CORRECTED FOURIER SERIES AND ITS APPLICATION TO FUNCTION APPROXIMATION

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Any quasismooth function $f(x)$ in a finite interval $[0,x_0]$, which has only a finite number of finite discontinuities and has only a finite number of extremes, can be approximated by a uniformly convergent Fourier series and a correction function. The correction function consists of algebraic polynomials and Heaviside step functions and is required by the aperiodicity at the endpoints (i.e., $f(0) \neq f(x_0)$) and the finite discontinuities in between. The uniformly convergent Fourier series and the correction function are collectively referred to as the corrected Fourier series. We prove that in order for the mth derivative of the Fourier series to be uniformly convergent, the order of the polynomial need not exceed $(m+1)$. In other words, including the no-more-than-$(m+1)$ polynomial has eliminated the Gibbs phenomenon of the Fourier series until its mth derivative. The corrected Fourier series is then applied to function approximation; the procedures to determine the coefficients of the corrected Fourier series are illustrated in detail using examples.

1. Introduction

The theory about the function approximation of finite functions in a finite interval by a Fourier series emerged as early as the nineteenth century [2]. In particular, for any *quasismooth function* in $[0,x_0]$, which is referred to as the single-valued finite function $f(x)$ that has only a finite number of finite discontinuities as well as a finite number of maxima and minima in the interval $[0,x_0]$, one could obtain the following Fourier series (partial sum):

$$S_N(x) = \sum_{|n|\leq N} A_n e^{i\alpha_n x}, \quad \alpha_n = \frac{2n\pi}{x_0}, \tag{1.1}$$

where the series coefficient A_n is the *Fourier projection* of $f(x)$ to the basic function $e^{i\alpha_n x}$, that is,

$$A_n = \mathcal{F}_1\langle f(x)\rangle_n \equiv \frac{1}{x_0}\int_0^{x_0} f(x)e^{-i\alpha_n x}dx. \tag{1.2}$$

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Notice that different definitions for smoothness of a function have been used with different meanings in different contexts, for example, in statistics [4]. The concern here is whether the Fourier series (1.1) converges to $f(x)$. The well-known Dirichlet theorem [2] states that for any quasismooth function,

$$\lim_{N\to\infty} S_N(x) = \lim_{N\to\infty} \sum_{|n|\le N} A_n e^{i\alpha_n x} = \begin{cases} \dfrac{[f(x-0)+f(x+0)]}{2} & x \in (0,x_0), \\ \dfrac{[f(0)+f(x_0)]}{2} & x = 0 \text{ or } x_0. \end{cases} \tag{1.3}$$

If $f(x)$ is aperiodic (i.e., $f(0) \neq f(x_0)$) or there are discontinuities in $(0,x_0)$, the Fourier series will not uniformly converge to $f(x)$. The so-called Gibbs phenomenon appears near the endpoints and those discontinuities [3]. On the other hand, if $f(x)$ is a periodic quasismooth function without discontinuities, the Fourier series (1.1) is uniformly convergent to $f(x)$ without the Gibbs phenomenon.

The Gibbs phenomenon is artificial oscillations near the discontinuities and aperiodic endpoints. They are numerical noise without any physical meaning, and should be eliminated if possible. In [1], the Fourier series with Gibbs oscillations is reexpanded into a Gegenbauer series. By doing so, they effectively filter out the Gibbs oscillation and obtain a rapidly convergent series. In this study, we will show that by using a correction function we directly obtain a uniformly convergent Fourier series without Gibbs oscillation (Section 2). We will refer to the correction function and Fourier series as the *corrected Fourier series*. Since the corrected Fourier series is uniformly convergent, we will apply it to the function approximation, as illustrated in Section 3. Section 4 then provides the concluding remarks.

2. Corrected Fourier series

As stated in the introduction, a *quasismooth function* $f(x)$ in $[0,x_0]$ has only a finite number of finite discontinuities as well as a finite number of maxima and minima in the interval $[0,x_0]$. Relevantly, a *quasismooth continuous function* is referred to as the quasismooth function without any discontinuity within the interval, but it can be either periodic or aperiodic.

Furthermore, two classes of functions are defined as follows. One is the class of *mth quasismooth functions* $\mathbf{Q_m}([0,x_0])$, of which the mth derivative of each member is a quasismooth function in $[0,x_0]$. Another is the class of *mth quasismooth continuous functions* $\mathbf{S_m}([0,x_0])$, of which the mth derivative of each member is a quasismooth continuous function in $[0,x_0]$. Here $m \geq 0$. In the case of $m = 0$, $\mathbf{Q_0}([0,x_0])$ has its member being a quasismooth function and $\mathbf{S_0}([0,x_0])$ being a quasismooth continuous function. The extension of $\mathbf{Q_m}([0,x_0])$ and $\mathbf{S_m}([0,x_0])$ to the cases with multiple variables is straightforward; for two variables they are denoted as $\mathbf{Q_m}([0,x_0],[0,y_0])$ and $\mathbf{S_m}([0,x_0],[0,y_0])$, respectively.

In addition, an *mth uniformly convergent Fourier series* in an interval means that the Fourier series remains uniformly convergent until its mth derivative without Gibbs phenomenon.

For clarity, the following notations will be used:

$$\begin{aligned} F(x)\big|_0^{x_0} &\equiv F(x_0) - F(0), \\ F(x,y)\big|_0^{x_0}\big|_0^{y_0} &\equiv F(x_0,y_0) + F(0,0) - F(x_0,0) - F(0,y_0). \end{aligned} \tag{2.1}$$

Heaviside step function $\mathcal{H}(x-x_j)$ is defined as

$$\mathcal{H}(x-x_j) = \begin{cases} 1, & x \geq x_j, \\ 0, & x < x_j. \end{cases} \tag{2.2}$$

Lemma 2.1. *Any mth quasismooth function $g_m(x) \in \mathbf{Q_m}([0,x_0])$ can be partitioned into an mth quasismooth continuous function $f_m(x) \in \mathbf{S_m}([0,x_0])$ and mth integrals of a set of Heaviside step functions as follows:*

$$g_m(x) = f_m(x) + \sum_j b_j \frac{(x-x_j)^m}{m!}\mathcal{H}(x-x_j), \tag{2.3}$$

where x_j is one of the discontinuities of the mth derivative of $g_m(x)$, denoted as $g_m^{(m)}(x)$, with jump b_j, that is, $b_j = g_m^{(m)}(x)|_{x_{j-0}}^{x_{j+0}}$.

In (2.3), the mth derivative of $f_m(x)$, denoted as $f_m^{(m)}(x)$, is continuous at the discontinuities of $g_m^{(m)}(x)$. This can be easily shown:

$$\begin{aligned} f_m^{(m)}(x)\big|_{x_{l-0}}^{x_{l+0}} &= \left[g_m^{(m)}(x) - \sum_{j\neq l} b_j \mathcal{H}(x-x_j) - b_l \mathcal{H}(x-x_l) \right] \Bigg|_{x_{l-0}}^{x_{l+0}} \\ &= \left[g_m^{(m)}(x) - b_l \mathcal{H}(x-x_l) \right] \big|_{x_{l-0}}^{x_{l+0}} = g_m^{(m)}(x)\big|_{x_{l-0}}^{x_{l+0}} - b_l = 0. \end{aligned} \tag{2.4}$$

Therefore, $f_m(x) \in \mathbf{S_m}([0,x_0])$.

Theorem 2.2. *Any mth quasismooth continuous function $f_m(x) \in \mathbf{S_m}([0,x_0])$ can be approximated uniformly by the sum of an mth uniformly convergent Fourier series and a polynomial no more than $(m+1)$th order:*

$$f_m(x) = \sum_{|n|<\infty} A_n e^{i\alpha_n x} + \sum_{l=1}^{m+1} a_l \frac{x^l}{l!}, \quad \alpha_n = \frac{2n\pi}{x_0}. \tag{2.5}$$

Proof. We start with $m = 0$. Any function $f_0(x) \in \mathbf{S_0}([0,x_0])$ is a quasismooth continuous function. It can be expressed by a periodic quasismooth continuous function $h(x)$ and a

linear function as follows:

$$f_0(x) = h(x) + ax, \tag{2.6}$$

where $a = [f_0(x_0) - f_0(0)]/x_0$. It is easy to see the periodicity of h since $h(0) = h(x_0) = f_0(0)$. Because $h(x)$ is a periodic, quasismooth continuous function, it can be approximated by a Fourier series that is uniformly convergent:

$$h(x) = \sum_{|n|<\infty} A_n e^{i\alpha_n x}, \quad \alpha_n = \frac{2n\pi}{x_0}, \tag{2.7}$$

where

$$A_n = \mathcal{F}_1\langle h(x)\rangle_n = \frac{1}{x_0}\int_0^{x_0} h(x)e^{-i\alpha_n x}dx. \tag{2.8}$$

It follows that

$$f_0(x) = \sum_{|n|<\infty} A_n e^{i\alpha_n x} + ax. \tag{2.9}$$

Notice that the right-hand side of the above equation has two parts: one is the Fourier series and other is a 1st-order polynomial. In other words, the function $f_0(x)$ is represented by a corrected Fourier series. Hence, the theorem is true for $m = 0$.

In the next step, consider the 1st quasismooth continuous function $f_1(x) \in \mathbf{S_1}([0,x_0])$. Because its first derivative $f_1^{(1)}(x)$ is a quasismooth continuous function in the interval $[0,x_0]$, $f_1^{(1)}(x)$ can be uniformly approximated by a corrected Fourier series:

$$f_1^{(1)}(x) = \sum_{|n|<\infty} A_n e^{i\alpha_n x} + ax = \sum_{n\neq 0} A_n e^{i\alpha_n x} + A_0 + ax. \tag{2.10}$$

Its integration yields

$$f_1(x) = \sum_{n\neq 0} \frac{A_n}{i\alpha_n} e^{i\alpha_n x} + A_0 x + B_0 + a\frac{x^2}{2!} \equiv \sum_{|n|<\infty} B_n e^{i\alpha_n x} + b_1 x + b_2 \frac{x^2}{2!}, \tag{2.11}$$

where

$$B_n = \begin{cases} \dfrac{A_n}{i\alpha_n} & n \neq 0, \\ B_0 & n = 0, \end{cases} \qquad b_1 = A_0, \qquad b_2 = a. \tag{2.12}$$

Since the termwise integration of a series improves its convergence, the implied uniformly convergence in the above equation is valid. Thus, the theorem is valid for $m = 1$.

According to the axiom of the mathematical induction, our proof will be complete if the theorem can be proved true for any $f_{m+1}(x) \in \mathbf{S_{m+1}}([0,x_0])$ after it is assumed true for any $f_m(x) \in \mathbf{S_m}([0,x_0])$.

By definition, the 1st derivative of $f_{m+1}(x)$ is an mth quasismooth continuous function in $\mathbf{S_m}([0,x_0])$, for which the theorem has been assumed true. Consequently, we have

$$f_{m+1}^{(1)}(x) = \sum_{|n|<\infty} A_n e^{i\alpha_n x} + \sum_{l=1}^{m+1} a_l \frac{x^l}{l!} = \sum_{n\neq 0} A_n e^{i\alpha_n x} + A_0 + \sum_{l=1}^{m+1} a_l \frac{x^l}{l!}. \tag{2.13}$$

Its integration yields

$$\begin{aligned} f_{m+1}(x) &= \sum_{n\neq 0} \frac{A_n}{i\alpha_n} e^{i\alpha_n x} + A_0 x + B_0 + \sum_{l=1}^{m+1} a_l \frac{x^{l+1}}{(l+1)!} \\ &= \sum_{n\neq 0} \frac{A_n}{i\alpha_n} e^{i\alpha_n x} + A_0 x + B_0 + \sum_{l=2}^{m+2} a_{l-1} \frac{x^l}{l!} \\ &= \sum_{|n|<\infty} B_n e^{i\alpha_n x} + \sum_{l=1}^{m+2} b_l \frac{x^l}{l!}, \end{aligned} \tag{2.14}$$

where

$$B_n = \begin{cases} \dfrac{A_n}{i\alpha_n} & n \neq 0, \\ B_0 & n = 0, \end{cases} \qquad b_l = \begin{cases} A_0 & l = 1, \\ a_{l-1} & l \geq 2. \end{cases} \tag{2.15}$$

The right-hand side of (2.14) is the same as that of (2.5). This means that the theorem is true for any $f_{m+1}(x) \in \mathbf{S_{m+1}}([0,x_0])$ after it is assumed true for any $f_m(x) \in \mathbf{S_m}([0,x_0])$. □

Corollary 2.3. *Any mth quasismooth function $g_m(x) \in \mathbf{Q_m}([0,x_0])$ can be uniformly approximated by a corrected Fourier series consisting of three parts: an mth uniformly convergent Fourier series, a no-more-than $(m+1)$th-order polynomial, and an mth integral of the Heaviside step functions at the discontinuities*

$$g_m(x) = \sum_{|n|<\infty} A_n e^{i\alpha_n x} + \sum_{l=1}^{m+1} a_l \frac{x^l}{l!} + \sum_j b_j \frac{(x-x_j)^m}{m!} \mathcal{H}(x-x_j), \quad \alpha_n = \frac{2n\pi}{x_0}. \tag{2.16}$$

It is easily seen that the corollary holds right after Lemma 2.1 and Theorem 2.2.

3. Function approximation: examples

As shown in the preceding section, the corrected Fourier series is uniformly convergent. In this section, we will make use of the corrected Fourier series to uniformly approximate quasismooth functions. Two 2nd quasismooth continuous functions in $\mathbf{S_2}([0,x_0])$ and $\mathbf{S_2}([0,x_0],[0,y_0])$, respectively, have been chosen to show how to determine coefficients of the corrected Fourier series. Extending to higher-order functions should be straightforward but tedious.

3.1. 2nd quasismooth continuous function $f(x) \in \mathbf{S}_2([0,x_0])$. According to the theorem in Section 2, the uniformly convergent corrected Fourier series of $f(x)$ is in the form of

$$f(x) = \sum_{|n|<\infty} A_n e^{i\alpha_n x} + a_1 x + a_2 \frac{x^2}{2!} + a_3 \frac{x^3}{3!}, \tag{3.1}$$

and its 1st and 2nd derivatives are correspondingly

$$\begin{aligned} f'(x) &= \sum_{|n|<\infty} i\alpha_n A_n e^{i\alpha_n x} + a_1 + a_2 x + a_3 \frac{x^2}{2!}, \\ f''(x) &= \sum_{|n|<\infty} (i\alpha_n)^2 A_n e^{i\alpha_n x} + a_2 + a_3 x. \end{aligned} \tag{3.2}$$

As implied, both derivatives of the corrected Fourier series are uniformly convergent.

Based on the endpoints values of $f(x)$ and its derivatives (*endpoints effect*), we obtain the following linear equations for a_l $(l = 1,2,3)$:

$$\begin{aligned} f(x)\big|_0^{x_0} &= a_1 x_0 + a_2 \frac{x_0^2}{2!} + a_3 \frac{x_0^3}{3!}, \\ f'(x)\big|_0^{x_0} &= a_2 x_0 + a_3 \frac{x_0^2}{2!}, \\ f''(x)\big|_0^{x_0} &= a_3 x_0. \end{aligned} \tag{3.3}$$

Obviously, if $f(x)$ is periodic (i.e., $f(x)|_0^{x_0} = f'(x)|_0^{x_0} = f''(x)|_0^{x_0} = 0$), a_l $(l = 1,2,3)$ are zero. The corrected Fourier series is just the regular one, which is uniformly convergent in the interval $[0,x_0]$. The coefficients a_l $(l = 1,2,3)$ are easily obtained by solving (3.3):

$$\begin{aligned} a_3 &= \frac{1}{x_0} f''(x)\big|_0^{x_0}, \\ a_2 &= \frac{1}{x_0} f'(x)\big|_0^{x_0} - a_3 \frac{x_0}{2!}, \\ a_1 &= \frac{1}{x_0} f(x)\big|_0^{x_0} - a_2 \frac{x_0}{2!} - a_3 \frac{x_0^2}{3!}. \end{aligned} \tag{3.4}$$

The coefficient A_n in (3.1) is the Fourier project (1.2) of $f(x) - (a_1 x + a_2(x^2/2!) + a_3(x^3/3!))$ on the basic function $e^{i\alpha_n x}$, that is,

$$A_n = \mathcal{F}_1\langle f(x)\rangle_n - (a_1 I_{1n} + a_2 I_{2n} + a_3 I_{3n}), \tag{3.5}$$

where $I_{ln} = \mathcal{F}_1\langle x^l/l!\rangle_n$.

If any of a_l $(l = 1,2,3)$ is nonzero or $f(x)$ is aperiodic, then Gibbs oscillations are expected in $\sum_n \mathcal{F}_1\langle f(x)\rangle_n e^{i\alpha_n x}$. However, the same oscillations exist in $\sum_{l=1}^{3}\sum_n a_l I_{ln} e^{i\alpha_n x}$.

They cancel each other exactly in $\sum_n A_n e^{i\alpha_n x} = \sum_n \mathcal{F}_1 \langle f(x)\rangle_n e^{i\alpha_n x} - \sum_{l=1}^{3} \sum_n a_l I_{ln} e^{i\alpha_n x}$. As a result, $\sum_n A_n e^{i\alpha_n x}$ is free of Gibbs oscillations and is uniformly convergent.

3.2. 2nd quasismooth continuous function with two unknowns $f(x,y) \in \mathbf{S_2}([0,x_0],[0,y_0])$. At first, the function $f(x,y)$ is a 2nd quasismooth continuous function with respect to x. Therefore, for any $y \in [0,y_0]$,

$$f(x,y) = \sum_{|n|<\infty} A_n(y) e^{i\alpha_n x} + a_1(y)x + a_2(y)\frac{x^2}{2!} + a_3(y)\frac{x^3}{3!}. \tag{3.6}$$

In the above equation, $A_n(y)$ and $a_l(y)$ are 2nd quasismooth continuous functions with respect to y. Thus, $A_n(y)$ and $a_l(y)$ can further expand into the following corrected Fourier series:

$$\begin{aligned} A_n(y) &= \sum_{|m|<\infty} A_{nm} e^{i\beta_m y} + b_{1n} y + b_{2n}\frac{y^2}{2!} + b_{3n}\frac{y^3}{3!}, \\ a_l(y) &= \sum_{|m|<\infty} a_{lm} e^{i\beta_m y} + d_{l1} y + d_{l2}\frac{y^2}{2!} + d_{l3}\frac{y^3}{3!} \quad (l = 1,2,3), \end{aligned} \tag{3.7}$$

where $\beta_m = 2m\pi/y_0$. From (3.6) and (3.7), the corrected Fourier series for $f(x,y)$ is

$$\begin{aligned} f(x,y) = {} & \sum_{|n|<\infty}\sum_{|m|<\infty} A_{nm} e^{i(\alpha_n x + \beta_m y)} + \sum_{|m|<\infty}\left(a_{1m}x + a_{2m}\frac{x^2}{2!} + a_{3m}\frac{x^3}{3!}\right)e^{i\beta_m y} \\ & + \sum_{|n|<\infty}\left(b_{1n}y + b_{2n}\frac{y^2}{2!} + b_{3n}\frac{y^3}{3!}\right)e^{i\alpha_n x} + \sum_{l=1}^{3}\sum_{l_0=1}^{3} d_{ll_0}\cdot\frac{x^l}{l!}\cdot\frac{y^{l_0}}{l_0!}. \end{aligned} \tag{3.8}$$

In (3.8), nine unknowns d_{ll_0} $(l,l_0 = 1,2,3)$ are obtained by solving the following linear equations:

$$\begin{aligned} & \sum_{l=1}^{3}\sum_{l_0=1}^{3} d_{ll_0}\cdot\frac{x^{l-j}}{(l-j)!}\cdot\frac{y^{l_0-j_0}}{(l_0-j_0)!}\mathcal{H}(l-j)\mathcal{H}(l_0-j_0)\,|_0^{x_0}\,|_0^{y_0} \\ & \quad = \frac{\partial^{j+j_0} f(x,y)}{\partial x^j \partial y^{j_0}}\Bigg|_0^{x_0}\Bigg|_0^{y_0} \quad (j = 0,1,2;\ j_0 = 0,1,2), \end{aligned} \tag{3.9}$$

which depend on the boundary values of $f(x,y)$ and its 1st and 2nd partial derivatives only. Notice that the first three terms on the right-hand side of (3.8) are identically zero due to the periodicity of either $e^{i\alpha_n x}$ or $e^{i\beta_m y}$. If we arrange the nine unknowns into a vector ordered as $(d_{11},d_{12},d_{13},d_{21},d_{22},d_{23},d_{31},d_{32},d_{33})$ and the equations ordered as $j = 0$, $j_0 = 0,1,2$; $j = 1$, $j_0 = 0,1,2$; and $j = 2$, $j_0 = 0,1,2$, then the coefficient matrix of the linear equations is up-triangular and can be easily inverted.

The next task is to determine a_{lm}, b_{ln}, and A_{nm}. In addition to the Fourier projection $\mathcal{F}_1\langle\cdot\rangle_n$ (1.2), two additional Fourier projections are defined:

$$\begin{aligned}\mathcal{F}_2\langle\cdot\rangle_m &= \frac{1}{y_0}\int_0^{y_0}\langle\cdot\rangle e^{-i\beta_m y}dy,\\ \mathcal{F}_0\langle\cdot\rangle_{nm} &= \frac{1}{x_0y_0}\int_0^{y_0}\int_0^{x_0}\langle\cdot\rangle e^{-i(\alpha_n x+\beta_m y)}dx\,dy.\end{aligned}\tag{3.10}$$

With respect to x, the endpoints effects of $f(x,y)$ and its derivatives yield

$$\begin{aligned} f\,\big|_0^{x_0} &= \sum_{|m|<\infty}\left(a_{1m}x_0+a_{2m}\frac{x_0^2}{2!}+a_{3m}\frac{x_0^3}{3!}\right)e^{i\beta_m y}+\sum_{l=1}^{3}\sum_{l_0=1}^{3}d_{ll_0}\cdot\frac{x_0^l}{l!}\cdot\frac{y_0^{l_0}}{l_0!},\\ \frac{\partial f}{\partial x}\bigg|_0^{x_0} &= \sum_{|m|<\infty}\left(a_{2m}x_0+a_{3m}\frac{x_0^2}{2!}\right)e^{i\beta_m y}+\sum_{l=2}^{3}\sum_{l_0=1}^{3}d_{ll_0}\cdot\frac{x_0^{l-1}}{(l-1)!}\cdot\frac{y_0^{l_0}}{l_0!},\\ \frac{\partial^2 f}{\partial x^2}\bigg|_0^{x_0} &= \sum_{|m|<\infty}(a_{3m}x_0)e^{i\beta_m y}+\sum_{l_0=1}^{3}d_{3l_0}\cdot x_0\cdot\frac{y_0^{l_0}}{l_0!}.\end{aligned}\tag{3.11}$$

Notice that the first and third terms in (3.8) are exactly zero due to the periodicity of $e^{i\alpha_n x}$. After the Fourier projection to $e^{-i\beta_m y}$ (i.e., $\mathcal{F}_2\langle\cdot\rangle_m$), the following equations are resulted and are solved for a_{lm} ($l=1,2,3$) for each m:

$$\begin{aligned} a_{1m}+a_{2m}\frac{x_0^2}{2!}+a_{3m}\frac{x_0^3}{3!} &= \mathcal{F}_2\left\langle f\,\big|_0^{x_0}\right\rangle_m-\sum_{l=1}^{3}\sum_{l_0=1}^{3}d_{ll_0}\cdot\frac{x_0^l}{l!}J_{l_0m},\\ a_{2m}x_0+a_{3m}\frac{x_0^2}{2!} &= \mathcal{F}_2\left\langle\frac{\partial f}{\partial x}\bigg|_0^{x_0}\right\rangle_m-\sum_{l=2}^{3}\sum_{l_0=1}^{3}d_{ll_0}\cdot\frac{x_0^{l-1}}{(l-1)!}J_{l_0m},\\ a_{3m}x_0 &= \mathcal{F}_2\left\langle\frac{\partial^2 f}{\partial x^2}\bigg|_0^{x_0}\right\rangle_m-\sum_{l_0=1}^{3}d_{3l_0}\cdot x_0J_{l_0m},\end{aligned}\tag{3.12}$$

where $J_{l_0m}=\mathcal{F}_2\langle y^{l_0}/l_0!\rangle_m$.

Similarly, the endpoints effect of $f(x,y)$ and its partial derivative with respect to y result in the following linear equations for b_{ln} for each n:

$$\begin{aligned} b_{1n}+b_{2n}\frac{y_0^2}{2!}+b_{3n}\frac{y_0^3}{3!} &= \mathcal{F}_1\left\langle f\,\big|_0^{y_0}\right\rangle_n-\sum_{l=1}^{3}\sum_{l_0=1}^{3}d_{ll_0}\cdot\frac{y_0^{l_0}}{l_0!}I_{ln},\\ b_{2n}y_0+b_{3n}\frac{y_0^2}{2!} &= \mathcal{F}_1\left\langle\frac{\partial f}{\partial y}\bigg|_0^{y_0}\right\rangle_n-\sum_{l=1}^{3}\sum_{l_0=2}^{3}d_{ll_0}\cdot\frac{y_0^{l_0-1}}{(l_0-1)!}I_{ln},\\ b_{3n}y_0 &= \mathcal{F}_1\left\langle\frac{\partial^2 f}{\partial y^2}\bigg|_0^{y_0}\right\rangle_n-\sum_{l=1}^{3}d_{l3}\cdot y_0I_{ln},\end{aligned}\tag{3.13}$$

where $I_{ln}=\mathcal{F}_1\langle x^l/l!\rangle_n$.

After d_{ll_0}, a_{lm}, and b_{ln} are obtained, the rest of the coefficients A_{nm} are readily obtained by the following Fourier projection:

$$A_{nm} = \mathcal{F}_0 \langle f(x,y) \rangle_{nm} - \left[\sum_{l=1}^{3} a_{lm} I_{ln} + \sum_{l=1}^{3} b_{ln} J_{lm} + \sum_{l=1}^{3} \sum_{l_0=1}^{3} d_{ll_0} I_{ln} J_{l_0 m} \right]. \tag{3.14}$$

Just as in the first example, any possible Gibbs oscillations associated with the partial sum of the first term in the above equation will be canceled exactly by those in the square bracket.

After all, each coefficient in (3.8) has been determined. This completes the function approximation of $f(x,y)$ by a corrected Fourier series.

4. Concluding remarks

Any quasismooth function can be uniformly approximated by a corrected Fourier series, which consists of a uniformly convergent Fourier series and a correction function. The corrected Fourier series is free of the Gibbs phenomenon, although the quasismooth function can be aperiodic and have discontinuities in general.

The correction function consists of algebraic polynomials and Heaviside step functions. The orders of the polynomials are no more than $(m+1)$, demanding that the mth derivative of the corrected Fourier series be uniformly convergent. The corrected Fourier series will not be overconstrained, if the function to be approximated has defined its derivation only until mth order. The solution of an mth-order ordinary or partial differential equation is one such function whose $(m+1)$ derivative is not necessarily defined. Applications of the corrected Fourier series to linear ordinary differential equations with varying coefficients and to linear partial differential equations on irregular region will be the subject of our future studies.

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References

[1] D. Gottlieb, C.-W. Shu, A. Solomonoff, and H. Vandeven, *On the Gibbs phenomenon. I. Recovering exponential accuracy from the Fourier partial sum of a nonperiodic analytic function*, J. Comput. Appl. Math. **43** (1992), no. 1-2, 81–98.

[2] M. Kline, *Mathematical Thought from Ancient to Modern Times*, Oxford University Press, New York, 1972.

[3] C. C. Lin and L. A. Segel, *Mathematics Applied to Deterministic Problems in the Natural Sciences*, Macmillan Publishing, New York, 1974.

[4] M. Woodroofe, *On choosing a delta-sequence*, Ann. Math. Statist. **41** (1970), no. 5, 1665–1671.

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