

# NOTE ON WEIGHTED CARLEMAN-TYPE INEQUALITY

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A double inequality involving the constant  $e$  is proved by using an inequality between the logarithmic mean and arithmetic mean. As an application, we generalize the weighted Carleman-type inequality.

## 1. Introduction

Let  $p > 1$  and  $a_n \geq 0$  with  $0 < \sum_{n=1}^{\infty} a_n^p < \infty$ . Then

$$\sum_{n=1}^{\infty} \left( \frac{a_1 + a_2 + \cdots + a_n}{n} \right)^p < \left( \frac{p}{p-1} \right)^p \sum_{n=1}^{\infty} a_n^p. \quad (1.1)$$

The constant  $(p/(p-1))^p$  is the best possible.

Inequality (1.1) is due to Hardy [6, page 239].

Replacing  $a_n$  in (1.1) by  $a_n^{1/p}$  for  $n \in \mathbb{N}$ , we obtain

$$\sum_{n=1}^{\infty} \left( \frac{a_1^{1/p} + a_2^{1/p} + \cdots + a_n^{1/p}}{n} \right)^p < \left( \frac{p}{p-1} \right)^p \sum_{n=1}^{\infty} a_n. \quad (1.2)$$

In (1.2), letting  $p \rightarrow \infty$ , then the following Carleman inequality [6, page 249] is deduced:

$$\sum_{n=1}^{\infty} (a_1 a_2 \cdots a_n)^{1/n} < e \sum_{n=1}^{\infty} a_n, \quad (1.3)$$

where  $a_n \geq 0$  for  $n \in \mathbb{N}$  and  $0 < \sum_{n=1}^{\infty} a_n < \infty$ . The constant  $e$  is the best possible.

Carleman's inequality (1.3) was generalized in [6, page 256] by Hardy as follows. Let  $a_n \geq 0$ ,  $\lambda_n > 0$ ,  $\Lambda_n = \sum_{m=1}^n \lambda_m$  for  $n \in \mathbb{N}$ , and  $0 < \sum_{n=1}^{\infty} \lambda_n a_n < \infty$ , then

$$\sum_{n=1}^{\infty} \lambda_n (a_1^{\lambda_1} a_2^{\lambda_2} \cdots a_n^{\lambda_n})^{1/\Lambda_n} < e \sum_{n=1}^{\infty} \lambda_n a_n. \quad (1.4)$$

Note that inequality (1.4) is usually referred to as a Carleman-type inequality or weighted Carleman-type inequality. In his original paper [5], Hardy himself said that it was Pólya who pointed out this inequality to him.

In several recent papers [2, 4, 11, 12, 13, 14, 15], some strengthened and generalized results of (1.3) and (1.4) have been given by estimating the weight coefficient  $(1 + 1/n)^n$ .

For information about the history of both Hardy's inequality and Carleman-type inequalities, please refer to [7, 9].

In this note, we will give a generalization of (1.4) as follows.

**THEOREM 1.1.** *Let  $0 < \lambda_{n+1} \leq \lambda_n$  with  $\Lambda_n = \sum_{m=1}^n \lambda_m \geq 1$  and  $\lim_{n \rightarrow \infty} \Lambda_n = \infty$ , and let  $a_n \geq 0$  for  $n \in \mathbb{N}$  satisfying  $0 < \sum_{n=1}^{\infty} \lambda_n a_n < \infty$ . Then for  $0 < p \leq 1$ ,*

$$\begin{aligned} & \sum_{n=1}^{\infty} \lambda_{n+1} (a_1^{\lambda_1} a_2^{\lambda_2} \cdots a_n^{\lambda_n})^{1/\Lambda_n} \\ & \leq \frac{1}{p} \sum_{n=1}^{\infty} \left[ \left( 1 + \frac{1}{\Lambda_n/\lambda_n} \right)^{p\Lambda_n/\lambda_n} \lambda_n a_n^p \Lambda_n^{p-1} \left( \sum_{k=1}^n \lambda_k (c_k a_k)^p \right)^{(1-p)/p} \right], \end{aligned} \quad (1.5)$$

in particular,

$$\begin{aligned} & \sum_{n=1}^{\infty} \lambda_{n+1} (a_1^{\lambda_1} a_2^{\lambda_2} \cdots a_n^{\lambda_n})^{1/\Lambda_n} \\ & < \frac{e^p}{p} \sum_{n=1}^{\infty} \left[ \left( 1 - \frac{1-2/e}{\Lambda_n/\lambda_n} \right)^p \lambda_n a_n^p \Lambda_n^{p-1} \left( \sum_{k=1}^n \lambda_k (c_k a_k)^p \right)^{(1-p)/p} \right], \end{aligned} \quad (1.6)$$

where

$$c_k^{\lambda_k} = \frac{(\Lambda_{k+1})^{\Lambda_k}}{(\Lambda_k)^{\Lambda_{k-1}}}. \quad (1.7)$$

**Remark 1.2.** In particular, taking in (1.6)  $p = 1$ , we obtain the following strengthened Hardy's inequality:

$$\sum_{n=1}^{\infty} \lambda_{n+1} (a_1^{\lambda_1} a_2^{\lambda_2} \cdots a_n^{\lambda_n})^{1/\Lambda_n} < e \sum_{n=1}^{\infty} \left( 1 - \frac{1-2/e}{\Lambda_n/\lambda_n} \right) \lambda_n a_n. \quad (1.8)$$

Taking in (1.8)  $\lambda_n \equiv 1$ , we obtain the following strengthened Carleman's inequality:

$$\sum_{n=1}^{\infty} (a_1 a_2 \cdots a_n)^{1/n} < e \sum_{n=1}^{\infty} \left( 1 - \frac{1-2/e}{n} \right) a_n. \quad (1.9)$$

## 2. Lemma

The well-known arithmetic mean  $A(a, b)$  and logarithmic mean  $L(a, b)$  of two positive numbers  $a$  and  $b$  are defined, respectively, for  $a = b$  by  $A(a, b) = L(a, b) = a$  and for  $a \neq b$

by

$$A(a,b) = \frac{a+b}{2}, \quad L(a,b) = \frac{b-a}{\ln b - \ln a}. \quad (2.1)$$

For  $a \neq b$ , we have

$$L(a,b) < A(a,b). \quad (2.2)$$

See [1] and the references therein.

LEMMA 2.1. *Let  $x \geq 1$  be a real number. Then*

$$e\left(1 - \frac{1/2}{x}\right) < \left(1 + \frac{1}{x}\right)^x \leq e\left(1 - \frac{1-2/e}{x}\right). \quad (2.3)$$

*The constants  $1/2$  and  $1 - 2/e$  are best possible.*

*Proof.* Inequality (2.3) is equivalent to

$$1 - \frac{2}{e} \leq x \left[ 1 - \frac{1}{e} \left(1 + \frac{1}{x}\right)^x \right] < \frac{1}{2}. \quad (2.4)$$

Define a function  $f$  for  $x > 0$  by

$$f(x) = x \left[ 1 - \frac{1}{e} \left(1 + \frac{1}{x}\right)^x \right]. \quad (2.5)$$

In order to prove (2.4), it is sufficient to show that the function  $f$  is strictly increasing on  $[1, \infty)$  and with

$$f(1) = 1 - \frac{2}{e}, \quad \lim_{x \rightarrow \infty} f(x) = \frac{1}{2}. \quad (2.6)$$

The following proof shows that in fact  $f'(x) > 0$  holds on  $(0, \infty)$ .

Easy computation yields

$$ef'(x) = e - [1 + xg(x)] \left(1 + \frac{1}{x}\right)^x, \quad (2.7)$$

where

$$g(x) = \ln \left(1 + \frac{1}{x}\right) - \frac{1}{x+1} = \frac{1}{L(x, x+1)} - \frac{1}{x+1}. \quad (2.8)$$

Now we are in a position to prove  $f'(x) > 0$ , which is equivalent to

$$h(x) = [1 + xg(x)] \left(1 + \frac{1}{x}\right)^x < e. \quad (2.9)$$

Differentiation yields

$$h'(x) = \left[ xg^2(x) + 2g(x) - \frac{1}{(x+1)^2} \right] \left(1 + \frac{1}{x}\right)^x. \quad (2.10)$$

In the following we show  $h'(x) > 0$ . Clearly, the equation

$$xt^2 + 2t - \frac{1}{(x+1)^2} = 0 \quad (2.11)$$

has two roots

$$t_{1,2} = \frac{-(x+1) \pm \sqrt{(x+1)^2 + x}}{x(x+1)}. \quad (2.12)$$

To prove  $h'(x) > 0$ , it is sufficient to show that

$$\frac{-(x+1) + \sqrt{(x+1)^2 + x}}{x(x+1)} = t_2 < g(x) = \frac{1}{L(x, x+1)} - \frac{1}{x+1}, \quad (2.13)$$

which is equivalent to

$$\frac{\sqrt{(x+1)^2 + x} - 1}{x(x+1)} < \frac{1}{L(x, x+1)}. \quad (2.14)$$

Inequality (2.14) holds based on the following fact:

$$\frac{\sqrt{(x+1)^2 + x} - 1}{x(x+1)} < \frac{2}{2x+1} = \frac{1}{A(x, x+1)} < \frac{1}{L(x, x+1)}. \quad (2.15)$$

Hence, the function  $h$  is increasing on  $(0, \infty)$ , and then  $h(x) < \lim_{x \rightarrow \infty} h(x) = e$ . This means  $f'(x) > 0$ , and then

$$1 - \frac{2}{e} = f(1) < \lim_{x \rightarrow \infty} f(x). \quad (2.16)$$

Using Maclaurin formula

$$(1+t)^{1/t} = e - \frac{e}{2}t + o(t), \quad (2.17)$$

we have

$$\lim_{n \rightarrow \infty} f(n) = \lim_{x \rightarrow \infty} f(x) = \lim_{t \rightarrow 0^+} f\left(\frac{1}{t}\right) = \lim_{t \rightarrow 0^+} \frac{(et)/2 + o(t)}{et} = \frac{1}{2}. \quad (2.18)$$

The proof of Lemma 2.1 is complete.  $\square$

*Remark 2.2.* There are other very sharp estimates of the crucial factor  $(1 + 1/n)^n$  in [8] and the references therein.

### 3. Proof of Theorem 1.1

By the power mean inequality, we have

$$\prod_{m=1}^n \alpha_m^{q_m} \leq \left( \sum_{m=1}^n q_m \alpha_m^p \right)^{1/p}, \quad (3.1)$$

where  $p \geq 0$ ,  $\alpha_m \geq 0$ , and  $q_m > 0$  for  $m \in \mathbb{N}$  with  $\sum_{m=1}^n q_m = 1$ .

Let  $c_m > 0$ ,  $\alpha_m = c_m a_m$ , and  $q_m = \lambda_m / \Lambda_m$ , then we obtain

$$(c_1 a_1)^{\lambda_1 / \Lambda_n} (c_2 a_2)^{\lambda_2 / \Lambda_n} \cdots (c_n a_n)^{\lambda_n / \Lambda_n} \leq \left( \frac{1}{\Lambda_n} \sum_{m=1}^n \lambda_m (c_m a_m)^p \right)^{1/p}. \quad (3.2)$$

Further, we have

$$\begin{aligned} & \sum_{n=1}^{\infty} \lambda_{n+1} (a_1^{\lambda_1} a_2^{\lambda_2} \cdots a_n^{\lambda_n})^{1/\Lambda_n} \\ &= \sum_{n=1}^{\infty} \lambda_{n+1} \frac{(c_1 a_1)^{\lambda_1 / \Lambda_n} (c_2 a_2)^{\lambda_2 / \Lambda_n} \cdots (c_n a_n)^{\lambda_n / \Lambda_n}}{(c_1^{\lambda_1} c_2^{\lambda_2} \cdots c_n^{\lambda_n})^{1/\Lambda_n}} \\ &\leq \sum_{n=1}^{\infty} \frac{\lambda_{n+1}}{(c_1^{\lambda_1} c_2^{\lambda_2} \cdots c_n^{\lambda_n})^{1/\Lambda_n}} \left( \frac{1}{\Lambda_n} \sum_{m=1}^n \lambda_m (c_m a_m)^p \right)^{1/p}. \end{aligned} \quad (3.3)$$

By the following inequality (see [3, 10])

$$\left( \sum_{m=1}^n z_m \right)^t \leq t \sum_{m=1}^n z_m \left( \sum_{k=1}^m z_k \right)^{t-1}, \quad (3.4)$$

where  $t \geq 1$  is constant and  $z_m \geq 0$  for  $m \in \mathbb{N}$ , it is easy to see that

$$\begin{aligned} & \left( \frac{1}{\Lambda_n} \sum_{m=1}^n \lambda_m (c_m a_m)^p \right)^{1/p} \leq \frac{1}{\Lambda_n} \left( \sum_{m=1}^n \lambda_m (c_m a_m)^p \right)^{1/p} \\ &\leq \frac{1}{p \Lambda_n} \sum_{m=1}^n \lambda_m (c_m a_m)^p \left( \sum_{k=1}^m \lambda_k (c_k a_k)^p \right)^{(1-p)/p}, \end{aligned} \quad (3.5)$$

where  $\Lambda_n \geq 1$  and  $0 < p \leq 1$ . Thus, we obtain from (3.3) and (3.5) that

$$\begin{aligned} & \sum_{n=1}^{\infty} \lambda_{n+1} (a_1^{\lambda_1} a_2^{\lambda_2} \cdots a_n^{\lambda_n})^{1/\Lambda_n} \\ &\leq \frac{1}{p} \sum_{n=1}^{\infty} \frac{\lambda_{n+1}}{\Lambda_n (c_1^{\lambda_1} c_2^{\lambda_2} \cdots c_n^{\lambda_n})^{1/\Lambda_n}} \sum_{m=1}^n \lambda_m (c_m a_m)^p \left( \sum_{k=1}^m \lambda_k (c_k a_k)^p \right)^{(1-p)/p} \\ &= \frac{1}{p} \sum_{m=1}^{\infty} \lambda_m (c_m a_m)^p \sum_{n=m}^{\infty} \left( \frac{\lambda_{n+1}}{\Lambda_n (c_1^{\lambda_1} c_2^{\lambda_2} \cdots c_n^{\lambda_n})^{1/\Lambda_n}} \right) \left( \sum_{k=1}^m \lambda_k (c_k a_k)^p \right)^{(1-p)/p}. \end{aligned} \quad (3.6)$$

Choosing  $c_1^{\lambda_1} c_2^{\lambda_2} \cdots c_n^{\lambda_n} = (\Lambda_{n+1})^{\Lambda_n}$  for  $n \in \mathbb{N}$  and setting  $\Lambda_0 = 0$ , we get from  $0 < \lambda_{n+1} \leq \lambda_n$  that

$$c_n = \left[ \frac{(\Lambda_{n+1})^{\Lambda_n}}{(\Lambda_n)^{\Lambda_{n-1}}} \right]^{1/\lambda_n} = \left( 1 + \frac{\lambda_{n+1}}{\Lambda_n} \right)^{\Lambda_n/\lambda_n} \Lambda_n \leq \left( 1 + \frac{\lambda_n}{\Lambda_n} \right)^{\Lambda_n/\lambda_n} \Lambda_n. \quad (3.7)$$

This implies that

$$\begin{aligned} & \sum_{n=1}^{\infty} \lambda_{n+1} (a_1^{\lambda_1} a_2^{\lambda_2} \cdots a_n^{\lambda_n})^{1/\Lambda_n} \\ & \leq \frac{1}{p} \sum_{m=1}^{\infty} \lambda_m (c_m a_m)^p \sum_{n=m}^{\infty} \frac{\lambda_{n+1}}{\Lambda_n \Lambda_{n+1}} \left( \sum_{k=1}^m \lambda_k (c_k a_k)^p \right)^{(1-p)/p} \\ & = \frac{1}{p} \sum_{m=1}^{\infty} \lambda_m (c_m a_m)^p \sum_{n=m}^{\infty} \left( \frac{1}{\Lambda_n} - \frac{1}{\Lambda_{n+1}} \right) \left( \sum_{k=1}^m \lambda_k (c_k a_k)^p \right)^{(1-p)/p} \\ & = \frac{1}{p} \sum_{m=1}^{\infty} \lambda_m (c_m a_m)^p \frac{1}{\Lambda_m} \left( \sum_{k=1}^m \lambda_k (c_k a_k)^p \right)^{(1-p)/p} \\ & \leq \frac{1}{p} \sum_{m=1}^{\infty} \left( 1 + \frac{1}{\Lambda_m/\lambda_m} \right)^{p\Lambda_m/\lambda_m} \lambda_m a_m^p \Lambda_m^{p-1} \left( \sum_{k=1}^m \lambda_k (c_k a_k)^p \right)^{(1-p)/p}. \end{aligned} \quad (3.8)$$

Hence, we obtain from the above inequality and Lemma 2.1 that

$$\begin{aligned} & \sum_{n=1}^{\infty} \lambda_{n+1} (a_1^{\lambda_1} a_2^{\lambda_2} \cdots a_n^{\lambda_n})^{1/\Lambda_n} \\ & < \frac{e^p}{p} \sum_{n=1}^{\infty} \left( 1 - \frac{1-2/e}{\Lambda_n/\lambda_n} \right)^p \lambda_n a_n^p \Lambda_n^{p-1} \left( \sum_{k=1}^n \lambda_k (c_k a_k)^p \right)^{(1-p)/p}. \end{aligned} \quad (3.9)$$

The last inequality holds strictly since the right-hand inequality of (2.3) is valid if and only if  $n = 1$ . The proof is complete.

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