2} \quad \text{for } i = 1, 2, \ldots, n. \tag{1.11}$$

Mikhaĭlov [10] calculated the constant C of Makabe and found out that

$$\Delta_n \leq \frac{2(\sigma+3)}{\sigma^3} \tag{1.12}$$

under the condition that

$$\sigma^2 \geq 100. \tag{1.13}$$

In 1995, Volkova showed that

$$\begin{aligned}\Delta_n \leq{}& \frac{2}{\sigma^2}\left(0.05\beta_4 + 0.1\beta_3^2 + 0.08\right) + \frac{2}{\sigma^3}\left(0.05\beta_3 + 0.17\beta_3\beta_4 + 0.056\right)\\ &+ \frac{2}{\sigma^4}\left(0.06\beta_4^2 + 0.27\beta_4 + 0.002\right),\end{aligned} \tag{1.14}$$

where β_3 and β_4 are the third and fourth semi-invariants of S_n, respectively. The bound of Volkova is valid under the condition (1.13).

In the present work, we improve the bounds in (1.6), (1.8), (1.10), (1.12), and (1.14) under the condition (1.13). Here is our main results.

Theorem 1.1. *For $\sigma^2 \geq 100$,*

$$\Delta_n \leq \frac{0.1618}{\sigma^2}. \tag{1.15}$$

In the proof of the main theorem, we use the idea of Uspensky which uses only Taylor's formula without using any high-power analytical tools.

Remarks 1.2. (1) In Theorem 1.1, we estimate Δ_n in case of $\sigma^2 \geq 100$. In fact, the bound is valid in the range $0 < \sigma^2 < 100$ as well. For example, if $\sigma^2 \in [25, 100)$, by using the argument of Theorem 1.1, one can get the bound of the form

$$\Delta_n \leq \frac{0.3056}{\sigma^2}. \tag{1.16}$$

In this case, Volkova [14] showed that

$$\Delta_n \leq \frac{2(2\sigma+5)}{5\sigma^3}, \tag{1.17}$$

which is larger than our result.

(2) The bound in Theorem 1.1 is correct in order (see Deheuvels et al. [4]) and in the case $p_1 = p_2 = \cdots = p_n$, the order of the bound is $1/n$.

2. Proof of main result

Let $\varphi_1, \varphi_2, \ldots, \varphi_n$ and φ be the characteristic functions of $X_1, X_2, \ldots, X_n$ and S_n, respectively. Hence

$$\varphi_j(t) = q_j + p_j e^{it} \quad \text{for } j = 1, 2, \ldots, n, \qquad \varphi(t) = \prod_{j=1}^{n} \left(q_j + p_j e^{it}\right), \tag{2.1}$$

where $i = \sqrt{-1}$. We note that the complex number $\varphi(t)$ can be represented in the form

$$\varphi_j(t) = \rho_j(t)e^{i\Theta_j(t)}, \tag{2.2}$$

where

$$\rho_j(t) := \left|\varphi_j(t)\right| = \left(p_j^2 + q_j^2 + 2p_jq_j\cos t\right)^{1/2} = \left(1 - 4p_jq_j\sin^2\frac{t}{2}\right)^{1/2}, \tag{2.3}$$

$$\Theta_j(t) := \text{argument of } \varphi_j(t) = \arctan\left(\frac{p_j\sin t}{q_j + p_j\cos t}\right). \tag{2.4}$$

Hence

$$\varphi(t) = \rho(t)e^{i\Theta(t)}, \tag{2.5}$$

where $\rho(t) = \prod_{j=1}^{n}\rho_j(t)$ and $\Theta(t) = \sum_{j=1}^{n}\Theta_j(t)(\mathrm{mod}\,2\pi)$.

Let

$$\begin{aligned}\alpha(t) &:= \Theta(t) - \mu t,\\ R(x) &:= \frac{1}{2\pi}\int_0^{\pi}\frac{\rho(t)\sin\left(\sigma xt - \alpha(t)\right)}{\sin(t/2)}dt.\end{aligned} \tag{2.6}$$

From Uspensky [12], we know that

$$P(a \le S_n \le b) = R(x_2) - R(x_1), \tag{2.7}$$

where x_1 and x_2 are defined in (1.5).

Lemma 2.1. *For $j = 1,2,\ldots,n$,*

(1) $\rho_j(t) \le e^{-(2/\pi^2)p_jq_jt^2}$ *for* $t \in [0,\pi)$,

(2) $\rho_j(t) \le e^{-(1/2)p_jq_jt^2+(1/24)p_jq_jt^4}$ *for* $t \in [0,\pi]$,

(3) $\rho_j(t) \ge e^{-(1/2)p_jq_jt^2-(1/4)p_j^2q_j^2t^4}$ *for* $t \in [0,\pi/2]$.

Proof. (1) By (2.3) and the fact that $|4p_jq_j\sin^2(t/2)| < 1$, we have

$$\ln\rho_j(t) = \frac{1}{2}\ln\left(1 - 4p_jq_j\sin^2\frac{t}{2}\right) \tag{2.8}$$

$$= -\frac{1}{2}\sum_{k=1}^{\infty}\frac{1}{k}\left(4p_jq_j\sin^2\frac{t}{2}\right)^k \tag{2.9}$$

$$\le -2p_jq_j\sin^2\frac{t}{2} \tag{2.10}$$

$$\le -\frac{2}{\pi^2}p_jq_jt^2, \tag{2.11}$$

where we have used the fact that

$$\sin\frac{t}{2} \ge \frac{t}{\pi} \quad \text{on } [0,\pi) \tag{2.12}$$

in the last inequality. Hence we have (1).

(2) By Taylor's formula, we know that $\sin^2(t/2) \geq t^2/4 - t^4/48$. So (2) follows by this fact and (2.10).

(3) Let $t \in [0, \pi/2]$. Using Taylor's formula for $f(t) = (1/3)\sin^4(t/2) + \sin^2(t/2)$, we can show that

$$\frac{t^2}{4} - \sin^2\frac{t}{2} \geq \frac{1}{3}\sin^4\frac{t}{2}. \tag{2.13}$$

Hence, by (2.9), (2.13) and the fact that $0 \leq \sin(t/2) \leq t/2$, we have

$$\begin{aligned}
\ln\rho_j(t) &= \left\{-\frac{1}{2}p_jq_jt^2 - \frac{1}{4}\left(4p_jq_j\right)^2\sin^4\frac{t}{2}\right\} + 2p_jq_j\left\{\frac{t^2}{4} - \sin^2\frac{t}{2}\right\} \\
&\quad - \frac{1}{2}\sum_{k=3}^{\infty}\frac{1}{k}\left(4p_jq_j\sin^2\frac{t}{2}\right)^k \\
&\geq \left\{-\frac{1}{2}p_jq_jt^2 - \frac{1}{4}p_j^2q_j^2t^4\right\} + \frac{2}{3}p_jq_j\sin^4\frac{t}{2} - \frac{1}{6}\sum_{k=3}^{\infty}\left(4p_jq_j\sin^2\frac{t}{2}\right)^k \\
&= \left\{-\frac{1}{2}p_jq_jt^2 - \frac{1}{4}p_j^2q_j^2t^4\right\} + \frac{2}{3}p_jq_j\sin^4\frac{t}{2} - \frac{(1/6)\left(4p_jq_j\right)^3\sin^6(t/2)}{1 - 4p_jq_j\sin^2(t/2)} \\
&\geq \left\{-\frac{1}{2}p_jq_jt^2 - \frac{1}{4}p_j^2q_j^2t^4\right\} + \frac{2}{3}p_jq_j\sin^4\frac{t}{2} - \frac{1}{3}\left(4p_jq_j\right)^3\sin^6\frac{t}{2} \\
&= \left\{-\frac{1}{2}p_jq_jt^2 - \frac{1}{4}p_j^2q_j^2t^4\right\} + \frac{2}{3}p_jq_j\sin^4\frac{t}{2}\left(1 - 32p_j^2q_j^2\sin^2\frac{t}{2}\right) \\
&\geq -\frac{1}{2}p_jq_jt^2 - \frac{1}{4}p_j^2q_j^2t^4
\end{aligned} \tag{2.14}$$

which implies that $\rho_j(t) \geq e^{-(1/2)p_jq_jt^2 - (1/4)p_j^2q_j^2t^4}$. □

We are now ready to prove the main result of this section. For convenience, we assume $\sigma^2 \geq 100$ and divide the proof into 5 steps as follows.

Step 1. We will show that $|\rho(t) - e^{-(1/2)\sigma^2t^2}| \leq (1/16)\sigma^2t^4e^{-(1/2)\sigma^2t^2}$ for $t \in [0, \sqrt{3/\sigma}]$.

From Lemma 2.1(2), we have

$$\rho(t) = \prod_{j=1}^{n}\rho_j(t) \leq e^{-(1/2)\sigma^2t^2 + (1/24)\sigma^2t^4}, \tag{2.15}$$

which implies that

$$\rho(t) - e^{-(1/2)\sigma^2t^2} \leq e^{-(1/2)\sigma^2t^2}\left(e^{(1/24)\sigma^2t^4} - 1\right) \leq \frac{1}{24}\sigma^2t^4e^{-(1/2)\sigma^2t^2 + (1/24)\sigma^2t^4}, \tag{2.16}$$

where we have used the fact that $e^x - 1 \leq xe^x$ for $x > 0$ in the last inequality.

By Lemma 2.1(3) and the fact that $0 \le p_j q_j \le 1/4$, we have

$$\begin{aligned}\rho(t) - e^{-(1/2)\sigma^2 t^2} &\ge e^{-(1/2)\sigma^2 t^2}\left(e^{-(1/4)\sum_{j=1}^{n} p_j^2 q_j^2 t^4} - 1\right)\\ &\ge e^{-(1/2)\sigma^2 t^2}\left(e^{-(1/16)\sigma^2 t^4} - 1\right)\\ &\ge -\frac{1}{16}\sigma^2 t^4 e^{-(1/2)\sigma^2 t^2},\end{aligned} \tag{2.17}$$

where we have used the fact that $e^{-x} - 1 > -x$ for $x > 0$ in the last inequality.

From (2.16), (2.17) and the fact that $e^{(1/24)\sigma^2 t^4} \le 3/2$ for $t \in [0, \sqrt{3/\sigma}]$, we have

$$\left|\rho(t) - e^{-(1/2)\sigma^2 t^2}\right| \le \frac{1}{16}\sigma^2 t^4 e^{-(1/2)\sigma^2 t^2} \quad \text{on } \left[0, \sqrt{\frac{3}{\sigma}}\right]. \tag{2.18}$$

Step 2. We will show that

$$\sin\left(\sigma x t - \alpha(t)\right) = \sin(\sigma x t) - \frac{1}{6}\sum_{j=1}^{n} p_j q_j (p_j - q_j) t^3 \cos(\sigma x t) + \triangle_1, \tag{2.19}$$

where $|\triangle_1| \le 0.0285 t^5 + 0.0035 t^6$ and $t \in [0, \sqrt{3/\sigma}]$.

From Uspensky [12, page 124], we see that

$$\Theta_j^{(1)}(0) = p_j, \qquad \Theta_j^{(2)}(0) = 0, \qquad \Theta_j^{(3)}(0) = p_j q_j (p_j - q_j), \tag{2.20}$$

and for $t \in [0, \pi/2]$,

$$\begin{aligned}\left|\Theta_j^{(3)}(t)\right| &\le \frac{9}{8} p_j q_j \left|p_j - q_j\right| \left(1 - 4 p_j q_j \sin^2 \frac{t}{2}\right)^{-3},\\ \left|\Theta_j^{(4)}(t)\right| &\le 2 p_j q_j \left|p_j - q_j\right| \left(1 - 4 p_j q_j \sin^2 \frac{t}{2}\right)^{-4} t.\end{aligned} \tag{2.21}$$

Hence, for $t \in [0, \sqrt{3/\sigma}]$ and $\sigma^2 \ge 100$, we have

$$\begin{aligned}\left|\Theta_j^{(3)}(t)\right| &\le \frac{9 p_j q_j}{8(1 - 3/4\sigma)^3} \le 1.4215 p_j q_j,\\ \left|\Theta_j^{(4)}(t)\right| &\le \frac{2 p_j q_j t}{(1 - 3/4\sigma)^4} \le 2.7319 p_j q_j t.\end{aligned} \tag{2.22}$$

Hence

$$\begin{aligned}\alpha(t) &= \frac{1}{6}\sum_{j=1}^{n} p_j q_j (p_j - q_j) t^3 + M_1(t) t^5,\\ \alpha(t) &= M_2(t) t^3,\end{aligned} \tag{2.23}$$

where $|M_1(t)| \le 0.0285$ and $|M_2(t)| \le 0.0593$. So

$$\begin{aligned}
&\sin(\sigma xt - \alpha(t)) \\
&\quad = \sin(\sigma xt)\cos(\alpha(t)) - \sin(\alpha(t))\cos(\sigma xt) \\
&\quad = \sin(\sigma xt)\left(1 - \frac{1}{2}\cos(t_0)\alpha^2(t)\right) \\
&\qquad - \left(\alpha(t) - \frac{1}{2}\sin(t_1)\alpha^2(t)\right)\cos(\sigma xt), \quad \text{for some } t_0 \text{ and } t_1, \\
&\quad = \sin(\sigma xt) - \frac{1}{6}\sum_{j=1}^{n} p_j q_j (p_j - q_j) t^3 \cos(\sigma xt) + \triangle_1,
\end{aligned} \tag{2.24}$$

where $|\triangle_1| \le |M_1(t)|t^5 + |\alpha^2(t)| \le 0.0285t^5 + 0.0035t^6$.

Step 3. We will show that

$$R(x) = \frac{1}{\pi}\int_0^{\sqrt{3/\sigma}} e^{-(1/2)\sigma^2 t^2} \frac{\sin(\sigma xt - \alpha(t))}{t}\,dt + \triangle_2, \tag{2.25}$$

where $|\triangle_2| \le 0.0713/\sigma^2$.

By Lemma 2.1(1) and (2.12), we have

$$\begin{aligned}
\frac{1}{2\pi}\int_{\sqrt{(3/4\sigma)}\pi}^{\pi} \frac{\rho(t)\sin(\sigma xt - \alpha(t))}{\sin(t/2)}\,dt &\le \frac{1}{2}\int_{\sqrt{(3/4\sigma)}\pi}^{\infty} \frac{e^{-(2/\pi^2)\sigma^2 t^2}}{t}\,dt \\
&= \frac{1}{2}\int_{\sqrt{3\sigma/2}}^{\infty} \frac{e^{-t^2}}{t}\,dt \\
&\le \frac{1}{3\sigma}\int_{\sqrt{3\sigma/2}}^{\infty} te^{-t^2}\,dt \\
&= \frac{1}{6\sigma}e^{-3\sigma/2} \\
&\le 0.0167e^{-3\sigma/2}.
\end{aligned} \tag{2.26}$$

By (2.12) and the fact that $\rho(t)$ is decreasing on $[0,\pi/2]$, we have

$$\begin{aligned}
&\frac{1}{2\pi}\int_{\sqrt{3/\sigma}}^{\sqrt{(3/4\sigma)}\pi} \frac{\rho(t)\sin(\sigma xt - \alpha(t))}{\sin(t/2)}\,dt \\
&\quad \le \frac{1}{2}\int_{\sqrt{3/\sigma}}^{\sqrt{(3/4\sigma)}\pi} \frac{\rho(t)}{t}\,dt \\
&\quad \le \frac{1}{2}\rho\left(\sqrt{\frac{3}{\sigma}}\right)\int_{\sqrt{3/\sigma}}^{\sqrt{(3/4\sigma)}\pi} \frac{1}{t}\,dt \quad \text{which by Lemma 2.1(2)} \\
&\quad \le \frac{3}{4}e^{-(3/2)\sigma}\ln\frac{\pi}{2} \\
&\quad = 0.3383e^{-(3/2)\sigma}.
\end{aligned} \tag{2.27}$$

From (2.26), (2.27) and the fact that

$$e^{(3/2)\sigma} \geq \frac{1}{10!}\left(\frac{3}{2}\sigma\right)^{10} \geq 1589\sigma^2 \quad \text{for } \sigma^2 \geq 100, \tag{2.28}$$

we have

$$R(x) = \frac{1}{2\pi}\int_0^{\sqrt{3/\sigma}} \frac{\rho(t)\sin\left(\sigma xt - \alpha(t)\right)}{\sin(t/2)}dt + \triangle_{21}, \tag{2.29}$$

where $|\triangle_{21}| \leq 0.00024/\sigma^2$.

Since $\sin x = x - \cos(x_0)(x^3/6)$ for some x_0, we have

$$\left|\frac{1}{\sin x} - \frac{1}{x}\right| = \left|\frac{\sin x - x}{x\sin x}\right| \leq \frac{x^2}{6\sin x}, \tag{2.30}$$

which implies that

$$\frac{1}{2\pi}\int_0^{\sqrt{3/\sigma}} \frac{\rho(t)\sin\left(\sigma xt - \alpha(t)\right)}{\sin(t/2)}dt = \frac{1}{\pi}\int_0^{\sqrt{3/\sigma}} \frac{\rho(t)\sin\left(\sigma xt - \alpha(t)\right)}{t}dt + \triangle_{22}, \tag{2.31}$$

where

$$\begin{aligned} |\triangle_{22}| &\leq \frac{1}{48\pi}\left|\int_0^{\sqrt{3/\sigma}} \frac{\rho(t)\sin\left(\sigma xt - \alpha(t)\right)t^2}{\sin(t/2)}dt\right| \quad \text{which by (2.12)} \\ &\leq \frac{1}{48}\int_0^{\sqrt{3/\sigma}} t\rho(t)dt \quad \text{which by Lemma 2.1(2)} \\ &\leq \frac{1}{32}\int_0^{\infty} te^{-(1/2)\sigma^2t^2}dt \\ &= \frac{1}{32\sigma^2}. \end{aligned} \tag{2.32}$$

By Step 1, we see that

$$\frac{1}{\pi}\int_0^{\sqrt{3/\sigma}} \frac{\rho(t)\sin\left(\sigma xt - \alpha(t)\right)}{t}dt = \frac{1}{\pi}\int_0^{\sqrt{3/\sigma}} \frac{e^{-(1/2)\sigma^2t^2}\sin\left(\sigma xt - \alpha(t)\right)}{t}dt + \triangle_{23}, \tag{2.33}$$

where

$$\begin{aligned} |\triangle_{23}| &\leq \frac{\sigma^2}{16\pi}\int_0^{\sqrt{3/\sigma}} t^3e^{-(1/2)\sigma^2t^2}dt \\ &\leq \frac{\sigma^2}{16\pi}\int_0^{\infty} t^3e^{-(1/2)\sigma^2t^2}dt \\ &= \frac{0.0398}{\sigma^2}. \end{aligned} \tag{2.34}$$

Hence, by (2.29)–(2.34), we have the conclusion of Step 3.

Step 4. We will show that

$$\frac{1}{\pi}\int_0^{\sqrt{3/\sigma}} e^{-(1/2)\sigma^2t^2}\frac{\sin(\sigma xt)}{t}dt = \frac{1}{\sqrt{2\pi}}\int_0^x e^{-(1/2)t^2}dt + \triangle_3, \tag{2.35}$$

where $|\triangle_3| \le (6.68\times 10^{-6})/\sigma^2$.

Note that

$$\frac{1}{\pi}\int_0^{\sqrt{3/\sigma}} e^{-(1/2)\sigma^2t^2}\frac{\sin(\sigma xt)}{t}dt = \frac{1}{\pi}\int_0^{\infty} e^{-(1/2)\sigma^2t^2}\frac{\sin(\sigma xt)}{t}dt + \triangle_3, \tag{2.36}$$

where

$$\begin{aligned} |\triangle_3| &\le \frac{1}{\pi}\int_{\sqrt{3/\sigma}}^{\infty}\frac{e^{-(1/2)\sigma^2t^2}}{t}dt \\ &\le \frac{\sigma}{3\pi}\int_{\sqrt{3/\sigma}}^{\infty} te^{-(1/2)\sigma^2t^2}dt \\ &= \frac{0.1061}{\sigma}e^{-(3/2)\sigma} \quad \text{which by (2.28)} \\ &\le \frac{6.68\times 10^{-6}}{\sigma^2}. \end{aligned} \tag{2.37}$$

Let $L(x) = \int_0^{\infty} e^{-(1/2)t^2}(\sin(xt)/t)dt$. From the well-known integral

$$\int_0^{\infty} e^{-at^2}\cos(bt)dt = \frac{1}{2}\sqrt{\frac{\pi}{a}}e^{-b^2/4a} \quad \text{for } a>0, \tag{2.38}$$

we have

$$L'(x) = \int_0^{\infty} e^{-(1/2)t^2}\cos(xt)dt = \frac{\sqrt{\pi}}{\sqrt{2}}e^{-x^2/2} \tag{2.39}$$

which implies that

$$L(x) = \frac{\sqrt{\pi}}{\sqrt{2}}\int_0^x e^{-(1/2)t^2}dt. \tag{2.40}$$

Hence

$$\begin{aligned} \frac{1}{\pi}\int_0^{\infty} e^{-(1/2)\sigma^2t^2}\frac{\sin(\sigma xt)}{t}dt &= \frac{1}{\pi}\int_0^{\infty} e^{-(1/2)t^2}\frac{\sin(xt)}{t}dt \\ &= \frac{1}{\sqrt{2\pi}}\int_0^x e^{-(1/2)t^2}dt. \end{aligned} \tag{2.41}$$

From (2.36) and (2.41), Step 4 is proved.

Step 5. We will show that

$$\frac{1}{\pi}\int_0^{\sqrt{3/\sigma}} e^{-(1/2)\sigma^2t^2}\frac{\sin\left(\sigma xt-\alpha(t)\right)}{t}dt=\frac{1}{\sqrt{2\pi}}\int_0^x e^{-(1/2)t^2}dt-G(x)+\triangle_4, \tag{2.42}$$

where $|\triangle_4|\le 0.0218/\sigma^2$.

Differentiating (2.38) twice with respect to b, we have

$$\int_0^\infty t^2e^{-at^2}\cos(bt)dt=\frac{1}{4a}\sqrt{\frac{\pi}{a}}\left(1-\frac{b^2}{2a}\right)e^{-b^2/4a}. \tag{2.43}$$

Putting $a=1/2$ and $b=x$, we have

$$\int_0^\infty t^2e^{-(1/2)t^2}\cos(xt)dt=\sqrt{\frac{\pi}{2}}(1-x^2)e^{-x^2/2}. \tag{2.44}$$

Hence

$$\begin{aligned}
&\frac{1}{6\pi}\sum_{j=1}^n p_jq_j(p_j-q_j)\int_0^{\sqrt{3/\sigma}} t^2e^{-(1/2)\sigma^2t^2}\cos(\sigma xt)dt\\
&\quad=\frac{1}{6\pi\sigma^3}\sum_{j=1}^n p_jq_j(p_j-q_j)\int_0^\infty t^2e^{-(1/2)t^2}\cos(xt)dt\\
&\qquad-\frac{1}{6\pi\sigma^3}\sum_{j=1}^n p_jq_j(p_j-q_j)\int_{\sqrt{3\sigma}}^\infty t^2e^{-(1/2)t^2}\cos(xt)dt\\
&\quad=\frac{1}{6\sqrt{2\pi}\sigma^3}(1-x^2)\sum_{j=1}^n p_jq_j(p_j-q_j)e^{-x^2/2}+\triangle_{41},
\end{aligned} \tag{2.45}$$

where

$$\begin{aligned}
|\triangle_{41}|&\le\frac{1}{6\pi\sigma}\int_{\sqrt{3\sigma}}^\infty t^2e^{-(1/2)t^2}dt\\
&\le\frac{1}{6\sqrt{3}\pi\sigma^{3/2}}\int_{\sqrt{3\sigma}}^\infty t^3e^{-(1/2)t^2}dt\\
&=\frac{2}{3\sqrt{3}\pi\sqrt{\sigma}}e^{-(3/2)\sigma}\quad\text{which by (2.28)}\\
&\le\frac{2.4\times10^{-5}}{\sigma^2}.
\end{aligned} \tag{2.46}$$

From (2.45), Steps 2 and 4,

$$\begin{aligned}
&\frac{1}{\pi}\int_0^{\sqrt{3/\sigma}} e^{-(1/2)\sigma^2t^2}\frac{\sin\left(\sigma xt-\alpha(t)\right)}{t}dt\\
&\quad=\frac{1}{\sqrt{2\pi}}\int_0^x e^{-(1/2)t^2}dt-\frac{1}{6\sqrt{2\pi}\sigma^3}(1-x^2)\sum_{j=1}^n p_jq_j(p_j-q_j)e^{-x^2/2}+\triangle_4,
\end{aligned} \tag{2.47}$$

where

$$\begin{aligned} |\triangle_4| &\leq \frac{1}{\pi}\int_0^{\sqrt{3/\sigma}} \left(0.0285t^4 + 0.0035t^5\right) e^{-(1/2)\sigma^2 t^2} dt + \frac{6.68\times 10^{-6}}{\sigma^2} + \frac{2.4\times 10^{-5}}{\sigma^2} \\ &\leq \frac{0.0095}{\sigma^2} + \frac{3.068\times 10^{-5}}{\sigma^2} \leq \frac{0.0096}{\sigma^2}. \end{aligned} \tag{2.48}$$

From Steps 3 and 5,

$$R(x) = \frac{1}{\sqrt{2\pi}}\int_0^x e^{-(1/2)t^2} dt - G(x) + \triangle_5, \tag{2.49}$$

where $|\triangle_5| < |\triangle_2| + |\triangle_4| \leq 0.0809/\sigma^2$. Hence, by (2.7), we have Theorem 1.1.

3. Example

We will demonstrate a possible application of approximation in Theorem 1.1 with the problem of estimating the distribution function of the number of empty cells in an equiprobable scheme for group allocation of particles introduced by Vatutin and Mikhaĭlov [13] as follows.

Suppose that n groups of s particles are allocated independently in N cells labelled by the numbers $1, 2, \ldots, N$. It is assumed that these particles are allocated one to a cell. Let

$$S_n := \text{number of cells remaining empty after } n \text{ groups are allocated.} \tag{3.1}$$

Vatutin and Mikhaĭlov [13] showed that the distribution function of S_n coincides with that for a sum of independent Bernoulli random variables with

$$\begin{aligned} \mu &= N\left(1 - \frac{s}{N}\right)^n, \\ \sigma^2 &= N\left(1 - \frac{s}{N}\right)^n \left[1 - N\left(1 - \frac{s}{N}\right)^n + (N-1)\left(1 - \frac{s}{N-1}\right)^n\right]. \end{aligned} \tag{3.2}$$

From Theorem 1.1, we see that

$$\Delta_n \leq \frac{0.1618}{\sigma^2}, \tag{3.3}$$

where σ^2 is defined in (3.2). We note that our bound is simpler than that in Volkova [14] and easy to evaluate.

Acknowledgments

The author would like to thank the referees for useful comments and acknowledge Thailand Research Fund for a financial support.

References

[1] R. N. Bhattacharya and R. Ranga Rao, *Normal Approximation and Asymptotic Expansions*, John Wiley & Sons, New York, 1976.

[2] L. H. Y. Chen and Q.-M. Shao, *A non-uniform Berry-Esseen bound via Stein's method*, Probab. Theory Related Fields **120** (2001), no. 2, 236–254.

[3] A. De Moivre, *The Doctrine of Chances: or, a Method of Calculating the Probability of Events in Play*, W. Pearson, London, 1718.

[4] P. Deheuvels, M. L. Puri, and S. S. Ralescu, *Asymptotic expansions for sums of nonidentically distributed Bernoulli random variables*, J. Multivariate Anal. **28** (1989), no. 2, 282–303.

[5] W. Feller, *On the Berry-Esseen theorem*, Z. Wahrscheinlichkeitstheorie und Verw. Gebiete **10** (1968), 261–268.

[6] J. E. Kolassa, *Series Approximation Methods in Statistics*, Lecture Notes in Statistics, vol. 88, Springer-Verlag, New York, 1994.

[7] P.-S. Laplace, *Theórie Analytique des Probabilités*, Mme Ve Courcier, Paris, 1812.

[8] H. Makabe, *A normal approximation to binomial distribution*, Rep. Statist. Appl. Res. Un. Japan. Sci. Engrs. **4** (1955), 47–53.

[9] ———, *The Approximations to the Poisson Binomial Distribution with Their Applications to the Sampling Inspection Theory. II*, Memo No. 19610602, Kawada Branch of Union of Japanese Scientists and Engineers, Kawada, Japan, 1961.

[10] V. G. Mikhaĭlov, *On a refinement of the central limit theorem for sums of independent random indicators*, Theory Probab. Appl. **38** (1993), no. 3, 479–489.

[11] V. V. Petrov, *Limit Theorems of Probability Theory. Sequences of Independent Random Variables*, Oxford Studies in Probability, vol. 4, The Clarendon Press, Oxford University Press, New York, 1995.

[12] J. V. Uspensky, *Introduction to Mathematical Probability*, McGraw-Hill, New York, 1937.

[13] V. A. Vatutin and V. G. Mikhaĭlov, *Limit theorems for the number of empty cells in an equiprobable scheme for group allocation of particles*, Theory Probab. Appl. **27** (1982), 734–743.

[14] A. Yu. Volkova, *A refinement of the central limit theorem for sums of independent random indicators*, Theory Probab. Appl. **40** (1995), no. 4, 791–794 .

K. Neammanee: Department of Mathematics, Faculty of Science, Chulalongkorn University, Bangkok 10330, Thailand

E-mail address: kritsana.n@chula.ac.th

Mathematical Problems in Engineering

Special Issue on Time-Dependent Billiards

Call for Papers

This subject has been extensively studied in the past years for one-, two-, and three-dimensional space. Additionally, such dynamical systems can exhibit a very important and still unexplained phenomenon, called as the Fermi acceleration phenomenon. Basically, the phenomenon of Fermi acceleration (FA) is a process in which a classical particle can acquire unbounded energy from collisions with a heavy moving wall. This phenomenon was originally proposed by Enrico Fermi in 1949 as a possible explanation of the origin of the large energies of the cosmic particles. His original model was then modified and considered under different approaches and using many versions. Moreover, applications of FA have been of a large broad interest in many different fields of science including plasma physics, astrophysics, atomic physics, optics, and time-dependent billiard problems and they are useful for controlling chaos in Engineering and dynamical systems exhibiting chaos (both conservative and dissipative chaos).

We intend to publish in this special issue papers reporting research on time-dependent billiards. The topic includes both conservative and dissipative dynamics. Papers discussing dynamical properties, statistical and mathematical results, stability investigation of the phase space structure, the phenomenon of Fermi acceleration, conditions for having suppression of Fermi acceleration, and computational and numerical methods for exploring these structures and applications are welcome.

To be acceptable for publication in the special issue of Mathematical Problems in Engineering, papers must make significant, original, and correct contributions to one or more of the topics above mentioned. Mathematical papers regarding the topics above are also welcome.

Authors should follow the Mathematical Problems in Engineering manuscript format described at http://www.hindawi.com/journals/mpe/. Prospective authors should submit an electronic copy of their complete manuscript through the journal Manuscript Tracking System at http://mts.hindawi.com/ according to the following timetable:

Manuscript Due	December 1, 2008
First Round of Reviews	March 1, 2009
Publication Date	June 1, 2009

Guest Editors

Edson Denis Leonel, Departamento de Estatística, Matemática Aplicada e Computação, Instituto de Geociências e Ciências Exatas, Universidade Estadual Paulista, Avenida 24A, 1515 Bela Vista, 13506-700 Rio Claro, SP, Brazil ; edleonel@rc.unesp.br

Alexander Loskutov, Physics Faculty, Moscow State University, Vorob'evy Gory, Moscow 119992, Russia; loskutov@chaos.phys.msu.ru