# ON A HIGHER-ORDER EVOLUTION EQUATION
# WITH A STEPANOV-BOUNDED SOLUTION

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We study strong solutions $u : \mathbb{R} \to X$, a Banach space $X$, of the $n$th-order evolution equation $u^{(n)} - Au^{(n-1)} = f$, an infinitesimal generator of a strongly continuous group $A : D(A) \subseteq X \to X$, and a given forcing term $f : \mathbb{R} \to X$. It is shown that if $X$ is reflexive, $u$ and $u^{(n-1)}$ are Stepanov-bounded, and $f$ is Stepanov almost periodic, then $u$ and all derivatives $u', \dots, u^{(n-1)}$ are strongly almost periodic. In the case of a general Banach space $X$, a corresponding result is obtained, proving weak almost periodicity of $u, u', \dots, u^{(n-1)}$.

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**1. Introduction.** In this paper, we are concerned with an $n$th-order evolution equation of the form

$$u^{(n)} - Au^{(n-1)} = f. \tag{1.1}$$

Here $A : D(A) \subseteq X \to X$ is an infinitesimal generator of a strongly continuous group, $f : \mathbb{R} \to X$ a given forcing term, $X$ a Banach space with scalar field $C$, $n$ a positive integer, and $\mathbb{R}$ denotes the set of reals. We will give suitable assumptions to ensure that almost periodicity of the forcing term $f$ carries over to the solution $u$ and its derivatives up to order $(n-1)$.

The reason for studying this rather special evolution equation may be classified as a first pilot study of the issue of higher-order evolution equations, which probably has not been studied before.

We first recall the relevant concepts. A continuous function $f : \mathbb{R} \to X$ is said to be strongly (or Bochner) almost periodic if, for every given $\varepsilon > 0$, there is an $r > 0$ such that any interval in $\mathbb{R}$ of length $r$ contains a point $\tau$ for which

$$\sup_{t \in \mathbb{R}} \|f(t+\tau) - f(t)\| \le \varepsilon. \tag{1.2}$$

Here $\| \cdot \|$ denotes the norm in $X$.

A function $f : \mathbb{R} \to X$ is called weakly almost periodic if $x^* f(\cdot) : \mathbb{R} \to C$ is continuous and almost periodic for every $x^*$ in the dual space $X^*$ of $X$.

We will call a function $f \in L^1_{\mathrm{loc}}(\mathbb{R}, X)$ Stepanov-bounded or briefly $S$-bounded if

$$\|f\|_S := \sup_{t \in \mathbb{R}} \int_t^{t+1} \|f(s)\| ds < \infty. \tag{1.3}$$

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$$\sup_{t \in \mathbb{R}} \int_t^{t+1} \|f(s+\tau) - f(s)\| ds \leq \varepsilon. \tag{1.4}$$

We denote by $L(X, X)$ the set of all bounded linear operators on $X$ into itself. An operator-valued function $T : \mathbb{R} \to L(X, X)$ will be called a strongly continuous group if

$$T(t_1 + t_2) = T(t_1)T(t_2) \quad \forall t_1, t_2 \in \mathbb{R}, \tag{1.5}$$

$$T(0) = I = \text{the identity operator on } X, \tag{1.6}$$

$$T(\cdot)x : \mathbb{R} \to X \text{ is continuous for every } x \in X. \tag{1.7}$$

We recall (e.g., from Dunford and Schwartz [4]) that the infinitesimal generator $A : D(A) \subseteq X \to X$ of a strongly continuous group $T : \mathbb{R} \to L(X, X)$ is a densely defined, closed linear operator.

An operator-valued function $T : \mathbb{R} \to L(X, X)$ is said to be strongly (weakly) almost periodic if $T(\cdot)x : \mathbb{R} \to X$ is strongly (weakly) almost periodic for every $x \in X$.

Suppose $A : D(A) \subseteq X \to X$ is a densely defined, closed linear operator, and $f : \mathbb{R} \to X$ is a continuous function. Then a strong solution of the evolution equation

$$u^{(n)}(t) - Au^{(n-1)}(t) = f(t) \quad \text{a.e. for } t \in \mathbb{R} \tag{1.8}$$

is an $n$ times strongly differentiable function $u : \mathbb{R} \to X$ with $u^{(n-1)}(t) \in D(A)$ for all $t \in \mathbb{R}$, and satisfies problem (1.8).

Our first result is as follows (see Zaidman [7, 8] for first-order evolution equations).

**THEOREM 1.1.** *Let $X$ be reflexive, $f : \mathbb{R} \to X$ continuous, $S$-almost periodic, $A$ infinitesimal generator of a strongly almost periodic group $T : \mathbb{R} \to L(X, X)$. In this case, if, for the strong solution $u : \mathbb{R} \to X$ of problem (1.8), both $u$ and $u^{(n-1)}$ are $S$-bounded on $\mathbb{R}$, then $u, u', \dots, u^{(n-1)}$ are all strongly almost periodic.*

Our second result refers to a weak variant of our first theorem in the case of a general—not necessarily reflexive—Banach space $X$.

**THEOREM 1.2.** *Suppose $f : \mathbb{R} \to X$ is an $S$-almost periodic (or a weakly almost periodic) continuous function, $A$ an infinitesimal generator of a strongly continuous group $T : \mathbb{R} \to L(X, X)$ such that the conjugate operator group $T^* : \mathbb{R} \to L(X^*, X^*)$ is strongly almost periodic. If, for the strong solution $u : \mathbb{R} \to X$ of problem (1.8), both $u$ and $u^{(n-1)}$ are $S$-bounded on $\mathbb{R}$, then $u, u', \dots, u^{(n-1)}$ are all weakly almost periodic.*

**REMARK 1.3.** For some examples of first-order and higher-order evolution equations with strongly almost periodic solutions, the reader may wish to consult Cooke [3] and Zaidman [9].

## 2. Lemmas

**LEMMA 2.1.** *If $A$ is the infinitesimal generator of a strongly continuous group $G : \mathbb{R} \to L(X,X)$, then the $(n-1)$th derivative of any solution of (1.8) has the representation*

$$u^{(n-1)}(t) = G(t)u^{(n-1)}(0) + \int_0^t G(t-s)f(s)ds \quad \text{for } t \in \mathbb{R}. \qquad (2.1)$$

**PROOF.** For an arbitrary but fixed $t \in \mathbb{R}$, we have

$$\begin{aligned} \frac{d}{ds}\big[G(t-s)u^{(n-1)}(s)\big] &= G(t-s)\big[u^{(n)}(s) - Au^{(n-1)}(s)\big] \\ &= G(t-s)f(s) \quad \text{a.e. for } s \in \mathbb{R}, \text{ by (1.8).} \end{aligned} \qquad (2.2)$$

Now, integrating (2.2) from 0 to $t$, we obtain

$$\int_0^t \frac{d}{ds}\big[G(t-s)u^{(n-1)}(s)\big]ds = \int_0^t G(t-s)f(s)ds, \qquad (2.3)$$

which gives the desired representation, by (1.6). $\qquad \square$

**LEMMA 2.2.** *If $g : \mathbb{R} \to X$ is a strongly almost periodic function, and $G : \mathbb{R} \to L(X,X)$ is a strongly (weakly) almost periodic operator-valued function, then $G(\cdot)g(\cdot) : \mathbb{R} \to X$ is a strongly (weakly) almost periodic function.*

For the proof of Lemma 2.2, see [6, Theorem 1] for weak almost periodicity.

**LEMMA 2.3.** *If $g : \mathbb{R} \to X$ is an $S$-almost periodic continuous function, and $G : \mathbb{R} \to L(X,X)$ is a weakly almost periodic operator-valued function, then $x^*G(\cdot)g(\cdot) : \mathbb{R} \to C$ is an $S$-almost periodic continuous function for every $x^* \in X^*$.*

**PROOF.** By our assumption, for an arbitrary but fixed $x^* \in X^*$, the function $x^*G(\cdot)x : \mathbb{R} \to C$ is almost periodic, and so is bounded on $\mathbb{R}$, for every $x \in X$. Hence, by the uniform-boundedness principle,

$$\sup_{t \in \mathbb{R}} \|x^*G(t)\| = K < \infty. \qquad (2.4)$$

We note that the function $x^*G(\cdot)g(\cdot)$ is continuous on $\mathbb{R}$ (see [6, proof of Theorem 1]).

Consider the functions $g_\eta$ given by

$$g_\eta(t) = \frac{1}{\eta} \int_0^\eta g(t+s)ds \quad \text{for } \eta > 0, \ t \in \mathbb{R}. \qquad (2.5)$$

Since $g$ is $S$-almost periodic from $\mathbb{R}$ to $X$, $g_\eta$ is strongly almost periodic from $\mathbb{R}$ to $X$ for every fixed $\eta > 0$. Further, as shown for $C$-valued functions in [2, pages 80-81], we can prove that $g_\eta \to g$ as $\eta \to 0+$ in the $S$-sense, that is,

$$\sup_{t \in \mathbb{R}} \int_t^{t+1} \|g(s) - g_\eta(s)\|ds \longrightarrow 0 \quad \text{as } \eta \longrightarrow 0+. \qquad (2.6)$$

Now we have

$$x^*G(s)g(s) = x^*G(s)\big[g(s) - g_\eta(s)\big] + x^*G(s)g_\eta(s) \quad \text{for } s \in \mathbb{R}, \qquad (2.7)$$

and, by (2.4) and (2.6),

$$
\begin{aligned}
\sup_{t \in \mathbb{R}} \int_t^{t+1} & \left| x^* G(s) \left[ g(s) - g_\eta(s) \right] \right| ds \\
& \leq K \sup_{t \in \mathbb{R}} \int_t^{t+1} \| g(s) - g_\eta(s) \| ds \longrightarrow 0 \quad \text{as } \eta \longrightarrow 0+.
\end{aligned}
\tag{2.8}
$$

By Lemma 2.2, the functions $x^* G(\cdot) g_\eta(\cdot)$ are almost periodic from $\mathbb{R}$ to $C$. Therefore, it follows from (2.7)-(2.8) that $x^* G(\cdot) g(\cdot)$ is $S$-almost periodic from $\mathbb{R}$ to $C$. $\qquad \square$

**LEMMA 2.4.** *If $g : \mathbb{R} \to X$ is an $S$-almost periodic continuous function, and $G : \mathbb{R} \to L(X,X)$ is a strongly almost periodic operator-valued function, then $G(\cdot) g(\cdot) : \mathbb{R} \to X$ is an $S$-almost periodic continuous function.*

The proof of this lemma parallels that of Lemma 2.3 and may therefore be safely omitted.

**LEMMA 2.5.** *In a reflexive space $X$, assume $h : \mathbb{R} \to X$ is an $S$-almost periodic continuous function, and*

$$
H(t) = \int_0^t h(s) ds \quad \text{for } t \in \mathbb{R}.
\tag{2.9}
$$

*If $H$ is $S$-bounded, then it is strongly almost periodic from $\mathbb{R}$ to $X$.*

For the proof of Lemma 2.5, see [5, Notes (ii)].

**LEMMA 2.6.** *For an operator-valued function $G : \mathbb{R} \to L(X,X)$, suppose $G^*(t)$ is the conjugate (adjoint) of the operator $G(t)$ for $t \in \mathbb{R}$. If $G^* : \mathbb{R} \to L(X^*, X^*)$ is strongly almost periodic, and $g : \mathbb{R} \to X$ is weakly almost periodic, then $G(\cdot) g(\cdot) : \mathbb{R} \to X$ is weakly almost periodic.*

For the proof of Lemma 2.6, see [6, Remarks (iii)].

**3. Proof of Theorem 1.1.** By (2.1), we have

$$
T(-t) u^{(n-1)}(t) = u^{(n-1)}(0) + \int_0^t T(-s) f(s) ds \quad \text{for } t \in \mathbb{R}.
\tag{3.1}
$$

Evidently, $T(-\cdot) : \mathbb{R} \to L(X,X)$ is a strongly almost periodic group. Therefore, $T(-\cdot)x : \mathbb{R} \to X$ is strongly almost periodic, and so is bounded on $\mathbb{R}$, for every $x \in X$. Hence, by the uniform-boundedness principle,

$$
\sup_{t \in \mathbb{R}} \| T(-t) \| < \infty.
\tag{3.2}
$$

Consequently, $T(-\cdot) u^{(n-1)}(\cdot)$ is $S$-bounded on $\mathbb{R}$ (by our assumption, $u^{(n-1)}$ is $S$-bounded on $\mathbb{R}$).

Moreover, by Lemma 2.4, $T(-\cdot) f(\cdot) : \mathbb{R} \to X$ is an $S$-almost periodic continuous function. So, by Lemma 2.5, $T(-\cdot) u^{(n-1)}(\cdot)$ is strongly almost periodic from $\mathbb{R}$ to $X$. Hence, by Lemma 2.2, $u^{(n-1)}(\cdot) = T(\cdot)[T(-\cdot) u^{(n-1)}(\cdot)]$ is strongly almost periodic from $\mathbb{R}$ to $X$.

Now consider a sequence $(\alpha_k)_{k=1,2,\dots}$ of infinitely differentiable nonnegative functions on $\mathbb{R}$ such that

$$\alpha_k(t) = 0 \quad \text{for } |t| \ge \frac{1}{k}, \qquad \int_{-1/k}^{1/k} \alpha_k(t)\,dt = 1. \tag{3.3}$$

The convolution of $u$ and $\alpha_k$ is defined by

$$(u^*\alpha_k)(t) = \int_{\mathbb{R}} u(t-s)\alpha_k(s)\,ds = \int_{\mathbb{R}} u(s)\alpha_k(t-s)\,ds \quad \text{for } t \in \mathbb{R}. \tag{3.4}$$

We set

$$C_{\alpha_k} = \max_{|t| \le 1/k} \alpha_k(t). \tag{3.5}$$

Then we have

$$\begin{aligned}
\left\| (u^*\alpha_k)(t) \right\| = \left\| \int_{-1}^{1} u(t-s)\alpha_k(s)\,ds \right\| &\le C_{\alpha_k} \int_{t-1}^{t+1} \left\| u(\rho) \right\| d\rho \\
&\le 2C_{\alpha_k} \|u\|_S \quad \text{for } t \in \mathbb{R}, \text{ by } (1.3).
\end{aligned} \tag{3.6}$$

That is, $u^*\alpha_k$ is bounded on $\mathbb{R}$.

We note that, for $m = 1,2,\dots,n-1$ and $k = 1,2,\dots,$

$$(u^*\alpha_k)^{(m)}(t) = (u^{(m)*}\alpha_k)(t) \quad \text{for } t \in \mathbb{R}. \tag{3.7}$$

Further, since $u^{(n-1)}$ is strongly almost periodic from $\mathbb{R}$ to $X$, $(u^*\alpha_k)^{(n-1)} = (u^{(n-1)*}\alpha_k)$ is strongly almost periodic from $\mathbb{R}$ to $X$. Consequently, by [3, corollary to Lemma 5], $u^*\alpha_k, u'^*\alpha_k, \dots, u^{(n-2)*}\alpha_k$ are all strongly almost periodic from $\mathbb{R}$ to $X$.

With $u^{(n-1)}$ being bounded on $\mathbb{R}$, $u^{(n-2)}$ is uniformly continuous on $\mathbb{R}$. Therefore, the sequence of convolutions $(u^{(n-2)*}\alpha_k)(t) \to u^{(n-2)}(t)$ as $k \to \infty$, uniformly for $t \in \mathbb{R}$. Hence $u^{(n-2)}$ is strongly almost periodic from $\mathbb{R}$ to $X$. We thus conclude successively that $u^{(n-2)}, \dots, u', u$ are all strongly almost periodic from $\mathbb{R}$ to $X$, completing the proof of the theorem.

**4. Proof of Theorem 1.2.** By our assumption, for an arbitrary but fixed $x^* \in X^*$, $x^*T(\cdot) = T^*(\cdot)x^* : \mathbb{R} \to X^*$ is strongly almost periodic, and so $x^*T(\cdot)x : \mathbb{R} \to C$ is almost periodic for every $x \in X$. Therefore, it follows that $T : \mathbb{R} \to L(X,X)$ is a weakly almost periodic group.

By (3.1), we have

$$x^*T(-t)u^{(n-1)}(t) = x^*u^{(n-1)}(0) + \int_0^t x^*T(-s)f(s)\,ds \quad \text{for } t \in \mathbb{R}. \tag{4.1}$$

By Lemma 2.3, $x^*T(-\cdot)f(\cdot) : \mathbb{R} \to C$ is an $S$-almost periodic continuous function. By (2.4), $x^*T(-\cdot)u^{(n-1)}(\cdot)$ is $S$-bounded on $\mathbb{R}$, and so, by Lemma 2.5, is almost periodic from $\mathbb{R}$ to $C$. That is, $T(-\cdot)u^{(n-1)}(\cdot)$ is weakly almost periodic from $\mathbb{R}$ to $X$. Consequently, by Lemma 2.6, $u^{(n-1)}(\cdot) = T(\cdot)[T(-\cdot)u^{(n-1)}(\cdot)]$ is weakly almost periodic from $\mathbb{R}$ to $X$.

For the sequence $(\alpha_k)_{k=1,2,\dots}$ defined by (3.3), $(x^* u^* \alpha_k) = x^*(u^* \alpha_k)$ is bounded on $\mathbb{R}$ (by (3.6)). Further, for $m = 1, 2, \dots, n-1$ and $k = 1, 2, \dots$, we have

$$(x^* u^* \alpha_k)^{(m)}(t) = (x^* u^{(m)*} \alpha_k)(t) \quad \text{for } t \in \mathbb{R}. \tag{4.2}$$

Now the rest of the proof is obvious.

If $f : \mathbb{R} \to X$ is weakly almost periodic, then by Lemma 2.6, $T(-\cdot)f(\cdot) : \mathbb{R} \to X$ is weakly almost periodic.

**REMARK 4.1.** If $T(t) \equiv I$ for $t \in \mathbb{R}$, and so $A = 0$, then problem (1.8) reduces to

$$u^{(n)}(t) = f(t) \quad \text{a.e. for } t \in \mathbb{R}. \tag{4.3}$$

(i) In a reflexive space $X$, suppose $f$ is defined as in Theorem 1.1. If $u : \mathbb{R} \to X$ is an $S$-bounded strong solution of problem (4.3), then $u, u', \dots, u^{(n-1)}$ are all strongly almost periodic from $\mathbb{R}$ to $X$.

(ii) Assume $f : \mathbb{R} \to X$ is a weakly almost periodic continuous function. If $u : \mathbb{R} \to X$ is an $S$-bounded strong solution of problem (4.3), then $u, u', \dots, u^{(n-1)}$ are all weakly almost periodic from $\mathbb{R}$ to $X$.

These statements are clearly special cases of Theorems 1.1 and 1.2 if we take into account that the assumption $u^{(n-1)}$ $S$-bounded can be omitted, since, by (4.3), $u^{(n)}$ is $S$-almost periodic, and so $u^{(n-1)}$ is strongly (weakly) uniformly continuous on $\mathbb{R}$ (by Amerio and Prouse [1, Theorem 8, page 79]).

## REFERENCES

[1]    L. Amerio and G. Prouse, *Almost-Periodic Functions and Functional Equations*, Van Nostrand Reinhold, New York, 1971.

[2]    A. S. Besicovitch, *Almost Periodic Functions*, Dover Publications, New York, 1955.

[3]    R. Cooke, *Almost periodicity of bounded and compact solutions of differential equations*, Duke Math. J. **36** (1969), 273–276.

[4]    N. Dunford and J. T. Schwartz, *Linear Operators. I. General Theory*, Pure and Applied Mathematics, vol. 7, Interscience Publishers, New York, 1958.

[5]    A. S. Rao, *On differential operators with Bohr-Neugebauer type property*, J. Differential Equations **13** (1973), 490–494.

[6]    ——, *On weakly almost-periodic families of linear operators*, Canad. Math. Bull. **18** (1975), no. 1, 81–85.

[7]    S. Zaidman, *Some asymptotic theorems for abstract differential equations*, Proc. Amer. Math. Soc. **25** (1970), 521–525.

[8]    ——, *Some remarks on almost-periodicity*, Atti Accad. Sci. Torino Cl. Sci. Fis. Mat. Natur. **106** (1972), 63–67.

[9]    ——, *Abstract differential equations with almost-periodic solutions*, J. Math. Anal. Appl. **107** (1985), no. 1, 291–299.

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