IJMMS 2004:12, 599–605
PII. S0161171204307180
http://ijmms.hindawi.com

q-RIEMANN ZETA FUNCTION

TAEKYUN KIM

Received 19 July 2003

We consider the modified q-analogue of Riemann zeta function which is defined by $\zeta_q(s) = \sum_{n=1}^{\infty}(q^{n(s-1)}/[n]^s)$, $0 < q < 1$, $s \in \mathbb{C}$. In this paper, we give q-Bernoulli numbers which can be viewed as interpolation of the above q-analogue of Riemann zeta function at negative integers in the same way that Riemann zeta function interpolates Bernoulli numbers at negative integers. Also, we will treat some identities of q-Bernoulli numbers using non-Archimedean q-integration.

2000 Mathematics Subject Classification: 11S80, 11B68.

1. Introduction. Throughout this paper, $\mathbb{Z}_p$, $\mathbb{Q}_p$, $\mathbb{C}$, and $\mathbb{C}_p$ will respectively denote the ring of p-adic rational integers, the field of p-adic rational numbers, the complex number field, and the completion of algebraic closure of $\mathbb{Q}_p$.

The p-adic absolute value in $\mathbb{C}_p$ is normalized so that $|p|_p = 1/p$. When one talks of q-extension, q is considered in many ways such as an indeterminate, a complex number $q \in \mathbb{C}$, or a p-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$, we normally assume $|q| < 1$. If $q \in \mathbb{C}_p$, then we normally assume $|q-1|_p < p^{-1/(p-1)}$ so that $q^x = \exp(x \log q)$ for $|x|_p \le 1$. We use the notation

$$[x] = [x:q] = \frac{1-q^x}{1-q} = 1+q+q^2+\cdots+q^{x-1}. \tag{1.1}$$

Note that $\lim_{q\to 1}[x] = x$ for $x \in \mathbb{Z}_p$ in the p-adic case.

Let $UD(\mathbb{Z}_p)$ be denoted by the set of uniformly differentiable functions on $\mathbb{Z}_p$.

For $f \in UD(\mathbb{Z}_p)$, we start with the expression

$$\frac{1}{[p^N]}\sum_{0\le j<p^N} q^j f(j) = \sum_{0\le j<p^N} f(j)\mu_q(j+p^N\mathbb{Z}_p) \tag{1.2}$$

representing the analogue of Riemann's sums for f (cf. [4]).

The integral of f on $\mathbb{Z}_p$ will be defined as the limit ($N \to \infty$) of these sums, which exists. The p-adic q-integral of a function $f \in UD(\mathbb{Z}_p)$ is defined by (see [4])

$$\int_{\mathbb{Z}_p} f(x)\,d\mu_q(x) = \lim_{N\to\infty}\frac{1}{[p^N]}\sum_{0\le j<p^N} f(j)q^j. \tag{1.3}$$

For d that is a fixed positive integer with $(p,d)=1$, let

$$X=X_d=\varprojlim_{N}\frac{\mathbb{Z}}{dp^N\mathbb{Z}},\quad X_1=\mathbb{Z}_p,$$
$$X^*=\bigcup_{\substack{0<a<dp\\(a,p)=1}}a+dp\mathbb{Z}_p,\tag{1.4}$$
$$a+dp^N\mathbb{Z}_p=\{x\in X\mid x\equiv a\,(\operatorname{mod} dp^N)\},$$

where $a\in\mathbb{Z}$ lies in $0\le a<dp^N$.

Let $\mathbb{N}$ be the set of positive integers. For $m,k\in\mathbb{N}$, the q-Bernoulli polynomials, $\beta_m^{(-m,k)}(x,q)$, of higher order for the variable x in $\mathbb{C}_p$ are defined using p-adic q-integral by (cf. [4])

$$\begin{aligned}\beta_m^{(-m,k)}&(x,q)\\&=\underbrace{\int_{\mathbb{Z}_p}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}}_{k\text{ times}}[x+x_1+x_2+\cdots+x_k]^m\\&\qquad\cdot q^{-x_1(m+1)-x_2(m+2)-\cdots-x_k(m+k)}d\mu_q(x_1)d\mu_q(x_2)\cdots d\mu_q(x_k).\end{aligned}\tag{1.5}$$

Now, we define the q-Bernoulli numbers of higher order as follows (cf. [2, 4, 7]):

$$\beta_m^{(-m,k)}\left(=\beta_m^{(-m,k)}(q)\right)=\beta_m^{(-m,k)}(0,q).\tag{1.6}$$

By (1.5), it is known that (cf. [4])

$$\begin{aligned}\beta_m^{(-m,k)}&=\lim_{N\to\infty}\frac{1}{[p^N]^k}\sum_{x_1=0}^{p^N-1}\cdots\sum_{x_k=0}^{p^N-1}[x_1+\cdots+x_k]^m q^{-x_1m-x_2(m+1)+\cdots-x_k(m+k-1)}\\&=\frac{1}{(1-q)^m}\sum_{i=0}^{m}\binom{m}{i}(-1)^i\frac{(i-m)(i-m-1)\cdots(i-m-k+1)}{[i-m][i-m-1]\cdots[i-m-k+1]},\end{aligned}\tag{1.7}$$

where $\binom{m}{i}$ are the binomial coefficients.

Note that $\lim_{q\to1}\beta_m^{(-m,k)}=B_m^{(k)}$, where $B_m^{(k)}$ are ordinary Bernoulli numbers of order k (cf. [2, 3, 5, 7, 9]). By (1.5) and (1.7), it is easy to see that

$$\begin{aligned}\beta_m^{(-m,1)}(x,q)&=\sum_{i=0}^{m}\binom{m}{i}q^{xi}\beta_i^{(-m,1)}[x]^{m-i}\\&=\frac{1}{(1-q)^m}\sum_{j=0}^{m}q^{jx}\binom{m}{j}(-1)^j\frac{j-m}{[j-m]}.\end{aligned}\tag{1.8}$$

We modify the q-analogue of Riemann zeta function which is defined in [1] as follows: for $q\in\mathbb{C}$ with $0<q<1$, $s\in\mathbb{C}$, define

$$\zeta_q(s)=\sum_{n=1}^{\infty}\frac{q^{(s-1)n}}{[n]^s}.\tag{1.9}$$

The numerator ensures the analytic continuation for $\Re(s) > 1$. In (1.9), we can consider the following problem.

"Are there q-Bernoulli numbers which can be viewed as interpolation of $\zeta_q(s)$ at negative integers in the same way that Riemann zeta function interpolates Bernoulli numbers at negative integers?"

In this paper, we give the value $\zeta_q(-m)$ for $m \in \mathbb{N}$, which is the answer of the above problem, and construct a new complex q-analogue of Hurwitz's zeta function and q-L-series. Also, we will treat some interesting identities of q-Bernoulli numbers.

2. Some identities of q-Bernoulli numbers $\beta_m^{(-m,1)}$. In this section, we assume $q \in \mathbb{C}_p$ with $|1-q|_p < p^{-1/(p-1)}$. By (1.5), we have

$$\begin{aligned} \beta_n^{(-n,1)}(x,q) &= \int_X q^{-(n+1)t}[x+t]^n d\mu_q(t) \\ &= [d]^{n-1}\sum_{i=0}^{d-1} q^{-ni}\int_{\mathbb{Z}_p} q^{-(n+1)dx}\left[\frac{x+i}{d}:q^d\right]^n d\mu_{q^d}(x). \end{aligned} \tag{2.1}$$

Thus, we have

$$\beta_n^{(-n,1)}(x,q) = [d]^{n-1}\sum_{i=0}^{d-1} q^{-ni}\beta_n^{(-n,1)}\left(\frac{x+i}{d},q^d\right), \tag{2.2}$$

where d, n are positive integers.

If we take $x=0$, then we have

$$[n]\beta_m^{(-m,1)} - n[n]^m\beta_m^{(-m,1)}(q^n) = \sum_{k=0}^{m-1}\binom{m}{k}[n]^k\beta_k^{(-m,1)}(q^n)\sum_{j=1}^{n-1} q^{-(m-j)k}[j]^{m-k}. \tag{2.3}$$

It is easy to see that $\lim_{q\to 1}\beta_m^{(-m,1)} = B_m$, where B_m are ordinary Bernoulli numbers (cf. [7]).

Remark 2.1. By (2.3), note that

$$n(1-n^m)B_m = \sum_{k=0}^{m-1}\binom{m}{k}n^k B_k\sum_{j=1}^{n-1} j^{m-k}. \tag{2.4}$$

Let $F_q(t)$ be the generating function of $\beta_n^{(-n,1)}$ as follows:

$$F_q(t) = \sum_{k=0}^{\infty}\beta_k^{(-k,1)}\frac{t^k}{k!}. \tag{2.5}$$

By (1.7) and (2.5), we easily see that

$$F_q(t) = -\sum_{m=0}^{\infty}\left(m\sum_{n=0}^{\infty} q^{-mn}[n]^{m-1}\right)\frac{t^m}{m!}. \tag{2.6}$$

Through differentiating both sides with respect to t in (2.5) and (2.6), and comparing coefficients, we obtain the following proposition.

Proposition 2.2. *For $m>0$, there exists*

$$-\frac{\beta_m^{(-m,1)}}{m}=\sum_{n=1}^{\infty}q^{-nm}[n]^{m-1}. \tag{2.7}$$

Moreover, $\beta_0^{(0,1)}=(q-1)/\log q$.

Remark 2.3. Note that Proposition 2.2 is a q-analogue of $\zeta(1-2m)$ for any positive integer m.

Let χ be a primitive Dirichlet character with conductor $f\in\mathbb{N}$.
For $m\in\mathbb{N}$, we define

$$\beta_{m,\chi}^{(-m,1)}=\int_X q^{-(m+1)x}\chi(x)[x]^m d\mu_q(x),\quad \text{for } m\geq 0. \tag{2.8}$$

Note that

$$\beta_{m,\chi}^{(-m,1)}=[d]^{m-1}\sum_{i=0}^{d-1}\chi(i)q^{-mi}\beta_m^{(-m,1)}\left(\frac{i}{d},q^d\right). \tag{2.9}$$

3. q-analogs of zeta functions. In this section, we assume $q\in\mathbb{C}$ with $|q|<1$. In [1], the q-analogue of Riemann zeta function was defined by (cf. [1])

$$\zeta_q^*(s)=\sum_{n=1}^{\infty}\frac{q^{ns}}{[n]^s},\quad \Re(s)>0. \tag{3.1}$$

Now, we modify the above q-analogue of Riemann zeta function as follows: for $q\in\mathbb{C}$ with $0<|q|<1$, $s\in\mathbb{C}$, define

$$\zeta_q(s)=\sum_{n=1}^{\infty}\frac{q^{(s-1)n}}{[n]^s}. \tag{3.2}$$

By (2.5), (2.6), and (2.7), we obtain the following proposition.

Proposition 3.1. *For $m\in\mathbb{N}$, there exists*
(i) $\zeta_q(1-m)=-\beta_m^{(-m,1)}/m$, *for $m\geq 1$;*
(ii) $\zeta_q(s)$ *having simple pole at $s=1$ with residue $(q-1)/\log q$.*

By (1.7) and (1.8), we see that

$$\beta_n^{(-n,1)}(x,q)=-n\sum_{k=0}^{\infty}\left([k]q^x+[x]\right)^{n-1}q^{-n(k+x)},\quad \text{where } 0\leq x<1. \tag{3.3}$$

Hence, we can define q-analogue of Hurwitz ζ-function as follows: for $s \in \mathbb{C}$, define

$$\zeta_q(s,x) = \sum_{n=0}^{\infty} \frac{q^{(s-1)(n+x)}}{([n]q^x + [x])^s}. \tag{3.4}$$

Note that $\zeta_q(s,x)$ has an analytic continuation in $\mathbb{C}$ with only one simple pole at $s = 1$.

By (3.3) and (3.4), we have the following theorem.

Theorem 3.2. *For any positive integer k, there exists*

$$\zeta_q(1-k,x) = -\frac{\beta_k^{(-k,1)}(x,q)}{k}. \tag{3.5}$$

Let χ be Dirichlet character with conductor $d \in \mathbb{N}$. By (2.9), the generalized q-Bernoulli numbers with χ can be defined by

$$\beta_{m,\chi}^{(-m,1)} = [d]^{m-1} \sum_{i=0}^{d-1} \chi(i) q^{-mi} \beta_m^{(-m,1)}\left(\frac{i}{d}, q^d\right). \tag{3.6}$$

For $s \in \mathbb{C}$, we define

$$L_q(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n) q^{(s-1)n}}{[n]^s}. \tag{3.7}$$

It is easy to see that

$$L_q(\chi,s) = [d]^{-s} \sum_{a=1}^{d} \chi(a) q^{(s-1)a} \zeta_{q^d}\left(s, \frac{a}{d}\right). \tag{3.8}$$

By (3.6), (3.7), and (3.8), we obtain the following theorem.

Theorem 3.3. *Let k be a positive integer. Then there exists*

$$L_q(1-k,\chi) = -\frac{\beta_{k,\chi}^{(-k,1)}}{k}. \tag{3.9}$$

Let a and F be integers with $0 < a < F$. For $s \in \mathbb{C}$, we consider the functions $H_q(s,a,F)$ as follows:

$$H_q(s,a,F) = \sum_{m \equiv a(F), m>0} \frac{q^{m(s-1)}}{[m]^s} = [F]^{-s} \zeta_{q^F}\left(s, \frac{a}{F}\right). \tag{3.10}$$

Then we have

$$H_q(1-n,a,F) = -\frac{[F]^{n-1}}{n} \beta_n^{(-n,1)}\left(\frac{a}{F}, q^F\right), \tag{3.11}$$

where n is any positive integer.

Therefore, we obtain the following theorem.

Theorem 3.4. *Let a and F be integers with $0 < a < F$. For $s \in \mathbb{C}$, there exists*
(i) $H_q(1-n,a,F) = -([F]^{n-1}/n)\beta_n^{(-n,1)}(a/F,q^F)$;
(ii) $H_q(s,a,F)$ *having a simple pole at $s = 1$ with residue* $(1/[F]F)((q^F-1)/\log q)$.

In a recent paper, the q-analogue of Riemann zeta function was studied by Cherednik (see [1]). In [1], we can consider the q-Bernoulli numbers which can be viewed as an interpolation of the q-analogue of Riemann zeta function at negative integers. In this paper, we have shown that the q-analogue of zeta function interpolates q-Bernoulli numbers at negative integers in the same way that Riemann zeta function interpolates Bernoulli numbers at negative integers (cf. [2, 5, 7]).

Remark 3.5. Let $q \in \mathbb{C}_p$ with $|1-q|_p < p^{-1/(p-1)}$. Then the p-adic q-gamma function was defined as (see [8])

$$\Gamma_{p,q}(n) = (-1)^n \prod_{1\le j<n,\,(j,p)=1} [j]. \tag{3.12}$$

For all $x \in \mathbb{Z}_p$, we have

$$\Gamma_{p,q}(x+1) = \epsilon_{p,q}(x)\Gamma_{p,q}(x), \tag{3.13}$$

where $\epsilon_{p,q}(x) = -[x]$ for $|x|_p = 1$, and $\epsilon_{p,q}(x) = -1$ for $|x|_p < 1$, (see [8]). By (3.13), we easily see that (cf. [6])

$$\log\Gamma_{p,q}(x+1) = \log\epsilon_{p,q}(x) + \log\Gamma_{p,q}(x). \tag{3.14}$$

By the differentiation of both sides in (3.14), we have (cf. [6])

$$\frac{\Gamma'_{p,q}(x+1)}{\Gamma_{p,q}(x+1)} = \frac{\Gamma'_{p,q}(x)}{\Gamma_{p,q}(x)} + \frac{\epsilon'_{p,q}(x)}{\epsilon_{p,q}(x)}. \tag{3.15}$$

By (3.15), we easily see that (cf. [6])

$$\frac{\Gamma'_{p,q}(x)}{\Gamma_{p,q}(x)} = \left(\sum_{j=1}^{x-1} \frac{q^j}{[j]}\right)\frac{\log q}{q-1} + \frac{\Gamma'_{p,q}(1)}{\Gamma_{p,q}(1)}. \tag{3.16}$$

Define

$$L_{p,q}(x) = \sum_{j=0}^{x-1} \frac{\epsilon'_{p,q}(j)}{\epsilon_{p,q}(j)}. \tag{3.17}$$

It is easy to check that $L_{p,q}(1) = 0$. By (3.15), we also see that

$$\frac{\Gamma'_{p,q}(x)}{\Gamma_{p,q}(x)} = L_{p,q}(x) + \frac{\Gamma'_{p,q}(1)}{\Gamma_{p,q}(1)}, \quad \text{for } x \in \mathbb{Z}_p, \tag{3.18}$$

where $L_{p,q}(x)$ denotes the indefinite sum of $\epsilon'_{p,q}(x)/\epsilon_{p,q}(x)$. By using (3.18) after substituting $x = 1$, we obtain $L_{p,q}(1) = 0$. The classical Euler constant was known as $\gamma = -\Gamma'(1)/\Gamma(1)$. In [8], Koblitz defined the p-adic q-Euler constant $\gamma_{p,q} = -\Gamma'_{p,q}(1)/\Gamma_{p,q}(1)$ (cf. [6, 8]). By using (3.16) and the congruence of Andrews (cf. [3]), we obtain the following congruence:

$$\frac{q-1}{\log q}\left(\frac{\Gamma'_{p,q}(p)}{\Gamma_{p,q}(p)} - \gamma_{p,q}\right) = \sum_{j=1}^{p-1}\frac{q^j}{[j]} \equiv \frac{p-1}{2}(q-1)(\operatorname{mod}[p]). \tag{3.19}$$

Acknowledgments. The author expresses his gratitude to the referees for their valuable suggestions and comments. This work was supported by the Korea Research Foundation Grant (KRF-2002-050-C00001).

References

[1] I. Cherednik, *On q-analogues of Riemann's zeta function*, Selecta Math. (N.S.) **7** (2001), no. 4, 447–491.

[2] T. Kim, *On explicit formulas of p-adic q-L-functions*, Kyushu J. Math. **48** (1994), no. 1, 73–86.

[3] ______, *On p-adic q-L-functions and sums of powers*, Discrete Math. **252** (2002), no. 1-3, 179–187.

[4] ______, *q-Volkenborn integration*, Russ. J. Math. Phys. **9** (2002), no. 3, 288–299.

[5] ______, *Non-archimedean q-integrals associated with multiple Changhee q-Bernoulli polynomials*, Russ. J. Math. Phys. **10** (2003), 91–98.

[6] T. Kim, L. C. Jang, K-H. Koh, and I.-S. Pyung, *A note on analogue of Γ-functions*, Proceedings of the Conference on 5th Transcendental Number Theory, vol. 5, no. 1, Gakushuin University, Tokyo, 1997, pp. 37–44.

[7] T. Kim and S.-H. Rim, *Generalized Carlitz's q-Bernoulli numbers in the p-adic number field*, Adv. Stud. Contemp. Math. (Pusan) **2** (2000), 9–19.

[8] N. Koblitz, *q-extension of the p-adic gamma function*, Trans. Amer. Math. Soc. **260** (1980), no. 2, 449–457.

[9] Q.-M. Luo, Z.-L. Wei, and F. Qi, *Lower and upper bounds of ζ(3)*, Adv. Stud. Contemp. Math. (Kyungshang) **6** (2003), no. 1, 47–51.

Taekyun Kim: Institute of Science Education, Kongju National University, Kongju 314-701, Korea

E-mail address: tkim@kongju.ac.kr

Mathematical Problems in Engineering

Special Issue on Modeling Experimental Nonlinear Dynamics and Chaotic Scenarios

Call for Papers

Thinking about nonlinearity in engineering areas, up to the 70s, was focused on intentionally built nonlinear parts in order to improve the operational characteristics of a device or system. Keying, saturation, hysteretic phenomena, and dead zones were added to existing devices increasing their behavior diversity and precision. In this context, an intrinsic nonlinearity was treated just as a linear approximation, around equilibrium points.

Inspired on the rediscovering of the richness of nonlinear and chaotic phenomena, engineers started using analytical tools from "Qualitative Theory of Differential Equations," allowing more precise analysis and synthesis, in order to produce new vital products and services. Bifurcation theory, dynamical systems and chaos started to be part of the mandatory set of tools for design engineers.

This proposed special edition of the *Mathematical Problems in Engineering* aims to provide a picture of the importance of the bifurcation theory, relating it with nonlinear and chaotic dynamics for natural and engineered systems. Ideas of how this dynamics can be captured through precisely tailored real and numerical experiments and understanding by the combination of specific tools that associate dynamical system theory and geometric tools in a very clever, sophisticated, and at the same time simple and unique analytical environment are the subject of this issue, allowing new methods to design high-precision devices and equipment.

Authors should follow the Mathematical Problems in Engineering manuscript format described at http://www.hindawi.com/journals/mpe/. Prospective authors should submit an electronic copy of their complete manuscript through the journal Manuscript Tracking System at http://mts.hindawi.com/ according to the following timetable:

Manuscript Due	December 1, 2008
First Round of Reviews	March 1, 2009
Publication Date	June 1, 2009

Guest Editors

José Roberto Castilho Piqueira, Telecommunication and Control Engineering Department, Polytechnic School, The University of São Paulo, 05508-970 São Paulo, Brazil; piqueira@lac.usp.br

Elbert E. Neher Macau, Laboratório Associado de Matemática Aplicada e Computação (LAC), Instituto Nacional de Pesquisas Espaciais (INPE), São Josè dos Campos, 12227-010 São Paulo, Brazil ; elbert@lac.inpe.br

Celso Grebogi, Center for Applied Dynamics Research, King's College, University of Aberdeen, Aberdeen AB24 3UE, UK; grebogi@abdn.ac.uk