CONTINUITY FOR MAXIMAL COMMUTATOR OF BOCHNER-RIESZ OPERATORS ON SOME WEIGHTED HARDY SPACES

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We show the boundedness for the commutator of Bochner-Riesz operator on some weighted H^1 space.

1. Introduction

Let b be a locally integrable function. The maximal operator $B_{*,b}^{\delta}$ associated with the commutator generated by the Bochner-Riesz operator is defined by

$$B_{*,b}^{\delta}(f)(x) = \sup_{r>0} \left| B_{r,b}^{\delta}(f)(x) \right|, \qquad (1.1)$$

where

$$B_{r,b}^{\delta}(f)(x) = \int_{R^n} B_r^{\delta}(x-y) f(y) (b(x) - b(y)) dy \qquad (1.2)$$

and $(B_r^{\delta}(\hat{f}))(\xi) = (1 - r^2 |\xi|^2)_+^{\delta} \hat{f}(\xi)$. We also define that

$$B_*^{\delta}(f)(x) = \sup_{r>0} \left| B_r^{\delta}(f)(x) \right|, \qquad (1.3)$$

which is the Bochner-Riesz operator (see [8]). Let E be the space $E = \{h : \|h\| = \sup_{r>0} |h(r)| < \infty\}$, then, for each fixed $x \in R^n$, $B_r^{\delta}(f)(x)$ may be viewed as a mapping from $[0, +\infty)$ to E, and it is clear that $B_*^{\delta}(f)(x) = \|B_r^{\delta}(f)(x)\|$ and $B_{*,b}^{\delta}(f)(x) = \|b(x) B_r^{\delta}(f)(x) - B_r^{\delta}(bf)(x)\|$.

As well known, a classical result of Coifman et al. [4] proved that the commutator $[b, T]$ generated by $\mathrm{BMO}(R^n)$ functions and the Calderón-Zygmund operator is bounded on $L^p(R^n)$ ($1 < p < \infty$). However, it was observed that $[b, T]$ is not bounded, in general, from $H^p(R^n)$ to $L^p(R^n)$ and from $L^1(R^n)$ to $L^{1,\infty}(R^n)$ for $p \leq 1$. But, if $H^p(R^n)$ is replaced by some suitable atomic space $H_b^p(R^n)$ and $H_B^1(R^n)$ (see [1, 6, 7, 9]), then $[b, T]$ maps continuously $H_b^p(R^n)$ into $L^p(R^n)$ and $H_B^1(R^n)$ into weak $L^1(R^n)$ for $p \in (n/(n+1), 1]$. The main purpose of this paper is to establish the weighted boundedness of the commutators

related to Bochner-Riesz operator and BMO(R^n) function on some weighted H^1 space. We first introduce some definitions (see [1, 6, 7, 9]).

Definition 1.1. Let b, w be locally integrable functions and $w \in A_1$ (i.e., $Mw(x) \le cw(x)$ a.e.). A bounded measurable function a on R^n is said to be (w,b)-atom if

  (i) supp $a \subset B = B(x_0,r)$,
  (ii) $\|a\|_{L^\infty} \le w(B)^{-1}$,
  (iii) $\int a(y)dy = \int a(y)b(y)dy = 0$.

A temperate distribution f is said to belong to $H_b^1(w)$ if, in the Schwartz distributional sense, it can be written as

$$f(x) = \sum_{j=1}^{\infty} \lambda_j a_j(x), \tag{1.4}$$

where a_j's are (w,b)-atoms, $\lambda_j \in \mathbb{C}$, and $\sum_{j=1}^{\infty} |\lambda_j| < \infty$. Moreover, $\|f\|_{H_b^1(w)} \sim \sum_{j=1}^{\infty} |\lambda_j|$.

Definition 1.2. Let $w \in A_1$. A function f is said to belong to weighted Block H^1 space $H_B^1(w)$ if f can be written as (1.4), where a_j's are w-atoms (i.e., a_j's satisfy Definition 1.1(i), (ii), and (iii)′ $\int a(y)dy = 0$) and $\lambda_j \in \mathbb{C}$ with

$$\sum_{j=1}^{\infty} |\lambda_j| \left(1 + \log^+ \frac{1}{|\lambda_j|}\right) < \infty. \tag{1.5}$$

Moreover, $\|f\|_{H_B^1(w)} \sim \sum_{j=1}^{\infty} |\lambda_j|(1 + \log^+((\sum_i |\lambda_i|)/|\lambda_j|))$.

Now, we formulate our results as follows.

THEOREM 1.3. *Let $b \in$ BMO(R^n) and $w \in A_1$. Then the maximal commutator $B_{*,b}^\delta$ is bounded from $H_b^1(w)$ to $L_w^1(R^n)$ when $\delta > (n-1)/2$.*

THEOREM 1.4. *Let $b \in$ BMO(R^n) and $w \in A_1$. Then the maximal commutator $B_{*,b}^\delta$ is bounded from $H_B^1(w)$ to $L_w^{1,\infty}(R^n)$ when $\delta > (n-1)/2$.*

THEOREM 1.5. *Let $b \in$ BMO(R^n) and $w \in A_1$. Then the maximal commutator $B_{*,b}^\delta$ is bounded from $H^1(w)$ to $L_w^{1,\infty}(R^n)$ when $\delta > (n-1)/2$.*

2. Proof of theorems

Proof of Theorem 1.3. It suffices to show that there exists a constant $C > 0$ such that for every (w,b)-atom a,

$$\left\| B_{*,b}^\delta(a) \right\|_{L_w^1} \le C. \tag{2.1}$$

Let a be a (w,b)-atom supported on a ball $B = B(x_0,R)$. We write

$$\int_{R^n} \left[B_{*,b}^\delta(a)(x) \right] w(x)dx$$
$$= \int_{|x-x_0| \le 2R} \left[B_{*,b}^\delta(a)(x) \right] w(x)dx + \int_{|x-x_0| > 2R} \left[B_{*,b}^\delta(a)(x) \right] w(x)dx \equiv I + II. \tag{2.2}$$

For I, taking $q > 1$, by Hölder's inequality and the L^q-boundedness of $B_{*,b}^\delta$ (see [2]), we see that

$$I \leq C\|B_{*,b}^\delta(a)\|_{L_w^q} \cdot w(2B)^{1-1/q} \leq C\|a\|_{L_w^q} w(B)^{1-1/q} \leq C. \tag{2.3}$$

For II, let $b_0 = |B(x_0,R)|^{-1} \int_{B(x_0,R)} b(y)dy$, then

$$
\begin{aligned}
II &\leq \sum_{k=1}^{\infty} \int_{2^{k+1}R \geq |x-x_0| > 2^k R} |b(x) - b_0| B_*^\delta(a)(x)w(x)dx \\
&\quad + \sum_{k=1}^{\infty} \int_{2^{k+1}R \geq |x-x_0| > 2^k R} B_*^\delta((b-b_0)a)(x)w(x)dx = II_1 + II_2.
\end{aligned}
\tag{2.4}
$$

For II_1, we choose δ_0 such that

$$\frac{n-1}{2} < \delta_0 < \min\left(\delta, \frac{n+1}{2}\right) \tag{2.5}$$

and consider the following two cases.

Case 1 $(0 < r \leq R)$. In this case, note that (see [8])

$$|B^\delta(z)| \leq C(1+|z|)^{-(\delta+(n+1)/2)}, \tag{2.6}$$

we have, for $|x - x_0| > 2|y - x_0|$,

$$
\begin{aligned}
|B_r^\delta(a)(x)| &\leq Cr^{-n} \int_{B(x_0,R)} \frac{|a(y)|}{(1+|x-y|/r)^{\delta+(n+1)/2}} dy \\
&\leq C|B|^{(\delta_0+(n+1)/2)/n} |2^{k+1}B|^{-(\delta_0+(n+1)/2)/n} w(B)^{-1}.
\end{aligned}
\tag{2.7}
$$

Case 2 $(r > R)$. In this case, note that

$$|\nabla^\beta B^\delta(z)| \leq C(1+|z|)^{-(\delta+(n+1)/2)} \tag{2.8}$$

for any $\beta = (\beta_1,\ldots,\beta_n) \in (\mathbb{N} \cup \{0\})^n$ and $|x - x_0| > 2|y - x_0|$, where

$$\nabla^\beta = \left(\frac{\partial}{\partial x_1}\right)^{\beta_1} \cdots \left(\frac{\partial}{\partial x_n}\right)^{\beta_n}, \tag{2.9}$$

by the vanishing condition of a, we gain

$$
\begin{aligned}
|B_r^\delta(a)(x)| &\leq Cr^{-(n+1)} \int_{B(x_0,R)} \frac{|a(y)||y-x_0|}{(1+|x-x_0|/r)^{\delta+(n+1)/2}} dy \\
&\leq C|B|^{(\delta_0+(n+1)/2)/n} |2^{k+1}B|^{-(\delta_0+(n+1)/2)/n} w(B)^{-1}.
\end{aligned}
\tag{2.10}
$$

Combining Case 1 with Case 2, we obtain

$$
\begin{aligned}
II_1 &\le C \sum_{k=1}^{\infty} \int_{2^{k+1}R \ge |x-x_0| > 2^k R} |b(x) - b_0| \, |B|^{(\delta_0 + (n+1)/2)/n} \\
&\quad \times |2^{k+1}B|^{-(\delta_0 + (n+1)/2)/n} w(B)^{-1} w(x) dx \\
&\le C \sum_{k=1}^{\infty} 2^{-k(\delta_0 + (n+1)/2)} w(B)^{-1} \int_{2^{k+1}R \ge |x-x_0| > 2^k R} |b(x) - b_0| \, w(x) dx .
\end{aligned}
\tag{2.11}
$$

Since $w \in A_1$, w satisfies the reverse of Hölder's inequality as follows:

$$
\left(\frac{1}{|B|} \int_B w(x)^p dx \right)^{1/p} \le \frac{C}{|B|} \int_B w(x) dx
\tag{2.12}
$$

for any ball B and some $1 < p < \infty$ (see[10]). Using the properties of $\mathrm{BMO}(R^n)$ functions (see [10]), and noting $w \in A_1$, then

$$
\frac{w(B_2)}{|B_2|} \cdot \frac{|B_1|}{w(B_1)} \le C
\tag{2.13}
$$

for all balls B_1, B_2 with $B_1 \subset B_2$. Thus, by Hölder's and reverse of Hölder's inequalities for $w \in A_1$, we get, for $1/p + 1/p' = 1$,

$$
\begin{aligned}
II_1 &\le C \sum_{k=1}^{\infty} 2^{-k(\delta_0 + (n+1)/2)} w(B)^{-1} |2^{k+1}B| \left(\frac{1}{|2^{k+1}B|} \int_{2^{k+1}B} |b(x) - b_0|^{p'} dx \right)^{1/p'} \\
&\quad \times \left(\frac{1}{|2^{k+1}B|} \int_{2^{k+1}B} w(x)^p dx \right)^{1/p} \\
&\le C \|b\|_{\mathrm{BMO}} \sum_{k=1}^{\infty} k 2^{-k(\delta_0 - (n-1)/2)} \left(\frac{w(2^k B)}{|2^k B|} \frac{|B|}{w(B)} \right) \le C .
\end{aligned}
\tag{2.14}
$$

For II_2, similar to the estimate of II_1, we obtain

$$
B_r^{\delta}((b - b_0)a)(x) \le C \|b\|_{\mathrm{BMO}} w(B)^{-1} |B|^{(\delta_0 + (n+1)/2)/n} |x - x_0|^{-(\delta_0 + (n+1)/2)} ,
\tag{2.15}
$$

thus

$$
\begin{aligned}
II_2 &\le C \|b\|_{\mathrm{BMO}} \sum_{k=1}^{\infty} w(B)^{-1} |B|^{(\delta_0 + (n+1)/2)/n} |2^k B|^{-(\delta_0 + (n+1)/2)/n} w(2^k B) \\
&\le C \|b\|_{\mathrm{BMO}} \sum_{k=1}^{\infty} 2^{-k(\delta_0 - (n-1)/2)} \left(\frac{w(2^k B)}{|2^k B|} \frac{|B|}{w(B)} \right) \le C .
\end{aligned}
\tag{2.16}
$$

This finishes the proof of Theorem 1.3.   □

To prove Theorem 1.4, we recall the following lemma (see [5, 10]).

LEMMA 2.1. *Let $w \geq 0$ and $\{g_k\}$ be a sequence of measurable functions satisfying*

$$||g_k||_{L_w^{1,\infty}} \leq 1. \tag{2.17}$$

Then, for every numerical sequence $\{\lambda_k\}$,

$$\left\|\sum_k \lambda_k g_k\right\|_{L_w^{1,\infty}} \leq C \sum_k |\lambda_k| \left(+ \log\left(\sum_j |\lambda_j| / |\lambda_k|\right)\right). \tag{2.18}$$

Proof of Theorem 1.4. By Lemma 2.1, it is enough to show that there exists a constant C such that

$$||B_{*,b}^{\delta}(a)||_{L_w^{1,\infty}} \leq C \quad \text{for each } w\text{-atom } a. \tag{2.19}$$

Let a be a w-atom supported on a ball $B = B(x_0, r)$. We write

$$
\begin{aligned}
w(\{x \in R^n &: B_{*,b}^{\delta}(a)(x) > \lambda\}) \\
&\leq w(\{x \in 2B : B_{*,b}^{\delta}(a)(x) > \lambda\}) + w(\{x \in (2B)^c : B_{*,b}^{\delta}(a)(x) > \lambda\}) = I + II.
\end{aligned} \tag{2.20}
$$

For I, by the L^q-boundedness of $B_{*,b}^{\delta}$ for $q > 1$, we gain

$$
\begin{aligned}
I &\leq \lambda^{-1} ||B_{*,b}^{\delta}(a)\chi_{2B}||_{L_w^1} \leq C\lambda^{-1} ||B_{*,b}^{\delta}(a)||_{L_w^q} \cdot w(B)^{1-1/q} \\
&\leq C\lambda^{-1} ||a||_{L_w^q} \cdot w(B)^{1-1/q} \leq C\lambda^{-1}.
\end{aligned} \tag{2.21}
$$

For II, let $b_0 = |B|^{-1} \int_B b(x)dx$, notice that

$$
\begin{aligned}
B_{*,b}^{\delta}(a)(x) &= ||b(x)B_r^{\delta}(a)(x) - B_r^{\delta}(ba)(x)|| \\
&= ||(b(x) - b_0)B_r^{\delta}(a)(x) - B_r^{\delta}((b - b_0)a)(x)|| \\
&\leq ||(b(x) - b_0)B_r^{\delta}(a)(x)|| + ||B_r^{\delta}((b - b_0)a)(x)|| \\
&\leq |b(x) - b_0| B_*^{\delta}(a)(x) + B_*^{\delta}((b - b_0)a)(x),
\end{aligned} \tag{2.22}
$$

we have

$$
\begin{aligned}
II &\leq w\left(\left\{x \in (2B)^c : |b(x) - b_0| g_\mu^*(a)(x) > \frac{\lambda}{2}\right\}\right) \\
&\quad + w\left(\left\{x \in (2B)^c : g_\mu^*((b - b_0)a)(x) > \frac{\lambda}{2}\right\}\right) = II_1 + II_2.
\end{aligned} \tag{2.23}
$$

Similar to the proof of Theorem 1.3, we get

$$
\begin{aligned}
II_1 &\leq C\lambda^{-1} \int_{(2B)^c} |b(x) - b_0| B_*^{\delta}(a)(x)w(x)dx \\
&= C\lambda^{-1} \sum_{k=1}^{\infty} \int_{2^{k+1}B \setminus 2^k B} |b(x) - b_0| B_*^{\delta}(a)(x)w(x)dx \leq C\lambda^{-1} ||b||_{\mathrm{BMO}}, \\
II_2 &\leq C\lambda^{-1} \int_{(2B)^c} B_*^{\delta}((b - b_0)a)(x)w(x)dx \leq C\lambda^{-1} ||b||_{\mathrm{BMO}}.
\end{aligned} \tag{2.24}
$$

Combining the estimate of I, II_1, and II_2, we gain

$$w(\{x \in R^n : B^\delta_{*,b}(a)(x) > \lambda\}) \leq C\lambda^{-1}\|b\|_{\mathrm{BMO}}. \tag{2.25}$$

This completes the proof of Theorem 1.4.   □

Proof of Theorem 1.5. . Given $f \in H^1(w)$, let $f = \sum_j \lambda_j a_j$ be the atomic decomposition for f. By a limiting argument, it suffices to show Theorem 1.5 for a finite sum of $f = \sum_Q \lambda_Q a_Q$ with $\sum_Q |\lambda_Q| \leq C\|f\|_{H^1(w)}$. We may assume that each Q (the supporting cube of a_Q) is dyadic. For $\lambda > 0$ by [3, Lemma 4.1], there exists a collection of pairwise disjoint dyadic cubes $\{S\}$ such that

$$\sum_{Q \subset S} |\lambda_Q| \leq C\lambda|S|, \quad \forall S,$$

$$\sum_S |S| \leq \lambda^{-1} \sum_Q |\lambda_Q|, \qquad \left\| \sum_{Q \not\subset S} \lambda_Q |Q|^{-1} \chi_Q \right\|_{L^\infty} \leq C\lambda. \tag{2.26}$$

Let $E = \bigcup_S \overline{S}$, where for a fixed cube Q, $\overline{Q}$ denotes the cube with the same center as Q but with the side-length $4\sqrt{n}$ times that of Q. Then, $|E| \leq C\lambda^{-1}\|f\|_{H^1}$. Set $M(x) = \sum_S \sum_{Q \subset S} \lambda_Q a_Q$, $N(x) = f(x) - M(x)$. By the L^2 boundedness of $B^\delta_{*,b}$ and the well-known argument, it suffices to show that

$$w(\{x \in E^c : B^\delta_{*,b}(M)(x) > \lambda\}) \leq C\lambda^{-1}\|f\|_{H^1(w)}. \tag{2.27}$$

Because $B^\delta_{*,b}(M)(x) \leq \sum_S \sum_{Q \subset S} |\lambda_Q| B^\delta_{*,b}(a_Q)(x)$, we have

$$w(\{x \in E^c : B^\delta_{*,b}(M)(x) > \lambda\})$$

$$\leq C\lambda^{-1} \int_{E^c} B^\delta_{*,b}(M)(x)w(x)dx$$

$$\leq C\lambda^{-1} \sum_S \sum_{Q \subset S} |\lambda_Q| \sum_{k=1}^\infty \int_{2^{k+1}\overline{Q} \setminus 2^k\overline{Q}} B^\delta_{*,b}(a_Q)(x)w(x)dx, \tag{2.28}$$

similar to the estimate of Theorem 1.3, we get, when $x \in E^c$,

$$B^\delta_{*,b}(a_Q)(x) \leq C\|b\|_{\mathrm{BMO}} w(B)^{-1} |Q|^{(\delta_0+(n+1)/2)/n} |x - x_0|^{-(\delta_0+(n+1)/2)}$$

$$+ C|b(x) - b_0| w(B)^{-1} 2^{-k(\delta_0+(n+1)/2)}, \tag{2.29}$$

thus, by Hölder's and reverse of Hölder's inequalities for $w \in A_1$, we obtain

$$w(\{x \in E^c : B_{*,b}^{\delta}(M)(x) > \lambda\})$$

$$\leq C\lambda^{-1}w(B)^{-1}\sum_{S}\sum_{Q \subset S}|\lambda_Q|\sum_{k=1}^{\infty}k2^{-k(\delta_0+(n+1)/2)}w(2^kQ)$$

$$\leq C\lambda^{-1}\sum_{S}\sum_{Q \subset S}|\lambda_Q|\sum_{k=1}^{\infty}k2^{-k(\delta_0-(n-1)/2)} \qquad (2.30)$$

$$\leq C\lambda^{-1}\sum_{S}\sum_{Q \subset S}|\lambda_Q| \leq C\lambda^{-1}\|f\|_{H^1(w)}.$$

This finishes the proof of Theorem 1.5. $\qquad\qquad\qquad\qquad\qquad\qquad\square$

Acknowledgment

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