

ON A THEOREM OF W. MEYER-KÖNIG AND H. TIETZ

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Let (u_n) be a sequence of real numbers and let L be an additive limitable method with some property. We prove that if the classical control modulo of the oscillatory behavior of (u_n) belonging to some class of sequences is a Tauberian condition for L , then convergence or subsequential convergence of (u_n) out of L is recovered depending on the conditions on the general control modulo of the oscillatory behavior of different order.

1. Introduction

In this paper, $O(1)$ or $o(1)$ means $O(1)$ as $n \rightarrow \infty$ or $o(1)$ as $n \rightarrow \infty$. A classical theorem of Tauber [12] asserts that an Abel's limitable sequence $u = (u_n)$ is convergent if

$$\omega_n^{(0)}(u) = n\Delta u_n = o(1). \quad (1.1)$$

To describe this, we say that (1.1) is a "Tauberian condition" for the Abel limitable method. Tauber [12] further proved that the weaker condition

$$\sigma_n^{(1)}(\omega^{(0)}(u)) = \frac{1}{n+1} \sum_{k=0}^n k\Delta u_k = o(1) \quad (1.2)$$

is also a Tauberian condition for the Abel limitable method. In [5], Meyer-König and Tietz gave the result that Tauber's passage from (1.1) to (1.2) is possible for a very general class of summability methods.

THEOREM 1.1 (Meyer-König and Tietz). *If (1.1) is a Tauberian condition for the regular and additive method L , then (1.2) is also a Tauberian condition for L .*

Both (1.1) and (1.2) are special cases of a concept introduced by Landau [3]. The definitions of slow oscillation given by Landau [3] and later by Schmidt [7] are rather cumbersome to use in the proofs. For this reason, we use a more suitable definition of slow oscillation given in [8]. Stanojević [10] proved that conditions (1.1) and (1.2) in

Tauber's theorem [12] can be replaced by the more general conditions that

$$(\omega_n^{(0)}(u)) \in S, \quad (1.3)$$

$$(\sigma_n^{(1)}(\omega_n^{(0)}(u))) \in S, \quad (1.4)$$

where S denotes the class of all slowly oscillating sequences introduced in [8]. Stanojević's passage from (1.3) to (1.4) is also possible for an additive method L , which need not to be regular, and satisfies some property.

The main objective of this paper is to obtain convergence or subsequential convergence of (u_n) by an additive method L with some property depending on the conditions on the general control modulo of the oscillatory behavior of different order if the classical control modulo of the oscillatory behavior of (u_n) belonging to some class of sequences is a Tauberian condition for L .

2. Notations and definitions

Throughout this paper, $u = (u_n)$ is a sequence of real numbers and λ_n denotes the integer part of λn . Denote by $\omega_n^{(0)}(u) = n\Delta u_n$ the classical control modulo of the oscillatory behavior of (u_n) . For each integer $m \geq 1$ and for all positive integers n , define recursively $\omega_n^{(m)}(u) = \omega_n^{(m-1)}(u) - \sigma_n^{(1)}(\omega_n^{(m-1)}(u))$ general control modulo of the oscillatory behavior of order m . For a sequence $u = (u_n)$ and for some integer $m \geq 0$, denote

$$\sigma_n^{(m)}(u) = \begin{cases} \frac{1}{n+1} \sum_{k=0}^n \sigma_k^{(m-1)}(u) = u_0 + \sum_{k=1}^n \frac{V_k^{(m-1)}(\Delta u)}{k} & \text{for } m \geq 1, \\ u_n & \text{for } m = 0, \end{cases} \quad (2.1)$$

where

$$V_n^{(m)}(\Delta u) = \begin{cases} \frac{1}{n+1} \sum_{k=0}^n V_k^{(m-1)}(\Delta u) & \text{for } m \geq 1, \\ \frac{1}{n+1} \sum_{k=0}^n k \Delta u_k & \text{for } m = 0, \end{cases} \quad (2.2)$$

$$\Delta u_n = \begin{cases} u_n - u_{n-1} & \text{for } n \geq 1, \\ u_0 & \text{for } n = 0, \end{cases}$$

and $\sigma_n^{(m)}(u) - \sigma_n^{(m+1)}(u) = V_n^{(m)}(\Delta u)$.

The Kronecker identity

$$u_n - \sigma_n^{(1)}(u) = V_n^{(0)}(\Delta u) \quad (2.3)$$

is well known and will be used extensively. A sequence (u_n) is Abel limitable to s if $\lim_{x \rightarrow 1^-} (1-x) \sum_{n=0}^{\infty} u_n x^n = s$ and Cesàro limitable to s if $\lim_n \sigma_n^{(1)}(u) = s$. If (u_n) is L limitable to s , we write $L - \lim_n u_n = s$. A limitation method L is called additive if $L - \lim_n u_n = s$ and $L - \lim_n v_n = t$ imply that $L - \lim_n (u_n + v_n) = s + t$. A sequence (u_n) is

slowly oscillating [8] if $\lim_{\lambda \rightarrow 1^+} \overline{\lim}_n \max_{n+1 \leq k \leq \lambda_n} |\sum_{j=n+1}^k \Delta u_j| = 0$. Note that every null sequence is slowly oscillating.

Since $\sigma_n^{(1)}(u) = u_0 + \sum_{k=1}^n (V_k^{(0)}(\Delta u)/k)$, from identity (2.3), we write (u_n) as

$$u_n = V_n^{(0)}(\Delta u) + \sum_{k=1}^n \frac{V_k^{(0)}(\Delta u)}{k} + u_0. \quad (2.4)$$

It is shown in [11] that if (u_n) is slowly oscillating, then $(V_n^{(0)}(\Delta u))$ is bounded. Therefore, the slow oscillation of (u_n) may be redefined in terms of its generating sequence $(V_n^{(0)}(\Delta u))$. By (2.4), it is clear that a sequence (u_n) is slowly oscillating if and only if $(V_n^{(0)}(\Delta u))$ is bounded and slowly oscillating [2].

A sequence (u_n) converges subsequentially [1, 9] if there exists a finite interval $I(u)$ such that all of the accumulation points of (u_n) are in $I(u)$ and every point of $I(u)$ is an accumulation point of $I(u)$. Notice that there are slowly oscillating sequences that do not converge subsequentially. For instance, the sequence $(\log n)$ is clearly slowly oscillating, but not subsequentially convergent.

3. Lemmas

We need the following lemmas to prove the theorems in the next section.

LEMMA 3.1 [9]. *Let (u_n) be Cesàro limitable to s . If (u_n) is slowly oscillating, then (u_n) converges to s .*

Proof. For $\lambda > 1$, we have

$$u_n - \sigma_n^{(1)}(u) = \frac{\lambda_n + 1}{\lambda_n - n} \left(\sigma_{\lambda_n}^{(1)}(u) - \sigma_n^{(1)}(u) \right) - \frac{1}{\lambda_n - n} \sum_{k=n+1}^{\lambda_n} \sum_{j=n+1}^k \Delta u_j. \quad (3.1)$$

From this identity, we have

$$\overline{\lim}_n |u_n - \sigma_n^{(1)}(u)| \leq \frac{\lambda}{\lambda - 1} \lim_n \left(\sigma_{\lambda_n}^{(1)}(u) - \sigma_n^{(1)}(u) \right) + \overline{\lim}_n \max_{n+1 \leq k \leq \lambda_n} \left| \sum_{j=n+1}^k \Delta u_j \right|. \quad (3.2)$$

Noticing that the first term on the right-hand side of (3.2) vanishes, we get $\overline{\lim}_n |u_n - \sigma_n^{(1)}(u)| \leq \overline{\lim}_n \max_{n+1 \leq k \leq \lambda_n} |\sum_{j=n+1}^k \Delta u_j|$. Finally letting $\lambda \rightarrow 1^+$, we obtain $\overline{\lim}_n |u_n - \sigma_n^{(1)}(u)| \leq 0$. This completes the proof. \square

LEMMA 3.2 [1]. *Let (u_n) be a bounded sequence. If $\Delta u_n = o(1)$, then every point of $[\underline{\lim}_n u_n, \overline{\lim}_n u_n]$ is an accumulation point of (u_n) .*

Proof. Let $\underline{\lim}_n u_n = l$, and $\overline{\lim}_n u_n = K$. If $l = K$, there is nothing to prove. Assume that (l, K) is not a singleton, and that $x \in (l, K)$ is not an accumulation point of (u_n) . Then, there exist distinct numbers b and c such that $l < b < x < c < K$ and there exists a positive integer n_1 such that for all $n \geq n_1$, in $[b, c]$ there is no point of (u_n) . From the assumption $\Delta u_n = o(1)$, it follows that there is a positive integer n_2 such that for all $n \geq n_2$, $|u_n - u_{n-1}| < c - b$. Since l and K are two distinct accumulation points, there is

a positive integer $m > \max(n_1, n_2)$ such that, $u_m < b$. Hence for some $n > m$, $u_n < b$ because there is no point of (u_n) in $[b, c]$. Then, $u_{n+1} \leq u_n + |u_{n+1} - u_n| < b + c - b = c$. Thus, $u_{n+1} < c$ but $u_{n+1} \notin [b, c]$. So $u_{n+1} < b$. By finite induction on n , for all $n > m$, $u_n < b$. Hence, $\lim_n u_n = K \leq b < c < K$, which is a contradiction. Consequently, every point of $[\underline{\lim}_n u_n, \overline{\lim}_n u_n]$ is an accumulation point of (u_n) . \square

4. Tauberian conditions for convergence

Throughout this paper, L will denote an additive limitation method with the following property: $L - \lim_n u_n = s$ implies that $L - \lim_n \sigma_n^{(1)}(u) = s$.

THEOREM 4.1. *If $(\omega_n^{(0)}(u)) \in S$ is a Tauberian condition for L , then $(\sigma_n^{(1)}(\omega^{(0)}(u))) \in S$ is also a Tauberian condition for L .*

Proof. Assume that $(\omega_n^{(0)}(u)) \in S$ is a Tauberian condition for L . Let the $L - \lim_n u_n = s$. For all nonnegative integers n , $\sigma_n^{(1)}(\omega^{(0)}(u)) = n\Delta\sigma_n^{(1)}(u)$. Since $L - \lim_n u_n = s$ implies that $L - \lim_n \sigma_n^{(1)}(u) = s$ and since $(\sigma_n^{(1)}(\omega^{(0)}(u))) \in S$, we conclude that $\lim_n \sigma_n^{(1)}(u) = s$. Using identity (2.3), it then follows that $(u_n) \in S$. Hence from Lemma 3.1, $\lim_n u_n = s$. \square

THEOREM 4.2. *If $(\omega_n^{(0)}(u)) \in S$ is a Tauberian condition for L , then $(\omega_n^{(1)}(u)) \in S$ is also a Tauberian condition for L .*

Proof. Assume that $(\omega_n^{(0)}(u)) \in S$ is a Tauberian condition for L . Let $L - \lim_n u_n = s$. For all nonnegative integers n , $\omega_n^{(1)}(u) = n\Delta V_n^{(0)}(\Delta u)$. By identity (2.3) and the additivity of L , we have $L - \lim_n V_n^{(0)}(\Delta u) = 0$. Together with $(\omega_n^{(1)}(u)) \in S$, we obtain that $V_n^{(0)}(\Delta u) = o(1)$. Since $(n\Delta\sigma_n^{(1)}(u)) = (V_n^{(0)}(\Delta u)) \in S$ and $L - \lim_n \sigma_n^{(1)}(u) = s$, it follows that (u_n) is Cesàro limitable to $L - \lim_n u_n = s$. By identity (2.3), we have $\lim_n u_n = s$. \square

Notice that in Theorem 4.2, the condition $(\omega_n^{(1)}(u)) \in S$ can be replaced by $(\omega_n^{(k)}(u)) \in S$ for any integer $k \geq 1$. Since every null sequence is slowly oscillating, in the above theorems the condition “belonging to S ” can be replaced by the condition “belonging to the class of all null sequences.” Hence, in particular, as an example of Theorem 4.1, we have the Meyer-König and Tietz theorem.

THEOREM 4.3. *If $(\omega_n^{(0)}(u)) \in S$ is a Tauberian condition for L , then $\omega_n^{(1)}(u) = O(1)$ is also a Tauberian condition for L .*

Proof. Assume that $(\omega_n^{(0)}(u)) \in S$ is a Tauberian condition for L . Let $L - \lim_n u_n = s$. Since $n\Delta V_n^{(0)}(\Delta u) = O(1)$, we have $(V_n^{(0)}(\Delta u) - V_n^{(1)}(\Delta u)) = (n\Delta V_n^{(1)}(\Delta u)) \in S$.

Since $L - \lim_n V_n^{(1)}(\Delta u) = 0$, it follows that $V_n^{(1)}(\Delta u) = o(1)$. By Lemma 3.1, we obtain $V_n^{(0)}(\Delta u) = o(1)$. From the identity $n\Delta\sigma_n^{(1)}(u) = V_n^{(0)}(\Delta u)$, and $L - \lim_n \sigma_n^{(1)}(u) = s$, it follows that $\lim_n \sigma_n^{(1)}(u) = s$. Hence from (2.3), we have $\lim_n u_n = s$. \square

The following theorems are proved in a similar manner.

THEOREM 4.4. *If $\omega_n^{(0)}(u) = O(1)$ is a Tauberian condition for L , then $\omega_n^{(1)}(u) = O(1)$ is also a Tauberian condition for L .*

THEOREM 4.5. *The following statements are equivalent.*

- (i) $\omega_n^{(0)}(u) = O(1)$ is a Tauberian condition for L .
- (ii) $(\omega_n^{(0)}(u)) \in S$ is a Tauberian condition for L .

5. Tauberian conditions for subsequential convergence

Littlewood [4] proved that

$$\omega_n^{(0)}(u) = O(1) \quad (5.1)$$

is a Tauberian condition for Abel limitable method. However, Rényi [6] noticed that

$$\sigma_n^{(1)}(\omega^{(0)}(u)) = O(1) \quad (5.2)$$

is not a Tauberian condition for Abel limitable method. We only recover convergence of the $(C, 1)$ -mean of the sequence (u_n) out of the Abel limitability of (u_n) and (5.2). Tauber's passage from (5.1) to (5.2) is also not possible for an additive limitation method L . Nevertheless, we can retrieve some information about the subsequential behavior of the sequence (u_n) by assuming an additional mild condition on (u_n) with condition (5.2). In the next theorem, we show that $\sigma_n^{(1)}(\omega^{(0)}(u)) = O(1)$ together with an additional condition on (u_n) yields subsequential convergence of (u_n) out of L -limitability of (u_n) if $\omega_n^{(0)}(u) = O(1)$ is a Tauberian condition for L .

THEOREM 5.1. *If $\omega_n^{(0)}(u) = O(1)$ is a Tauberian condition for L , then the conditions $\sigma_n^{(1)}(\omega^{(0)}(u)) = O(1)$ and $(\Delta V_n^{(0)}(\Delta u)) \in S$ are Tauberian conditions for subsequential convergence of (u_n) for L .*

Proof. Assume that $\omega_n^{(0)}(u) = O(1)$ is a Tauberian condition for L . Let $L - \lim_n u_n = s$. Since $n\Delta\sigma_n^{(1)}(u) = O(1)$ and $L - \lim_n \sigma_n^{(1)}(u) = s$, it follows that $\lim_n \sigma_n^{(1)}(u) = s$. Since $V_n^{(0)}(\Delta u) = O(1)$, from identity (2.3), (u_n) is bounded. From $\sigma_n^{(1)}(u) = \sum_{k=1}^n (V_k^{(0)}(\Delta u)/k)$, it follows that $V_n^{(0)}(\Delta u)/n = o(1)$. Since $(\Delta V_n^{(0)}(\Delta u)) \in S$, again by Lemma 3.1, $\Delta V_n^{(0)}(\Delta u) = o(1)$. By the identity $\Delta u_n - (V_n^{(0)}(\Delta u)/n) = \Delta V_n^{(0)}(\Delta u)$, we obtain $\Delta u_n = o(1)$. Therefore by Lemma 3.2, (u_n) converges subsequentially. \square

We end this section with the following result.

THEOREM 5.2. *If $(\omega_n^{(0)}(u)) \in S$ is a Tauberian condition for L , then the conditions $\sigma_n^{(1)}(\omega^{(0)}(u)) = O(1)$ and $(\Delta V_n^{(0)}(\Delta u)) \in S$ are Tauberian conditions for the subsequential convergence of (u_n) for L .*

Proof. Assume that $(\omega_n^{(0)}(u)) \in S$ is a Tauberian condition for L . Let $L - \lim_n u_n = s$. The boundedness of $(\sigma_n^{(1)}(\omega^{(0)}(u)))$ implies that $(V_n^{(1)}(\Delta u)) \in S$. Since $n\Delta\sigma_n^{(2)}(u) = V_n^{(1)}(\Delta u)$ and $L - \lim_n \sigma_n^{(2)}(u) = s$, by hypotheses, we get $\lim_n \sigma_n^{(2)}(u) = s$. Since $V_n^{(0)}(\Delta u) = O(1)$, $(\sigma_n^{(1)}(u)) \in S$. By Lemma 3.1, $\lim_n \sigma_n^{(1)}(u) = s$. By identity (2.3), (u_n) is bounded. The rest of the proof is as the proof in Theorem 5.1. \square

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