RADICAL APPROACH IN BCH-ALGEBRAS

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We define the notion of radical in BCH-algebra and investigate the structure of $[X;k]$, a viewpoint of radical in BCH-algebras.

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1. Introduction. In 1966, Imai and Iséki [8] and Iséki [9] introduced two classes of abstract algebras: BCK-algebras and BCI-algebras. It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras. In 1983, Hu and Li [5, 6] introduced a wide class of abstract algebras: BCH-algebras. They have shown that the class of BCI-algebras is a proper subclass of the class of BCH-algebras. They have studied some properties of these algebras.

As we know, the primary aim of the theory of BCH-algebras is to determine the structure of all BCH-algebras. The main task of a structure theorem is to find a complete system of invariants describing the BCH-algebra up to isomorphism, or to establish some connection with other mathematics branches. In addition, the ideal theory plays an important role in studying BCI-algebras, and some interesting results have been obtained by several authors [1, 2, 3, 4, 11, 14, 15]. In 1992, Huang [7] introduced nil ideals in BCI-algebras. In 1999, Roh and Jun [13] introduced nil ideals in BCH-algebras. They introduced the concept of nil subsets by using nilpotent elements, and investigated some related properties.

In this note, we define the notion of radical in BCH-algebra, and some fundamental results concerning this notion are proved.

2. Preliminaries. A BCH-*algebra* is a nonempty set X with a constant 0 and a binary operation "$*$" satisfying the following axioms:

(1) $x * x = 0$,

(2) $x * y = 0$ and $y * x = 0$ imply $x = y$,

(3) $(x * y) * z = (x * z) * y$

for all $x, y, z \in X$. A BCH-algebra X satisfying the identity $((x * y) * (x * z)) * (z * y) = 0$ and $0 * x = 0$ for all $x, y, z \in X$ is called a BCK-*algebra*. We define the relation $\leq$ by $x \leq y$ if and only if $x * y = 0$.

In any BCH-algebra X, the following hold: for all $x, y \in X$,

(4) $(x * (x * y)) \leq y$,

(5) $x \leq 0$ implies $x = 0$,

(6) $0 * (x * y) = (0 * x) * (0 * y)$,

(7) $x * 0 = x$,

(8) $0 * (0 * (0 * x)) = 0 * x$.

A nonempty subset S of BCH-algebra X is called a *subalgebra* of X if $x * y \in S$ whenever $x, y \in S$.

A nonempty subset I of BCH-algebra X is called an *ideal* of X if $0 \in I$ and if $x * y, y \in I$ imply that $x \in I$. It is possible that an ideal of a BCH-algebra may not be a subalgebra.

3. Main results. In what follows, the letter X denotes a BCH-algebra unless otherwise specified.

DEFINITION 3.1. For any $x \in X$ and any positive integer n, the *nth power x^n* of x is defined by

$$x^1 = x, \quad x^n = x * (0 * x^{n-1}). \tag{3.1}$$

Clearly $0^n = 0$.

THEOREM 3.2. *For any $x \in X$ and any positive integer n,*

$$(0 * x)^n = 0 * x^n. \tag{3.2}$$

PROOF. We argue by induction on the positive integer n. For $n = 1$ there is nothing to prove. Assume that the theorem is true for some positive integer n. Then using (6) we have

$$\begin{aligned}
(0 * x)^{n+1} &= (0 * x) * (0 * (0 * x)^n) \\
&= (0 * x) * (0 * (0 * x^n)) \\
&= 0 * (x * (0 * x^n)) = 0 * x^{n+1}. \quad \square
\end{aligned} \tag{3.3}$$

DEFINITION 3.3. [10] In a BCH-algebra X, the set $A^+ := \{x \in X \mid 0 \le x\}$ is called a *positive part* of X and the set $A(X) := \{x \in X \mid 0 * (0 * x) = x\}$ is called an *atom part* of X. Further an element of $A(X)$ will be called an *atom* of X.

In the following theorem we give some properties of BCK-algebras.

THEOREM 3.4. *If X is a BCH-algebra, then the positive part A^+ of X is a subset of the set $\{x \in X \mid x^2 = x\}$.*

PROOF. Let $x \in A^+$. Then we have $x^2 = x * (0 * x) = x * 0 = x$, and hence $A^+ \subseteq \{x \in X \mid x^2 = x\}$. $\square$

COROLLARY 3.5. *If X is a BCK-algebra, then $X = \{x \in X \mid x^2 = x\}$.*

In [10], Kim and Roh proved $A(X) = \{0 * (0 * x) \mid x \in X\} = \{0 * x \mid x \in X\}$.

Note that $A(X)$ is a subalgebra of X and $((x * y) * (x * z)) * (z * y) = 0$ for all $x, y, z \in A(X)$, and hence $A(X)$ is a p-semisimple BCI-algebra. Thus by [12] we have the following property: for any $a, b \in A(X)$ and any positive integer n, we have $(a * b)^n = a^n * b^n$.

Hence the following corollary is an immediate consequence of Theorem 3.2.

TABLE 3.1

$*$	0	a	b	c
0	0	c	0	a
a	a	0	a	c
b	b	c	0	a
c	c	a	c	0

COROLLARY 3.6. *For any x in a BCH-algebra X and any positive integer n,*
   (i) $0 * x^n \in A(X)$,
   (ii) $0 * (x * y)^n = (0 * x^n) * (0 * y^n)$.

DEFINITION 3.7. Let R be a nonempty subset of a BCH-algebra X and k a positive integer. Then define

$$[R;k] := \{x \in R \mid x^k = 0\}, \tag{3.4}$$

which is called the *radical* of R.

We know that, in general, the radical of an ideal in X may not be an ideal.

EXAMPLE 3.8. Let $X = \{0,a,b,c\}$ be a BCH-algebra in which $*$-operation is defined as in Table 3.1. Taking an ideal $R = X$, then $[R;3] = \{0,a,c\}$ is not an ideal of X since $b * a = c \in [R;3]$ and $b \notin [R;3]$.

THEOREM 3.9. *Let S be a subalgebra of a BCH-algebra X and k a positive integer. If $x \in [S;k]$, then $0 * x \in [S;k]$.*

PROOF. Let $x \in [S;k]$. Then $x^k = 0$ and $x \in S$. Thus by Theorem 3.2 we have

$$(0 * x)^k = 0 * x^k = 0, \quad 0 * x \in S, \tag{3.5}$$

and hence $0 * x \in [S;k]$. □

This leaves open question, if R is a subalgebra of X and $0 * x \in [R;k]$, then is x in $[R;k]$? The answer is negative. In Example 3.8, $[X;3]$ is a subalgebra of X and $0 * b \in [X;3]$, but $b \notin [X;3]$.

DEFINITION 3.10 [10]. For $e \in A(X)$, the set $\{x \in X \mid e * x = 0\}$ is called the *branch* of X determined by e and is denoted by $A(e)$.

THEOREM 3.11. *Let k be a positive integer and $A(X) = X$. Then*

$$A(e) \cap [X;k] \neq \varnothing \Longrightarrow A(e) \subseteq [X;k]. \tag{3.6}$$

PROOF. Suppose that $A(e) \cap [X;k] \neq \varnothing$, then there exists $x \in A(e) \cap [X;k]$. Thus by Theorem 3.9, we have

$$e = 0 * (0 * x) \in [X;k]. \tag{3.7}$$

Let $y \in A(e)$, then $y \in [X;k]$ since $0 * (0 * y) = e \in [X;k]$, and hence $A(e) \subseteq [X;k]$. □

Theorem 3.12. *For any positive integer k and $A(X) = X$,*

$$[X;k] = \underset{x \in [X;k]}{\cup} A\big(0 * (0 * x)\big) = \underset{e \in A(X) \cap [X;k]}{\cup} A(e) = \underset{e \in [A(X);k]}{\cup} A(e). \tag{3.8}$$

Proof. $A(X) \cap [X;k] = [A(X);k]$ is obvious. By Theorem 3.11, we have $x \in A(0 * (0 * x)) \subseteq [X;k]$ for all $x \in [X;k]$, and so there exists $e = 0 * (0 * x) \in A(X) \cap [X;k]$ such that $x \in A(e) \subseteq [X;k]$. Therefore we obtain

$$[X;k] = \underset{x \in [X;k]}{\cup} A\big(0 * (0 * x)\big) = \underset{e \in A(X) \cap [X;k]}{\cup} A(e) = \underset{e \in [A(X);k]}{\cup} A(e). \tag{3.9}$$

$\square$

References

[1] B. Ahmad, *On classification of BCH-algebras*, Math. Japon. **35** (1990), no. 5, 801–804.
[2] M. A. Chaudhry, *On BCH-algebras*, Math. Japon. **36** (1991), no. 4, 665–676.
[3] M. A. Chaudhry and H. Fakhar-Ud-Din, *Ideals and filters in BCH-algebras*, Math. Japon. **44** (1996), no. 1, 101–111.
[4] W. A. Dudek and J. Thomys, *On decompositions of BCH-algebras*, Math. Japon. **35** (1990), no. 6, 1131–1138.
[5] Q. P. Hu and X. Li, *On BCH-algebras*, Math. Sem. Notes Kobe Univ. **11** (1983), no. 2, part 2, 313–320.
[6] ———, *On proper BCH-algebras*, Math. Japon. **30** (1985), no. 4, 659–661.
[7] W. P. Huang, *Nil-radical in BCI-algebras*, Math. Japon. **37** (1992), no. 2, 363–366.
[8] Y. Imai and K. Iséki, *On axiom systems of propositional calculi. XIV*, Proc. Japan Acad. **42** (1966), 19–22.
[9] K. Iséki, *An algebra related with a propositional calculus*, Proc. Japan Acad. **42** (1966), 26–29.
[10] K. H. Kim and E. H. Roh, *The role of A^+ and $A(X)$ in BCH-algebras*, Math. Japon. **52** (2000), no. 2, 317–321.
[11] S. Y. Kim, Q. Zhang, and E. H. Roh, *On nil subsets in BCH-algebras*, Math. Japon. **52** (2000), no. 2, 285–288.
[12] J. Meng and S. M. Wei, *Periods of elements in BCI-algebras*, Math. Japon. **38** (1993), no. 3, 427–431.
[13] E. H. Roh and Y. B. Jun, *On properties of nil subsets in BCH-algebras*, submitted to Indian J. Pure Appl. Math.
[14] E. H. Roh, Y. B. Jun, and Q. Zhang, *Special subset in BCH-algebras*, Far East J. Math. Sci. (FJMS) **3** (2001), no. 5, 723–729.
[15] E. H. Roh, S. Y. Kim, and Y. B. Jun, *On a problem in BCH-algebras*, Math. Japon. **52** (2000), no. 2, 279–283.

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