

THE ORLICZ SPACE OF ENTIRE SEQUENCES

K. CHANDRASEKHARA RAO and N. SUBRAMANIAN

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Let Γ denote the space of all entire sequences and \wedge the space of all analytic sequences. This paper is devoted to the study of the general properties of Orlicz space Γ_M of Γ .

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1. Introduction. An Orlicz function is a function $M : [0, \infty) \rightarrow [0, \infty)$ which is continuous, nondecreasing, and convex with $M(0) = 0$, $M(x) > 0$ for $x > 0$, and $M(x) \rightarrow \infty$ as $x \rightarrow \infty$. If the convexity of Orlicz function M is replaced by $M(x + y) \leq M(x) + M(y)$, then this function is called a modulus function, defined and discussed by Ruckle [5] and Maddox [4].

An Orlicz function M is said to satisfy the Δ_2 -condition for all values of u if there exists a constant $K > 0$ such that $M(2u) \leq KM(u)$ ($u \geq 0$). The Δ_2 -condition is equivalent to $M(\ell u) \leq K \cdot \ell M(u)$, for all values of u and for $\ell > 1$.

An Orlicz function M can always be represented in the following integral form: $M(x) = \int_0^x q(t)dt$, where q , known as the kernel of M , is right-differentiable for $t \geq 0$, $q(0) = 0$, $q(t) > 0$ for $t > 0$, q is nondecreasing, and $q(t) \rightarrow \infty$ as $t \rightarrow \infty$. Lindenstrauss and Tzafriri [3] used the idea of Orlicz function to construct the Orlicz sequence space

$$\ell_M = \left\{ x \in w : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty, \text{ for some } \rho > 0 \right\}, \quad (1.1)$$

where $w = \{\text{all complex sequences}\}$.

The space ℓ_M with the norm

$$\|x\| = \inf \left\{ \rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \leq 1 \right\} \quad (1.2)$$

becomes a Banach space which is called an Orlicz sequence space.

2. A complex sequence whose k th term is x_k will be denoted by (x_k) or x . A sequence $x = (x_k)$ is said to be analytic if $\sup_{(k)} |x_k|^{1/k} < \infty$. The vector space of all analytic sequences will be denoted by \wedge . A sequence x is called an entire sequence if $\lim_{k \rightarrow \infty} |x_k|^{1/k} = 0$. The vector space of all entire sequences will be denoted by Γ .

DEFINITION 2.1. The space consisting of all sequences x in w such that $M(|x_k|^{1/k}/\rho) \rightarrow 0$ as $k \rightarrow \infty$ for some arbitrarily fixed $\rho > 0$ is denoted by Γ_M , with M being a modulus function. In other words, $\{M(|x_k|^{1/k}/\rho)\}$ is a null sequence. The space Γ_M is

a metric space with the metric

$$d(x, y) = \sup_{(k)} M \left(\frac{|x_k - y_k|^{1/k}}{\rho} \right) \quad (2.1)$$

for all $x = \{x_k\}$ and $y = \{y_k\}$ in Γ_M .

Given a sequence $x = \{x_k\}$ whose n th section is the sequence $x^{(n)} = \{x_1, x_2, \dots, x_n, 0, 0, \dots\}$, $\delta^{(n)} = (0, 0, \dots, 1, 0, 0, \dots)$, with 1 in the n th place and zeros elsewhere; let $\Phi = \{\text{all finite sequences}\}$.

An FK-space (or a metric space) X is said to have AK property if $(\delta^{(n)})$ is a Schauder basis for X . Or equivalently $x^{(n)} \rightarrow x$.

The space is said to have or be an AD space if Φ is dense in X .

We note that AK implies AD by [1].

If X is a sequence space, we give the following definitions:

- (i) $X' =$ the continuous dual of X ;
- (ii) $X^\alpha = \{a = (a_k) : \sum_{k=1}^\infty |a_k x_k| < \infty, \text{ for each } x \in X\}$;
- (iii) $X^\beta = \{a = (a_k) : \sum_{k=1}^\infty a_k x_k \text{ is convergent, for each } x \in X\}$;
- (iv) $X^\gamma = \{a = (a_k) : \sup_{(n)} |\sum_{k=1}^n a_k x_k| < \infty, \text{ for each } x \in X\}$;
- (v) let X be an FK-space $\supset \Phi$, then $X^f = \{f(\delta^{(n)}) : f \in X'\}$. X^α , X^β , and X^γ are called the α -(or Köthe-Toeplitz-) dual of X , β -(or generalized-Köthe-Toeplitz-) dual of X , and γ -dual of X , respectively.

Note that $X^\alpha \subset X^\beta \subset X^\gamma$. If $X \subset Y$, then $Y^\mu \subset X^\mu$, for $\mu = \alpha, \beta$, or γ .

LEMMA 2.2 (see [6, Theorem 7.2.7]). *Let X be an FK-space $\supset \Phi$. Then*

- (i) $X^\gamma \subset X^f$;
- (ii) if X has AK, $X^\beta = X^f$;
- (iii) if X has AD, $X^\beta = X^\gamma$.

We note that $\Gamma^\alpha = \Gamma^\beta = \Gamma^\gamma = \wedge$.

PROPOSITION 2.3. $\Gamma \subset \Gamma_M$, with the hypothesis that $M(|x_k|^{1/k}/\rho) \leq |x_k|^{1/k}$.

PROOF. Let $x \in \Gamma$. Then we have the following implications:

$$|x_k|^{1/k} \rightarrow 0 \quad \text{as } k \rightarrow \infty. \quad (2.2)$$

But $M(|x_k|^{1/k}/\rho) \leq |x_k|^{1/k}$, by our assumption, implies that

$$\begin{aligned} M \left(\frac{|x_k|^{1/k}}{\rho} \right) &\rightarrow 0 \quad \text{as } k \rightarrow \infty \text{ (by (2.2))} \\ &\Rightarrow x \in \Gamma_M \\ &\Rightarrow \Gamma \subset \Gamma_M. \end{aligned} \quad (2.3)$$

This completes the proof. □

PROPOSITION 2.4. Γ_M has AK where M is a modulus function.

PROOF. Let $x = \{x_k\} \in \Gamma_M$, but then $\{M(|x_k|^{1/k}/\rho)\} \in \Gamma$, and hence

$$\sup_{k \geq n+1} M\left(\frac{|x_k|^{1/k}}{\rho}\right) \rightarrow 0 \quad \text{as } n \rightarrow \infty. \quad (2.4)$$

By using (2.4), $d(x, x^{[n]}) = \sup_{k \geq n+1} M(|x_k|^{1/k}/\rho) \rightarrow 0$ as $n \rightarrow \infty$, which implies that $x^{[n]} \rightarrow x$ as $n \rightarrow \infty$, implying that Γ_M has AK. This completes the proof. \square

PROPOSITION 2.5. Γ_M is solid.

PROOF. Let $|x_k| \leq |y_k|$ and let $y = (y_k) \in \Gamma_M$. $M(|x_k|^{1/k}/\rho) \leq M(|y_k|^{1/k}/\rho)$, because M is nondecreasing. But $M(|y_k|^{1/k}/\rho) \in \Gamma$ because $y \in \Gamma_M$. That is, $M(|y_k|^{1/k}/\rho) \rightarrow 0$ as $k \rightarrow \infty$ and $M(|x_k|^{1/k}/\rho) \rightarrow 0$ as $k \rightarrow \infty$. Therefore $x = \{x_k\} \in \Gamma_M$. This completes the proof. \square

PROPOSITION 2.6. Let M be an Orlicz function which satisfies the Δ_2 -condition. Then $\Gamma \subset \Gamma_M$.

PROOF. Let

$$x \in \Gamma. \quad (2.5)$$

Then $|x_k|^{1/k} \leq \varepsilon$ for sufficiently large k and every $\varepsilon > 0$. But then by taking $\rho \geq 1/2$,

$$\begin{aligned} M\left(\frac{|x_k|^{1/k}}{\rho}\right) &\leq M\left(\frac{\varepsilon}{\rho}\right) \quad (\text{because } M \text{ is nondecreasing}) \\ &\leq M(2\varepsilon) \\ \Rightarrow M\left(\frac{|x_k|^{1/k}}{\rho}\right) &\leq KM(\varepsilon) \quad (\text{by the } \Delta_2\text{-condition, for some } K > 0) \\ &\leq \varepsilon \quad (\text{by defining } M(\varepsilon) < \frac{\varepsilon}{k}) \\ \Rightarrow M\left(\frac{|x_k|^{1/k}}{\rho}\right) &\rightarrow 0 \quad \text{as } k \rightarrow \infty. \end{aligned} \quad (2.6)$$

Hence $x \in \Gamma_M$.

From (2.5) and since

$$x \in \Gamma_M, \quad (2.7)$$

we get

$$\Gamma \subset \Gamma_M. \quad (2.8)$$

This completes the proof. \square

PROPOSITION 2.7. If M is a modulus function, then Γ_M is a linear set over the set of complex numbers \mathbb{C} .

PROOF. Let $x, y \in \Gamma_M$ and $\alpha, \beta \in \mathbb{C}$. In order to prove the result, we need to find some ρ_3 such that

$$M\left(\frac{|\alpha x_k + \beta y_k|^{1/k}}{\rho_3}\right) \rightarrow 0 \quad \text{as } k \rightarrow \infty. \quad (2.9)$$

Since $x, y \in \Gamma_M$, there exist some positive ρ_1 and ρ_2 such that

$$\begin{aligned} M\left(\frac{|x_k|^{1/k}}{\rho_1}\right) &\rightarrow 0 \quad \text{as } k \rightarrow \infty, \\ M\left(\frac{|y_k|^{1/k}}{\rho_2}\right) &\rightarrow 0 \quad \text{as } k \rightarrow \infty. \end{aligned} \quad (2.10)$$

Since M is a nondecreasing modulus function, we have

$$\begin{aligned} M\left(\frac{|\alpha x_k + \beta y_k|^{1/k}}{\rho_3}\right) &\leq M\left(\frac{|\alpha x_k|^{1/k}}{\rho_3} + \frac{|\beta y_k|^{1/k}}{\rho_3}\right) \\ &\leq M\left(\frac{|\alpha|^{1/k} |x_k|^{1/k}}{\rho_3} + \frac{|\beta|^{1/k} |y_k|^{1/k}}{\rho_3}\right) \\ &\leq M\left(\frac{|\alpha| |x_k|^{1/k}}{\rho_3} + \frac{|\beta| |y_k|^{1/k}}{\rho_3}\right). \end{aligned} \quad (2.11)$$

Take ρ_3 such that

$$\frac{1}{\rho_3} = \min\left\{\frac{1}{|\alpha|} \frac{1}{\rho_1}, \frac{1}{|\beta|} \frac{1}{\rho_2}\right\}. \quad (2.12)$$

Then

$$\begin{aligned} M\left(\frac{|\alpha x_k + \beta y_k|^{1/k}}{\rho_3}\right) &\leq M\left(\frac{|x_k|^{1/k}}{\rho_1} + \frac{|y_k|^{1/k}}{\rho_2}\right) \\ &\leq M\left(\frac{|x_k|^{1/k}}{\rho_1}\right) + M\left(\frac{|y_k|^{1/k}}{\rho_2}\right) \\ &\rightarrow 0 \quad (\text{by (2.10)}). \end{aligned} \quad (2.13)$$

Hence

$$M\left(\frac{|\alpha x_k + \beta y_k|^{1/k}}{\rho_3}\right) \rightarrow 0 \quad \text{as } k \rightarrow \infty. \quad (2.14)$$

So $(\alpha x + \beta y) \in \Gamma_M$. Therefore Γ_M is linear. This completes the proof. \square

DEFINITION 2.8. Let $p = (p_k)$ be any sequence of positive real numbers. Then

$$\Gamma_M(p) = \left\{ x = \{x_k\} : \left(M \left(\frac{|x_k|^{1/k}}{\rho} \right) \right)^{p_k} \rightarrow 0 \text{ as } k \rightarrow \infty \right\}. \quad (2.15)$$

Suppose that p_k is a constant for all k , then $\Gamma_M(p) = \Gamma_M$.

PROPOSITION 2.9. Let $0 \leq p_k \leq q_k$ and let $\{q_k/p_k\}$ be bounded. Then $\Gamma_M(q) \subset \Gamma_M(p)$.

PROOF. Let

$$x \in \Gamma_M(q), \quad (2.16)$$

$$\left(M \left(\frac{|x_k|^{1/k}}{\rho} \right) \right)^{q_k} \rightarrow 0 \text{ as } k \rightarrow \infty. \quad (2.17)$$

Let $t_k = (M(|x_k|^{1/k}/\rho))^{q_k}$ and $\lambda_k = p_k/q_k$. Since $p_k \leq q_k$, we have $0 \leq \lambda_k \leq 1$.

Take $0 < \lambda < \lambda_k$. Define

$$\begin{aligned} u_k &= \begin{cases} t_k & (t_k \geq 1) \\ 0 & (t_k < 1), \end{cases} \\ v_k &= \begin{cases} 0 & (t_k \geq 1) \\ t_k & (t_k < 1), \end{cases} \\ t_k &= u_k + v_k, \quad t_k^{\lambda_k} = u_k^{\lambda_k} + v_k^{\lambda_k}. \end{aligned} \quad (2.18)$$

Now it follows that

$$u_k^{\lambda_k} \leq u_k \leq t_k, \quad v_k^{\lambda_k} \leq v_k^{\lambda}. \quad (2.19)$$

Since $t_k^{\lambda_k} = u_k^{\lambda_k} + v_k^{\lambda_k}$, then $t_k^{\lambda_k} \leq t_k + v_k^{\lambda}$.

$$\begin{aligned} \left(M \left(\frac{|x_k|^{1/k}}{\rho} \right)^{q_k} \right)^{\lambda_k} &\leq \left(M \left(\frac{|x_k|^{1/k}}{\rho} \right) \right)^{q_k} \\ \Rightarrow \left(M \left(\frac{|x_k|^{1/k}}{\rho} \right)^{q_k} \right)^{p_k/q_k} &\leq \left(M \left(\frac{|x_k|^{1/k}}{\rho} \right) \right)^{q_k} \\ \Rightarrow \left(M \left(\frac{|x_k|^{1/k}}{\rho} \right) \right)^{p_k} &\leq \left(M \left(\frac{|x_k|^{1/k}}{\rho} \right) \right)^{q_k}. \end{aligned} \quad (2.20)$$

But

$$\left(M \left(\frac{|x_k|^{1/k}}{\rho} \right) \right)^{q_k} \rightarrow 0 \text{ (by (2.17))}. \quad (2.21)$$

Hence $(M(|x_k|^{1/k}/\rho))^{p_k} \rightarrow 0$ as $k \rightarrow \infty$. Hence

$$x \in \Gamma_M(p). \quad (2.22)$$

From (2.16) and (2.22), we get

$$\Gamma_M(q) \subset \Gamma_M(p). \quad (2.23)$$

This completes the proof. \square

PROPOSITION 2.10. (a) Let $0 < \inf p_k \leq p_k \leq 1$. Then $\Gamma_M(p) \subset \Gamma_M$.

(b) Let $1 \leq p_k \leq \sup p_k < \infty$. Then $\Gamma_M \subset \Gamma_M(p)$.

PROOF. (a) Let $x \in \Gamma_M(p)$,

$$\left(M \left(\frac{|x_k|^{1/k}}{\rho} \right) \right)^{p_k} \rightarrow 0 \quad \text{as } k \rightarrow \infty. \quad (2.24)$$

Since $0 < \inf p_k \leq p_k \leq 1$,

$$\left(M \left(\frac{|x_k|^{1/k}}{\rho} \right) \right) \leq \left(M \left(\frac{|x_k|^{1/k}}{\rho} \right) \right)^{p_k}, \quad (2.25)$$

From (2.24) and (2.25) it follows that

$$x \in \Gamma_M. \quad (2.26)$$

Thus

$$\Gamma_M(p) \subset \Gamma_M. \quad (2.27)$$

We have thus proven (a).

(b) Let $p_k \geq 1$ for each k and $\sup p_k < \infty$ and let $x \in \Gamma_M$.

$$M \left(\frac{|x_k|^{1/k}}{\rho} \right) \rightarrow 0 \quad \text{as } k \rightarrow \infty. \quad (2.28)$$

Since $1 \leq p_k \leq \sup p_k < \infty$, we have

$$\begin{aligned} \left(M \left(\frac{|x_k|^{1/k}}{\rho} \right) \right)^{p_k} &\leq \left(M \left(\frac{|x_k|^{1/k}}{\rho} \right) \right), \\ \left(M \left(\frac{|x_k|^{1/k}}{\rho} \right) \right)^{p_k} &\rightarrow 0 \quad \text{as } k \rightarrow \infty \quad (\text{by using (2.28)}). \end{aligned} \quad (2.29)$$

Therefore $x \in \Gamma_M(p)$. This completes the proof. \square

PROPOSITION 2.11. *Let $0 < p_k \leq q_k < \infty$ for each k . Then $\Gamma_M(p) \subseteq \Gamma_M(q)$.*

PROOF. Let $x \in \Gamma_M(p)$

$$\left(M \left(\frac{|x_k|^{1/k}}{\rho} \right) \right)^{p_k} \rightarrow 0 \quad \text{as } k \rightarrow \infty. \quad (2.30)$$

This implies that $(M(|x_k|^{1/k}/\rho)) \leq 1$ for sufficiently large k . Since M is nondecreasing, we get

$$\begin{aligned} \left(M \left(\frac{|x_k|^{1/k}}{\rho} \right) \right)^{q_k} &\leq \left(M \left(\frac{|x_k|^{1/k}}{\rho} \right) \right)^{p_k} \\ &\Rightarrow \left(M \left(\frac{|x_k|^{1/k}}{\rho} \right) \right)^{q_k} \rightarrow 0 \quad \text{as } k \rightarrow \infty \quad (\text{by using (2.30)}). \end{aligned} \quad (2.31)$$

Since $x \in \Gamma_M(q)$, hence $\Gamma_M(p) \subseteq \Gamma_M(q)$. This completes the proof. \square

PROPOSITION 2.12. $\Gamma_M(p)$ is r -convex for all r , where $0 \leq r \leq \inf p_k$. Moreover, if $p_k = p \leq 1$ for all k , then they are p -convex.

PROOF. We will prove the theorem for $\Gamma_M(p)$. Let $x \in \Gamma_M(p)$ and $r \in (0, \lim_{n \rightarrow \infty} \inf p_n)$. Then, there exists k_0 such that $r \leq p_k$ for all $k > k_0$.

Now, define

$$g^*(x) = \inf \left\{ \rho : M \left(\frac{|x_k - y_k|^{1/k}}{\rho} \right)^r + M \left(\frac{|x_k - y_k|^{1/k}}{\rho} \right)^{p_n} \right\}. \quad (2.32)$$

Since $r \leq p_k \leq 1$ for all $k > k_0$, g^* is subadditive. Further, for $0 \leq |\lambda| \leq 1$,

$$|\lambda|^{p_k} \leq |\lambda|^r \quad \forall k > k_0. \quad (2.33)$$

Therefore, for each λ , we have

$$g^*(\lambda x) \leq |\lambda|^r \cdot g^*(x). \quad (2.34)$$

Now, for $0 < \delta < 1$,

$$U = \{x : g^*(x) \leq \delta\}, \quad (2.35)$$

which is an absolutely r -convex set, for

$$|\lambda|^r + |\mu|^r \leq 1, \quad x, y \in U. \quad (2.36)$$

Now

$$\begin{aligned}
 g^*(\lambda x + \mu y) &\leq g^*(\lambda x) + g^*(\mu y) \\
 &\leq |\lambda|^r g^*(x) + |\mu|^r g^*(y) \\
 &\leq |\lambda|^r \delta + |\mu|^r \delta \quad (\text{using (2.34) and (2.35)}) \\
 &\leq (|\lambda|^r + |\mu|^r) \delta \\
 &\leq 1 \cdot \delta \quad (\text{by using (2.36)}) \\
 &\leq \delta.
 \end{aligned} \tag{2.37}$$

If $p_k = p \leq 1$ for all k , then for $0 < r < 1$, $U = \{x : g^*(x) \leq \delta\}$ is an absolutely p -convex set. This can be obtained by a similar analysis and therefore we omit the details. This completes the proof. \square

PROPOSITION 2.13. $(\Gamma_M)^\beta = \wedge$.

PROOF

STEP 1. $\Gamma \subset \Gamma_M$ by Proposition 2.3; this implies that $(\Gamma_M)^\beta \subset \Gamma^\beta = \wedge$. Therefore,

$$(\Gamma_M)^\beta \subset \wedge. \tag{2.38}$$

STEP 2. Let $y \in \wedge$. Then $|y_k| < M^k$ for all k and for some constant $M > 0$.

Let $x \in \Gamma_M$. Then $M(|x_k|^{1/k}/\rho) \rightarrow 0$ as $k \rightarrow \infty$. Hence $M(|x_k|^{1/k}/\rho) < \varepsilon$ for given $\varepsilon > 0$ for sufficiently large k .

Take $\varepsilon = 1/2M$ so that $M(|x_k|/\rho) < 1/(2M)^k$.

But then $M(|x_k y_k|/\rho) \leq 1/2^k$ so that $\sum_{k=1}^{\infty} M(|x_k y_k|/\rho)$ converges. Therefore $\sum_{k=1}^{\infty} M(x_k y_k/\rho)$ converges. Hence $\sum_{k=1}^{\infty} x_k y_k$ converges so that $y \in (\Gamma_M)^\beta$. Thus

$$\wedge \subset (\Gamma_M)^\beta. \tag{2.39}$$

STEP 3. From (2.38) and (2.39), we obtain

$$(\Gamma_M)^\beta = \wedge. \tag{2.40}$$

This completes the proof. \square

PROPOSITION 2.14. $(\Gamma_M)^\mu = \wedge$ for $\mu = \alpha, \beta, \gamma, f$.

PROOF

STEP 1. Γ_M has AK by Proposition 2.4. Hence, by Lemma 2.2(i), we get $(\Gamma_M)^\beta = (\Gamma_M)^f$. But $(\Gamma_M)^\beta = \wedge$. Hence

$$(\Gamma_M)^f = \wedge. \tag{2.41}$$

STEP 2. Since AK implies AD, hence by Lemma 2.2(iii) we get $(\Gamma_M)^\beta = (\Gamma_M)^\gamma$. Therefore

$$(\Gamma_M)^\gamma = \wedge. \tag{2.42}$$

STEP 3. Γ_M is normal by [Proposition 2.5](#). Hence, by [2, Proposition 2.7], we get

$$(\Gamma_M)^\alpha = (\Gamma_M)^\gamma = \wedge. \quad (2.43)$$

From (2.41), (2.42), and (2.43), we have

$$(\Gamma_M)^\alpha = (\Gamma_M)^\beta = (\Gamma_M)^\gamma = (\Gamma_M)^f = \wedge. \quad (2.44)$$

□

PROPOSITION 2.15. *The dual space of Γ_M is \wedge . In other words, $\Gamma_M^* = \wedge$.*

PROOF. We recall that δ^k has 1 in the k th place and zeros elsewhere, with

$$\begin{aligned} x = \delta^k, \quad \left\{ M\left(\frac{|x_k|^{1/k}}{\rho}\right) \right\} &= \left\{ \frac{M(0)^1}{\rho}, \frac{M(0)^{1/2}}{\rho}, \dots, \frac{M(1)^{1/k}}{\rho}, \frac{M(0)^{1/(k+1)}}{\rho}, \dots \right\} \\ &= \left\{ 0, 0, \dots, \frac{M(1)^{1/k}}{\rho}, 0, \dots \right\} \end{aligned} \quad (2.45)$$

which is a null sequence. Hence $\delta^k \in \Gamma_M$. $f(x) = \sum_{k=1}^{\infty} x_k y_k$ with $x \in \Gamma_M$ and $f \in \Gamma_M^*$, where Γ_M^* is the dual space of Γ_M . Take $x = \delta^k \in \Gamma_M$. Then

$$|y_k| \leq \|f\| d(\delta^k, 0) < \infty \quad \forall k. \quad (2.46)$$

Thus (y_k) is a bounded sequence and hence an analytic sequence. In other words, $y \in \wedge$. Therefore $\Gamma_M^* = \wedge$. This completes the proof. □

LEMMA 2.16 [6, Theorem 8.6.1]. $Y \supset X \Leftrightarrow Y^f \subset X^f$, where X is an AD-space and Y an FK-space.

PROPOSITION 2.17. *Let Y be any FK-space $\supset \Phi$. Then $Y \supset \Gamma_M$ if and only if the sequence $\delta^{(k)}$ is weakly analytic.*

PROOF. The following implications establish the result: since Γ_M has AD and by [Lemma 2.16](#),

$$\begin{aligned} Y \supset \Gamma_M &\Leftrightarrow Y^f \subset (\Gamma_M)^f \\ &\Leftrightarrow Y^f \subset \wedge \quad \left(\text{since } (\Gamma_M)^f = \wedge \right) \\ &\Leftrightarrow \text{for each } f \in Y', \text{ the topological dual of } Y \cdot f(\delta^{(k)}) \in \wedge \\ &\Leftrightarrow f(\delta^{(k)}) \text{ is analytic} \\ &\Leftrightarrow \delta^{(k)} \text{ is weakly analytic,} \end{aligned} \quad (2.47)$$

this completes the proof. □

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K. Chandrasekhara Rao: Srinivasa Ramanujan Centre, Shanmugha Arts, Science, Technology, and Research Academy (SASTRA), Kumbakonam 612 001, India

E-mail address: kcrao2008@yahoo.co.in

N. Subramanian: Department of Mathematics, Shanmugha Arts, Science, Technology, and Research Academy (SASTRA), Thanjavur 613 402, India

E-mail address: nsmaths@yahoo.com

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