ABELIAN SUBGROUP SEPARABILITY OF SOME ONE-RELATOR GROUPS

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We prove that any group in the class of one-relator groups given by the presentation $\langle a, b; [a^m, b^n] = 1\rangle$, where m and n are integers greater than 1, is cyclic subgroup separable (or π_c). We establish some suitable properties of these groups which enable us to prove that every finitely generated abelian subgroup of any of such groups is finitely separable.

1. Introduction

Following Malcev [5], we will call a subgroup M of a group G *finitely separable* if for any element $g \in G$, not belonging to M, there exists a homomorphism φ of group G onto some finite group X such that $g\varphi \notin M\varphi$. This is equivalent to the statement that for any element $g \in G \setminus M$, there exists a normal subgroup N of finite index in G such that $g \notin MN$.

A group G is *subgroup separable* if each of its finitely generated subgroups is finitely separable. If every cyclic (or finitely generated abelian) subgroup of a group G is finitely separable, then group G is said to be *cyclic (resp., abelian) subgroup separable*. Cyclic subgroup separable groups are also called π_c *groups*.

Evidently, every π_c group is residually finite (i.e., any of its one-element subsets is finitely separable). However, the converse is not true. So, cyclic subgroup separability strengthens residual finiteness. But subgroup separability is stronger; in fact, if G is a finitely presented subgroup separable group, then G has a solvable generalized word problem (just as if G is a finitely presented residually finite group, then G has a solvable word problem). So subgroup separability is so strong that very few classes of groups satisfy it. But cyclic subgroup separability is possessed by much larger classes of groups.

Hall in [1] showed that free groups are π_c. In [5], Malcev proved that finitely generated torsion-free nilpotent groups are π_c. Free products of π_c groups are π_c; further, it was established that finite extensions of π_c groups are π_c [6, 7]. Cyclic subgroup separability of certain generalized free products of π_c groups amalgamating a cyclic subgroup was proved in [10, 8]. Some one-relator groups happen to be π_c [10, 11].

In this paper, we enlarge the classes of one-relator π_c groups. We will prove the following theorem.

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Theorem 1.1. *Any group $G_{mn} = \langle a, b; [a^m, b^n] = 1\rangle$, where m and n are integers greater than 1, is π_c.*

As we will see, any of groups G_{mn} is a special type of generalized free product of π_c groups amalgamating a finitely generated abelian subgroup. We will establish the following property of these groups.

Theorem 1.2. *Every abelian subgroup of any of groups $G_{mn}(m, n > 1)$, which contains a cyclically reduced element of length greater than 1, is cyclic.*

Therefore, the next corollary, which is the aim of this paper, is obvious.

Corollary 1.3. *Every finitely generated abelian subgroup of G_{mn}, where m and n are integers greater than 1, is finitely separable.*

We note that the assertion of the corollary is also valid when $m = 1$ or $n = 1$. Indeed, we can use the results in [10] or [11]to see it.

Our interest in the class of groups G_{mn} is that these groups are "very well behaved." Their residual finiteness is well known; it follows, for example, from the results of [2]. Thus, groups G_{mn} are Hopfian (i.e., each of their surjective endomorphisms is an automorphism) since they are finitely generated and residually finite [4]. Some properties of these groups were considered in [9] where, in particular, the description of their endomorphisms was given.

The proofs in [2, 9] made use of the presentation of group G_{mn} as amalgamated free product. The same presentation will be the crucial tool in the proof of the results stated above.

2. Preliminaries on amalgamated free products

We recall some notions and properties concerned with the construction of free product $P = (X * Y; U)$ of groups X and Y amalgamating subgroup U.

Every element $g \in P$ can be written in the following form:

(I)

$$g = x_1 x_2 \cdots x_r \quad (r \geqslant 1), \tag{2.1}$$

where for any $i = 1, 2, \ldots, r$, element x_i belongs to one of the free factors X or Y and if $r > 1$, any consecutive x_i and x_{i+1} do not belong to the same factor X or Y (nor the amalgamated subgroup U). Any such form of element g is called *reduced*.

In general, an element of group $P = (X * Y; U)$ can have more than one reduced form. Further if $g = y_1 y_2 \cdots y_s$ is one more reduced form of g, then $r = s$ and, for any $i = 1, 2, \ldots, r$, x_i and y_i belong to the same factor X or Y; more precisely, there exist elements $u_0, u_1, \ldots, u_r$ of U such that

(II)

$$x_i = u_{i-1}^{-1} y_i u_i \quad (1 \leq i \leq r)\ u_0 = u_r = 1. \tag{2.2}$$

Thus, any two reduced forms of an element g of a group $P = (X * Y; U)$ have the same number of components, which we will call the *length* of element g and denote by $l(g)$.

Remark 2.1. The length of an element introduced here is different from that in [3]. Here, elements of length 1 are elements of the free factors, whereas elements of the amalgamated subgroup have length 0 according to [3].

An element g is called *cyclically reduced* if either $l(g) = 1$ or $l(g) > 1$ and the first and the last components of its reduced form do not belong to the same factor X or Y.

In the group P, any element g which is not cyclically reduced can be written as

(III)

$$g = u \cdot v \cdot u^{-1}, \tag{2.3}$$

where u and v are elements of P with reduced forms $u = x_1 x_2 \cdots x_r$ and $v = y_1 y_2 \cdots y_s$, and element v is cyclically reduced. In these forms, elements x_r and y_1 do not belong to the same subgroup X or Y and if $s > 1$, element $y_s x_r^{-1}$, which lies in the same subgroup X or Y as elements x_r and y_s, does not belong to subgroup U.

Since a given group G is π_c if and only if for every pair of its elements f and g either $f = g^k$ for some integer k or there exists a normal subgroup N of finite index in G such that $f \not\equiv g^t \pmod{N}$ for all integers t, we state the following tool, analogous to [10, Lemma 1].

Proposition 2.2. *Let $P = (X * Y; U)$ be the free product of groups X and Y amalgamating subgroup U. Let $f, g \in P$ such that $f = x_1 x_2 \cdots x_r$ and $g = y_1 y_2 \cdots y_s$. If f is reduced and g is cyclically reduced, then $f \in \langle g \rangle$ only under the following conditions.*

(1) *Elements f and g belong to the same free factor X or Y and the equality $f = g^k$ is satisfied in that free factor, for some integer $k \neq 0$.*

(2) *Elements f and g belong to different subgroups X and Y and the equality $f = g^k$ is satisfied in the amalgamated subgroup U, for some integer $k \neq 0$.*

(3) *$l(f)$, $l(g) > 1$ and $l(g)$ divides $l(f)$; in this case, if $f = g^k$ for some positive integer $k \neq 0$, then $r = ks$ and there exist elements $u_0, u_1, \ldots, u_{ks}$ of subgroup U such that*

(P)

$$\begin{gathered} x_i = u_{i-1}^{-1} y_i u_i, \quad \text{for } i = 1, 2, \ldots, ks, \text{ where} \\ y_j = y_{j+ls} \quad \text{for } j = 1, 2, \ldots, s,\ 0 \leq l \leq k-1,\ u_0 = u_{ks} = 1. \end{gathered} \tag{2.4}$$

If integer k is negative, then similar equations hold when replacing elements y_{ks}^{-1} by y_1, y_{ks-1}^{-1} by y_2, and so on.

Proof. Let $f = x_1 x_2 \cdots x_r$ be reduced and let $g = y_1 y_2 \cdots y_s$ be cyclically reduced. If $s = l(g) > 1$, then for any integer $n \neq 0$, $l(g^n) = |n| \cdot l(g)$. Therefore, if $f = g^k$ for some integer $k \neq 0$, then either $l(f) = l(g) = 1$ or $l(f) > 1$ and $l(g) > 1$.

Consequently, we have the following possibilities.

Case 1. $l(f) = l(g) = 1$.

Subcase i. Elements f and g can belong to the same free factor X or Y, and hence the equality $f = g^k$ takes place in this factor. This corresponds to case (2) in the proposition.

Subcase ii. Let elements f and g belong to different factors. For example, let $f \in X$ and $g \in Y$; then $g^k \in Y$ and $X \ni f = g^k \in Y$, thus $f = g^k \in X \cap Y = U$. The equality $f = g^k$

now holds in the amalgamated subgroup U of the group P. Hence, (2) in the formulation of the proposition is satisfied.

Case 2. $l(f) > 1$ and $l(g) > 1$.

Since g is cyclically reduced, g^k is cyclically reduced and $l(g^k) = ks$. So, the equality $f = g^k$ implies that $r = ks$ and

$$x_1 x_2 \cdots x_r = y_1 y_2 \cdots y_s y_1 y_2 \cdots y_s \cdots y_1 y_2 \cdots y_s, \tag{2.5}$$

which we can write as

$(*)$

$$x_1 x_2 \cdots x_r = y_1 y_2 \cdots y_s y_{s+1} y_{s+2} \cdots y_{ks}, \tag{2.6}$$

where $y_j = y_{j+ls}$ for $j = 1,\ldots,s$ and $0 \le l \le k-1$. So, by applying conditions of equalities (2.2), there exists a sequence of elements $u_0,\ldots,u_{ks}$ of subgroup U such that

$$x_i = u_{i-1}^{-1} y_i u_i, \quad \text{for } i = 1,\ldots,ks, \qquad u_0 = u_{ks} = 1. \tag{2.7}$$

Thus, $x_i = u_{i-1}^{-1} y_i u_i$ for $i = 1,2,\ldots,ks$, where $y_j = y_{j+ls}$, for $j = 1,2,\ldots,s$ and $0 \le l \le k-1$, and $u_0 = u_{ks} = 1$. So, case (3) in the formulation of the proposition is satisfied.

If integer k is negative, then replacing in (2.6) elements y_{ks}^{-1} by y_1, y_{ks-1}^{-1} by y_2, and so on, we obtain equations similar to (2.4). Hence, the proposition is proven. □

3. Structure and some subgroups and quotients of groups G_{mn}

We now establish some preliminary properties of groups of kind

$$G_{mn} = \langle a, b; [a^m, b^n] = 1 \rangle, \tag{3.1}$$

where $m > 1$ and $n > 1$. We begin by describing the construction of such groups as amalgamated free products.

Let

$$H = \langle c, d; cd = dc \rangle \tag{3.2}$$

be the free abelian group of a rank 2. We denote by $\langle x \rangle$ the infinite cycle, generated by element x.

We denote by A the free product of groups $\langle a \rangle$ and H, amalgamating subgroups $\langle a^m \rangle$ and $\langle c \rangle$, through the isomorphism carrying element a^m into element c. Thus, group A has the presentation

$$A = \langle a, c, d; cd = dc, a^m = c \rangle. \tag{3.3}$$

Similarly, we define the group

$$B = \langle b, c, d; cd = dc, b^n = d \rangle, \tag{3.4}$$

which is the free product of groups $\langle b \rangle$ and H, amalgamating subgroups $\langle b^n \rangle$ and $\langle d \rangle$.

By using Tietze transformations, one proves that the free product of groups A and B, amalgamating subgroup H, coincides with group G_{mn}. Thus,

$$G_{mn} = (A * B; H), \tag{3.5}$$

where

$$A = (\langle a \rangle * H; a^m = c), \qquad B = (\langle b \rangle * H; b^n = d). \tag{3.6}$$

These notations are fixed everywhere below. So the length of an element should be considered in the decomposition of the indicated group as free product with amalgamated subgroups.

From the decomposition of groups A, B and G_{mn} as free product with amalgamated subgroup and from the properties of this construction, it follows that these groups are torsion-free. Further, [3, Corollary 4.5] permits to prove, that the center of group A coincides with subgroup $\langle c \rangle$, the center of group B is subgroup $\langle d \rangle$, and the center of group G_{mn} is trivial.

Since the amalgamated subgroups in the decomposition of groups A and B are central in the free factors, it can be checked immediately that the following proposition holds.

Proposition 3.1. *For any element g of group A, not belonging to subgroup H, $g^{-1}Hg \cap H = \langle c \rangle$. Similarly, $g^{-1}Hg \cap H = \langle d \rangle$ for $g \in B \setminus H$. In particular, for any element g of A or B and any element $h \in H, g^{-1}hg \in H$ if and only if g and h commute.*

Our next goal is to build some subgroups and quotients of group G_{mn} and examine some of their properties.

Let t be an integer greater than 1. We denote by H^t the subgroup of H, consisting of elements h^t, where $h \in H$. It is easy to see that the factor group H/H^t, denoted by $H(t)$, is the direct product of two cyclic subgroups of order t, generated by elements cH^t and dH^t, respectively. Since the order of a^m in the group $\langle a; a^{mt} = 1 \rangle$ is also equal to t, we can construct the free product

$$A(t) = (\langle a; a^{mt} = 1 \rangle * H(t); a^m = cH^t) \tag{3.7}$$

of groups $\langle a; a^{mt} = 1 \rangle$ and H/H^t amalgamating subgroup $\langle a^m = cH^t \rangle$. Since the canonical homomorphisms of the group $\langle a \rangle$ onto the group $\langle a; a^{mt} = 1 \rangle$ and of H onto the factor group H/H^t coincide on the amalgamated subgroup, (relative to the decomposition of group A as amalgamated free product), there exists a homomorphism ρ_t of group A onto group $A(t)$, extending the indicated canonical mappings.

Similarly,

$$B(t) = (\langle b; b^{nt} = 1 \rangle * H(t); b^n = dH^t) \tag{3.8}$$

is the free product of cyclic group $\langle b; b^{nt} = 1\rangle$ of order nt and group H/H^t amalgamating subgroup $\langle b^n = dH^t\rangle$, and the homomorphism σ_t of B onto $B(t)$ extends the corresponding canonical mappings.

Now we introduce and examine the free product

$$G_{mn}(t) = \big(A(t) * B(t); H(t)\big) \tag{3.9}$$

of groups $A(t)$ and $B(t)$ amalgamating subgroup H/H^t. Since the homomorphisms $\rho_t : A \to A(t)$ and $\sigma_t : B \to B(t)$, clearly, coincide on the amalgamated subgroup H, then relative to the decomposition of group $G_{mn} = (A * B; H)$, there exists a homomorphism $\tau_t : G_{mn} \to G_{mn}(t)$, extending the mappings ρ_t and σ_t.

The following assertions can be easily checked.

Proposition 3.2. *For any homomorphism φ of G_{mn} onto a finite group X, there exists an integer $t > 1$ and a homomorphism ψ of the group $G_{mn}(t)$ onto X such that $\varphi = \tau_t\psi$.*

Also, for any integers $t > 1$ and $s > 1$ such that t divides s, there exists a homomorphism $\varphi : G_{mn}(s) \to G_{mn}(t)$ such that $\tau_t = \tau_s\varphi$. These assertions are also true for groups A and B.

Proposition 3.3. *For any element g of group A (or B), not belonging to subgroup H, there exists an integer $t_0 > 1$ such that for any positive integer t divisible by t_0, element $g\rho_t$ does not belong to subgroup $H\rho_t$ (resp., $g\sigma_t$ does not belong to subgroup $H\sigma_t$).*

Proof. We will give the proof only for $g \in A \setminus H$. The case where $g \in B \setminus H$ is similar.

Let element $g \in A \setminus H$ and let the form $g = x_1x_2\cdots x_r$ be reduced relative to the decomposition of group A as an amalgamated free product.

If $r = 1$, then g belongs to subgroup $\langle a\rangle$, that is, $g = a^k$ for some integer k and m does not divide k. Therefore, for any integer t, $g\rho_t \in \langle a; a^{mt} = 1\rangle$. Thus in group $A(t)$, element $g\rho_t$ cannot belong to the amalgamated subgroup (generated by element a^m) and consequently, it cannot belong to subgroup $H(t) = H\rho_t$.

Let $r > 1$. Every component x_i, $i = 1,\ldots,r$, of the reduced form of element g looks like a^{k_i}, where integer k_i is not divisible by m, or $c^{k_i}d^{l_i}$, where $l_i \neq 0$. If $x_i = c^{k_i}d^{l_i}$ $(l_i \neq 0)$, then let t_0 be an integer which does not divide l_i for any i. So, for any t divisible by t_0, $x_i\rho_t = c^{k_i}H^t d^{l_i}H^t$, where $d^{l_i} \notin H^t$. Hence, $x_i\rho_t$ will not belong to the amalgamated subgroup of group $A(t)$ (generated, we recall, by cH^t). Also for the components x_i such that $x_i = a^{k_i}$, their images $x_i\rho_t$ will not belong to the amalgamated subgroup. Therefore, the form

$$g\rho_t = (x_1\rho_t)(x_2\rho_t)\cdots(x_r\rho_t) \tag{3.10}$$

of element $g\rho_t$ is reduced in $A(t)$ and since $r > 1$, this element does not belong to its free factor $H(t) = H\rho_t$. The proposition is demonstrated. □

From Proposition 3.3, it follows that subgroup H is finitely separable in each of the groups A and B. We then have the following proposition.

Proposition 3.4. *For any element g of G_{mn}, there exists an integer $t_0 > 1$ such that for any positive integer t divisible by t_0, the length of element $g\tau_t$ in $G_{mn}(t)$ is equal to the length of g in G_{mn}.*

We will need finite separability of the double cosets of subgroup H.

Proposition 3.5. *For any elements f and g of A (or B) such that f does not belong to the double coset HgH, there exists an integer $t_0 > 1$ such that for any positive integer t divisible by t_0, $f\rho_t$ does not belong to the double coset $H(t)(g\rho_t)H(t)$.*

Proof. We again consider only the case when elements f and g belong to subgroup A. So, suppose that $f \in A$ and does not belong to the double coset HgH. In view of Proposition 3.2, it is enough to prove that there exists a homomorphism φ of A onto a finite group X such that element $f\varphi$ of X does not belong to the double coset $(H\varphi)(g\varphi)(H\varphi)$.

To this end, let $\overline{A} = A/C$ be the quotient group of group A by its (central) subgroup $C = \langle c \rangle$. The image xC of an element $x \in A$ in $\overline{A}$ will be denoted by $\overline{x}$.

It is obvious that $\overline{A}$ is the (ordinary) free product of the cyclic group X of order m generated by $\overline{a}$, and the infinite cycle Y generated by $\overline{d}$. The canonical homomorphism of A onto $\overline{A}$ maps subgroup H onto subgroup Y, and consequently the image of the double coset HgH is the double coset $Y\overline{g}Y$. Since $C \leqslant H$, element $\overline{f}$ does not belong to this coset $Y\overline{g}Y$.

We can assume, without loss of generality, that any of the elements $\overline{f}$ and $\overline{g}$, if it is different from the identity, has a reduced form, where the first and the last syllables do not belong to subgroup Y. Every Y-syllable of these reduced forms can be written as $\overline{d}^k$, for some integer $k \neq 0$. Since the set M of all such exponents k is finite, we can choose an integer $t > 0$ such that for any $k \in M$, the inequality $t > 2|k|$ holds. We denote by $\widetilde{A} = \overline{A}/(\overline{d}^t)^{\overline{A}}$ the factor group of group $\overline{A}$ by the normal closure of the element $\overline{d}^t$ in group $\overline{A}$. The group $\widetilde{A}$ is the free product of groups X and Y/Y^t. Since different integers from M are not congruent to one another and none are not congruent to zero modulo t, the reduced forms of the images $\widetilde{f}$ and $\widetilde{g}$ of elements $\overline{f}$ and $\overline{g}$ in $\widetilde{A}$ are the same as in $\overline{A}$. In particular, $\widetilde{f}$ does not belong to the double coset $Y/Y^t\,\widetilde{g}\,Y/Y^t$. Since this coset consists of a finite number of elements and the group $\widetilde{A}$ is residually finite, there exists a normal subgroup $\widetilde{N}$ of finite index of $\widetilde{A}$ such that

$$\widetilde{f} \notin (Y/Y^t\widetilde{g}Y/Y^t) \cdot \widetilde{N}. \tag{3.11}$$

If now θ is the product of the canonical homomorphisms of A onto $\overline{A}$ and of $\overline{A}$ onto $\widetilde{A}$ and N is the full preimage by θ of the subgroup $\widetilde{N}$, then N is a normal subgroup of finite index of A and $f \notin (HgH)N$. Thus, the canonical homomorphism φ of A onto the quotient group A/N has the required property and the proposition is proven. □

So we quickly establish Theorem 1.2 to end with properties of group G_{mn}.

4. Proof of Theorem 1.2

We prove the following Lemma.

Lemma 4.1. *Let $g \in G_{mn}$ such that $g^{-1}Hg \cap H \neq 1$. Then g belongs to one of the subgroups A or B. In particular, any element of G_{mn}, which commutes with a nonidentity element of subgroup H, belongs to A or B.*

Proof. We will prove that if the length of an element $g \in G_{mn}$ is greater than 1, then $g^{-1}Hg \cap H = 1$. Hence, obviously, the lemma follows.

So let $g \in G_{mn} = (A * B; H)$ with reduced form $g = g_1 g_2 \cdots g_r$ and let $r > 1$. Assume that $g_1 \in A$; the case when $g_1 \in B$ is similar. Let $g^{-1} h_1 g = h_2$ for some elements h_1 and h_2 of the subgroup H. We write this equality more precisely:

(IV)

$$g_r^{-1} \cdots g_2^{-1} g_1^{-1} h_1 g_1 g_2 \cdots g_r = h_2. \tag{4.1}$$

Since the expression of the left-hand side of equality (4.1) is not reduced in A, we have the inclusion $g_1^{-1} h_1 g_1 \in H$. Hence, it follows by Proposition 3.1 that, $h_1 \in \langle c \rangle$ and $g_1^{-1} h_1 g_1 = h_1$ because g_1 does not belong to H and $g_1^{-1} H g_1 \cap H = \langle c \rangle$. Therefore, equality (4.1) becomes

$$g_r^{-1} \cdots g_2^{-1} h_1 g_2 \cdots g_r = h_2, \tag{4.2}$$

and hence in any case $r = 2$ or $r > 2$, the inclusion $g_2^{-1} h_1 g_2 \in H$ takes place. Since g_2 belongs to subgroup B and also does not belong to subgroup H, $h_1 \in \langle d \rangle$ because $g_2^{-1} H g_2 \cap H = \langle d \rangle$ (see Proposition 3.1). Thus, $h_1 \in \langle c \rangle \cap \langle d \rangle = 1$. Consequently, $h_1 = 1$ as required. □

Proof of Theorem 1.2. Let U be an abelian subgroup of group G_{mn} and let U contain a cyclically reduced element $u = u_1 u_2 \cdots u_r$ of length greater than 1. It is not difficult to see that any other nonidentity element v of U is cyclically reduced of length greater than 1.

Indeed, suppose that $v = v_1 v_2 \cdots v_s$ is another nonidentity element of U. So $uv = vu$. Then certainly $v \notin H$, since $U \cap H = 1$. If $s = 1$, then v belongs to one of the subgroups A or B together with one of the elements u_1 or u_r. Let for example v and u_1 belong to the same subgroup A or B and let u_r and v belong to different subgroups A and B. From $uv = vu$, we have the equality

$$u_1 u_2 \cdots u_r v = v u_1 u_2 \cdots u_r, \tag{4.3}$$

from which the left-hand side has length $r + 1$, while the right-hand side has length $r - 1$ or r, depending on whether element $v u_1$ belongs to H or not. Hence, this inequality of lengths is a contradiction which proves that $s > 1$. Now suppose that v is not cyclically reduced. This means that elements v_1 and v_s belong to the same subgroup A or B and this subgroup contains one of the elements u_1 and u_r; let it be u_1. Then from the equality $uv = vu$, that is,

$$u_1 u_2 \cdots u_r v_1 v_2 \cdots v_s = v_1 v_2 \cdots v_s u_1 u_2 \cdots u_r, \tag{4.4}$$

the left-hand side has length $r + s$, while the right-hand side has length $r + s - 1$; this is again impossible. So, element v is cyclically reduced, and consequently any nonidentity element from U is cyclically reduced and has length greater than 1.

Let u be now a nonidentity element of minimum length from subgroup U and let $u = u_1 u_2 \cdots u_r$ be reduced. We claim that subgroup U is generated by u. Namely, for any nonidentity element $v \in U$, we will prove by induction on $l(v)$ that v is equal to some power of u.

Let $v = v_1 v_2 \cdots v_s$ be a reduced form of such an element v. We recall that by the selection of the element u, we have the inequality $s \geqslant r$. If we replace, when necessary, v by v^{-1}, we can, without loss of generality, consider that elements u_1 and v_1 belong to the same subgroup A or B. Then u_r and v_1, as well as elements u_1 and v_s, also belong to different subgroups, and consequently, the forms of both parts of the equality

$$u_1 u_2 \cdots u_r v_1 v_2 \cdots v_s = v_1 v_2 \cdots v_s u_1 u_2 \cdots u_r \tag{4.5}$$

are reduced. By the equalities in (2.2), there exist elements $h_0, h_1, \ldots, h_{r+s}$ of subgroup H such that

$$u_i = h_{i-1}^{-1} v_i h_i \quad (1 \leq i \leq r+s) \quad h_0 = h_{r+s} = 1. \tag{4.6}$$

Hence, $u = v_1 v_2 \cdots v_r h_r$, and consequently, if $s = r$, we have $u = v h_r$. But then, $h_r = v^{-1} u \in U \cap H$. Hence $h_r = 1$ and $v = u$. So, if $s = r$, our statement is satisfied.

If $s > r$, we have $v = uv'$, where $v' = h_r^{-1} v_{r+1} \cdots v_s$ and the length of element $v' \in U$ is less than s. Hence, by induction, $v' = u^k$ for some integer k. Therefore, $v = u^{k+1}$, and hence the proof is complete. □

We now proceed to the proof of Theorem 1.1.

5. Proof of Theorem 1.1

Before we go through this proof, we note that for any $t > 1$, each of the groups $A(t)$ and $B(t)$ is the amalgamated free product of two finite groups. Hence, $A(t)$ and $B(t)$ are free-by-finite [6], and therefore they are π_c [7].

We also see that A and B are free products of finitely generated abelian groups amalgamating cyclic subgroups. Thus by [8] or [10], groups A and B are π_c.

So, to prove Theorem 1.1, it is enough to show that for any two elements f and g of G_{mn}, where $f \notin \langle g \rangle$, there exists an integer $t > 1$ such that $f\tau_t \notin \langle g \rangle \tau_t$. Then it will be possible to find a homomorphism ψ of $G_{mn}(t)$ onto a finite group X such that $(f\tau_t)\psi \notin \langle g \rangle (\tau_t)\psi$, and consequently the product $\tau_t \psi$ will be the required homomorphism of group G_{mn} onto a finite group.

Now let f and g be a pair of elements of G_{mn} such that $f \notin \langle g \rangle$. We can assume that g is cyclically reduced. Group G_{mn} is residually finite as was mentioned. So we consider $g \neq 1$ and clearly $f \neq 1$.

If $f \notin \langle g \rangle$, then by Proposition 2.2, we have the following cases.

(1) Elements f and g belong to the same factor A or B and $f \neq g^k$ for all integers k.
(2) Elements f and g belong to different factors A and B and $f \neq g^k$ for all integers k.
(3) $l(g) = 1$ and $l(f) > 1$.
(4) $l(f) = 1$ and $l(g) > 1$.
(5) $l(f) > 1$, $l(g) > 1$ and $l(g)$ does not divide $l(f)$.
(6) $l(f) > 1$, $l(g) > 1$ and $l(g)$ divides $l(f)$, but (2.4) is not satisfied.

We prove separately the above cases.

Case (1). Assume first that both elements lie in one of the subgroups A or B; let it be A for instance. Since $f \notin \langle g \rangle$ and since A is π_c, there exists a homomorphism φ of A onto

some finite group X such that $f\varphi \notin \langle g\rangle\varphi$. By Proposition 3.2, there exist an integer $t > 1$ and a homomorphism ψ of $A(t)$ onto X such that $\varphi = \rho_t\psi$. This means that $f\rho_t \notin \langle g\rangle\rho_t$ in $A(t)$. Thus, by Proposition 2.2, for the given integer t, in group $G_{mn}(t)$, elements $f\tau_t$ and $g\tau_t$ belong to the free factor $A(t)$ and $f\tau_t \notin \langle g\rangle\tau_t$.

Case (2). Let elements f and g now lie in different subgroups A and B; for example let f belong to A, let g belong to B, and each of these elements does not belong to the subgroup H. By Proposition 3.3, there exists an integer $t_1 > 1$ such that for any positive integer t divisible by t_1, $f\rho_t$ does not belong to $H\rho_t$, and there exists an integer $t_2 > 1$ such that for any positive integer t divisible by t_2, $g\sigma_t$ does not belong to $H\sigma_t$. Thus, at $t = t_1t_2$, elements $f\tau_t$ and $g\tau_t$ belong to different free factors $A(t)$ and $B(t)$ of $G_{mn}(t)$ and also do not belong to the amalgamated subgroup $H(t)$ relative to the decomposition of $G_{mn}(t)$. Hence, $f\tau_t \notin \langle g\rangle\tau_t$ since otherwise, by Proposition 2.2, this would imply that elements $f\tau_t$ and $g\tau_t$ belong to the amalgamated subgroup $H(t)$.

Case (3). $l(g) = 1$ and $l(f) > 1$. thus, $f \notin \langle g\rangle$ by Proposition 2.2.

From Proposition 3.4, it follows that there exists an integer $t_1 > 1$ such that for any positive integer t divisible by t_1, the length of $f\tau_t$ in $G_{mn}(t)$ is equal to the length of f in G_{mn}; that is, $l(f\tau_t) > 1$. Similarly, there exists an integer $t_2 > 1$ such that for any positive integer t divisible by t_2, $l(g\tau_t) = 1$ in $G_{mn}(t)$. Thus, at $t = t_1t_2$, $l(f\tau_t) > 1$ and $l(g\tau_t) = 1$ in $G_{mn}(t)$. Hence, from Proposition 2.2, $f\tau_t \notin \langle g\rangle\tau_t$ and the required integer t is found.

Case (4). $l(g) > 1$ and $l(f) = 1$. The proof of this case is similar to the proof of case (3) above.

Case (5). $l(f) > 1$, $l(g) > 1$, and $l(g)$ does not divide $l(f)$.

From Proposition 3.4, there exists an integer $t_1 > 1$ such that for any positive integer t divisible by t_1, $l(f\tau_t) = l(f)$. Similarly, there exists an integer $t_2 > 1$ such that for any positive integer t divisible by t_2, $l(g\tau_t) = l(g)$. Thus at $t = t_1t_2$, elements $f\tau_t$ and $g\tau_t$ of $G_{mn}(t)$ are such that $l(f\tau_t) > 1$, $l(g\tau_t) > 1$, and $l(g\tau_t)$ does not divide $l(f\tau_t)$. Hence, from Proposition 2.2, $f\tau_t$ cannot belong to the subgroup $\langle g\rangle\tau_t$.

Case (6). $l(f) > 1$, $l(g) > 1$, and $l(g)$ divides $l(f)$ but (2.4) is not satisfied.

Let $f = x_1x_2\cdots x_r$ be reduced and let $g = y_1y_2\cdots y_s$ be cyclically reduced. By Proposition 2.2, if $f = g^k$ for some integer $k \neq 0$, then $r = ks$ and there exist elements $h_0, h_1, \ldots, h_{ks}$ of H such that property (2.4) is satisfied. Thus elements x_i and y_i $(1 \leq i \leq ks)$ belong to the same free factor and define the same double coset of H and $y_j = y_{j+ls}$ for $1 \leq j \leq s$ and $0 \leq l \leq k-1$.

So, if $f \notin \langle g\rangle$, three subcases arise.

Subcase (i). Suppose that, for some integer $i = 1, \ldots, ks$, elements y_i and x_i lie in different free factors A and B of group G_{mn} (and, certainly, are not in the subgroup H). Then as in case (2) above, there exists an integer $t_i > 1$ such that for any positive integer t divisible by t_i, elements $x_i\tau_t$ and $y_i\tau_t$ belong to the different free factors $A(t)$ and $B(t)$, respectively, and do not belong to the amalgamated subgroup $H(t)$ of $G_{mn}(t)$. Also by Proposition 3.4, there exist integers $t' > 1$ and $t'' > 1$ such that for any positive integer t divisible by t', the length of $f\tau_t$ in $G_{mn}(t)$ is equal to r and for any positive integer t divisible by t'', the

length of $g\tau_t$ in $G_{mn}(t)$ is equal to s. Thus, at $t = t_i t' t''$, elements $f\tau_t$ and $g\tau_t$ have reduced forms

$$(x_1\tau_t)(x_2\tau_t)\cdots(x_r\tau_t), \qquad (y_1\tau_t)(y_2\tau_t)\cdots(y_s\tau_t), \tag{5.1}$$

respectively, in $G_{mn}(t)$ and $x_i\tau_t$ and $y_i\tau_t$ belong to different free factors $A(t)$ and $B(t)$, respectively, and do not belong to the amalgamated subgroup $H(t)$. Hence, by Proposition 2.2, $f\tau_t$ cannot belong to subgroup $\langle g\rangle\tau_t$ in group $G_{mn}(t)$.

Subcase (ii). Let now, for all $i = 1,\ldots,ks$, elements y_i and x_i belong to the same subgroup A or B and for some j ($j = 1,\ldots,ks$), element x_j does not belong to the double coset Hy_jH. Suppose that x_j and y_j both belong to A. By Proposition 3.5, there exists an integer $t_j > 1$ such that for any positive integer t divisible by t_j, element $x_j\rho_t$ is not in the double coset $H/H^t(y_j\rho_t)H/H^t$. Also by Proposition 3.4, there exist integers $t' > 1$ and $t'' > 1$ such that, for any positive integer t divisible by t', the length of $f\tau_t$ in $G_{mn}(t)$ is equal to r and for any positive integer t divisible by t'', the length of $g\tau_t$ in $G_{mn}(t)$ is equal to s. Since $r = ks$, at $t = t_j t' t''$, elements $f\tau_t$ and $g\tau_t$ have reduced forms

$$(x_1\tau_t)(x_2\tau_t)\cdots(x_r\tau_t), \qquad (y_1\tau_t)(y_2\tau_t)\cdots(y_r\tau_t), \tag{5.2}$$

respectively, in $G_{mn}(t)$ and $x_j\tau_t$ does not lie in the double coset $H/H^t(y_j\tau_t)H/H^t$. Hence by Proposition 2.2, $f\tau_t$ cannot belong to subgroup $\langle g\rangle\tau_t$ in $G_{mn}(t)$, since (2.4) is not satisfied. Thus integer t is found.

Subcase (iii). We suppose now that for some i, $y_i \neq y_{i+ls}$, $1 \leq i \leq s$, and $1 \leq l \leq k-1$.

(a) If these elements y_i and y_{i+ls} belong to the same subgroup A or B, then as in case (1) above, since groups A and B are residually finite, there exists a homomorphism φ of group A (or B) onto some finite group X such that $y_i\varphi \neq y_{i+ls}\varphi$. But since by Proposition 3.2 any homomorphism of group A (group B) onto a finite group factors through some group $A(t)$ (resp., $B(t)$), for this t, there exists homomorphism ψ of group $A(t)$ (or $B(t)$) onto X such that $\varphi = \rho_t\psi$. Then, in the free factor and thus in $G_{mn}(t)$, $y_i\rho_t \neq y_{i+ls}\rho_t$, and so condition (2.4) for the inclusion of element $f\tau_t$ in subgroup $\langle g\rangle\tau_t$ will not be satisfied.

(b) If these elements y_i and y_{i+ls} belong to different subgroups A and B, then similar to case (2), for some value of t, elements $y_i\tau_t$ and $y_{i+ls}\tau_t$ belong to different free factors $A(t)$ and $B(t)$ of group $G_{mn}(t)$ and also do not belong to the amalgamated subgroup $H(t)$. Thus, in group $G_{mn}(t)$, $y_i\tau_t \neq y_{i+ls}\tau_t$ and condition (2.4) of Proposition 2.2 will not be satisfied. So, element $f\tau_t$ cannot be in subgroup $\langle g\rangle\tau_t$. Hence we have the proof of Theorem 1.1.

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