

## THE ORLICZ SPACE OF ENTIRE SEQUENCES

K. CHANDRASEKHARA RAO and N. SUBRAMANIAN

Received 24 November 2003

Let  $\Gamma$  denote the space of all entire sequences and  $\wedge$  the space of all analytic sequences. This paper is devoted to the study of the general properties of Orlicz space  $\Gamma_M$  of  $\Gamma$ .

2000 Mathematics Subject Classification: 46A45.

**1. Introduction.** An Orlicz function is a function  $M : [0, \infty) \rightarrow [0, \infty)$  which is continuous, nondecreasing, and convex with  $M(0) = 0$ ,  $M(x) > 0$  for  $x > 0$ , and  $M(x) \rightarrow \infty$  as  $x \rightarrow \infty$ . If the convexity of Orlicz function  $M$  is replaced by  $M(x + y) \leq M(x) + M(y)$ , then this function is called a modulus function, defined and discussed by Ruckle [5] and Maddox [4].

An Orlicz function  $M$  is said to satisfy the  $\Delta_2$ -condition for all values of  $u$  if there exists a constant  $K > 0$  such that  $M(2u) \leq KM(u)$  ( $u \geq 0$ ). The  $\Delta_2$ -condition is equivalent to  $M(\ell u) \leq K \cdot \ell M(u)$ , for all values of  $u$  and for  $\ell > 1$ .

An Orlicz function  $M$  can always be represented in the following integral form:  $M(x) = \int_0^x q(t)dt$ , where  $q$ , known as the kernel of  $M$ , is right-differentiable for  $t \geq 0$ ,  $q(0) = 0$ ,  $q(t) > 0$  for  $t > 0$ ,  $q$  is nondecreasing, and  $q(t) \rightarrow \infty$  as  $t \rightarrow \infty$ . Lindenstrauss and Tzafriri [3] used the idea of Orlicz function to construct the Orlicz sequence space

$$\ell_M = \left\{ x \in w : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty, \text{ for some } \rho > 0 \right\}, \quad (1.1)$$

where  $w = \{\text{all complex sequences}\}$ .

The space  $\ell_M$  with the norm

$$\|x\| = \inf \left\{ \rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \leq 1 \right\} \quad (1.2)$$

becomes a Banach space which is called an Orlicz sequence space.

**2.** A complex sequence whose  $k$ th term is  $x_k$  will be denoted by  $(x_k)$  or  $x$ . A sequence  $x = (x_k)$  is said to be analytic if  $\sup_{(k)} |x_k|^{1/k} < \infty$ . The vector space of all analytic sequences will be denoted by  $\wedge$ . A sequence  $x$  is called an entire sequence if  $\lim_{k \rightarrow \infty} |x_k|^{1/k} = 0$ . The vector space of all entire sequences will be denoted by  $\Gamma$ .

**DEFINITION 2.1.** The space consisting of all sequences  $x$  in  $w$  such that  $M(|x_k|^{1/k}/\rho) \rightarrow 0$  as  $k \rightarrow \infty$  for some arbitrarily fixed  $\rho > 0$  is denoted by  $\Gamma_M$ , with  $M$  being a modulus function. In other words,  $\{M(|x_k|^{1/k}/\rho)\}$  is a null sequence. The space  $\Gamma_M$  is

a metric space with the metric

$$d(x, y) = \sup_{(k)} M\left(\frac{|x_k - y_k|^{1/k}}{\rho}\right) \quad (2.1)$$

for all  $x = \{x_k\}$  and  $y = \{y_k\}$  in  $\Gamma_M$ .

Given a sequence  $x = \{x_k\}$  whose  $n$ th section is the sequence  $x^{(n)} = \{x_1, x_2, \dots, x_n, 0, \dots\}$ ,  $\delta^{(n)} = (0, 0, \dots, 1, 0, 0, \dots)$ , with 1 in the  $n$ th place and zeros elsewhere; let  $\Phi = \{\text{all finite sequences}\}$ .

An FK-space (or a metric space)  $X$  is said to have AK property if  $(\delta^{(n)})$  is a Schauder basis for  $X$ . Or equivalently  $x^{(n)} \rightarrow x$ .

The space is said to have or be an AD space if  $\Phi$  is dense in  $X$ .

We note that AK implies AD by [1].

If  $X$  is a sequence space, we give the following definitions:

- (i)  $X'$  = the continuous dual of  $X$ ;
- (ii)  $X^\alpha = \{a = (a_k) : \sum_{k=1}^{\infty} |a_k x_k| < \infty, \text{ for each } x \in X\}$ ;
- (iii)  $X^\beta = \{a = (a_k) : \sum_{k=1}^{\infty} a_k x_k \text{ is convergent, for each } x \in X\}$ ;
- (iv)  $X^\gamma = \{a = (a_k) : \sup_{(n)} |\sum_{k=1}^n a_k x_k| < \infty, \text{ for each } x \in X\}$ ;
- (v) let  $X$  be an FK-space  $\supset \Phi$ , then  $X^f = \{f(\delta^{(n)}) : f \in X'\}$ .  $X^\alpha, X^\beta$ , and  $X^\gamma$  are called the  $\alpha$ - (or Köthe-Toeplitz-) dual of  $X$ ,  $\beta$ - (or generalized-Köthe-Toeplitz-) dual of  $X$ , and  $\gamma$ -dual of  $X$ , respectively.

Note that  $X^\alpha \subset X^\beta \subset X^\gamma$ . If  $X \subset Y$ , then  $Y^\mu \subset X^\mu$ , for  $\mu = \alpha, \beta$ , or  $\gamma$ .

**LEMMA 2.2** (see [6, Theorem 7.2.7]). *Let  $X$  be an FK-space  $\supset \Phi$ . Then*

- (i)  $X^\gamma \subset X^f$ ;
- (ii) if  $X$  has AK,  $X^\beta = X^f$ ;
- (iii) if  $X$  has AD,  $X^\beta = X^\gamma$ .

We note that  $\Gamma^\alpha = \Gamma^\beta = \Gamma^\gamma = \wedge$ .

**PROPOSITION 2.3.**  $\Gamma \subset \Gamma_M$ , with the hypothesis that  $M(|x_k|^{1/k}/\rho) \leq |x_k|^{1/k}$ .

**PROOF.** Let  $x \in \Gamma$ . Then we have the following implications:

$$|x_k|^{1/k} \rightarrow 0 \quad \text{as } k \rightarrow \infty. \quad (2.2)$$

But  $M(|x_k|^{1/k}/\rho) \leq |x_k|^{1/k}$ , by our assumption, implies that

$$\begin{aligned} M\left(\frac{|x_k|^{1/k}}{\rho}\right) &\rightarrow 0 \quad \text{as } k \rightarrow \infty \text{ (by (2.2))} \\ \implies x &\in \Gamma_M \\ \implies \Gamma &\subset \Gamma_M. \end{aligned} \quad (2.3)$$

This completes the proof. □

**PROPOSITION 2.4.**  $\Gamma_M$  has AK where  $M$  is a modulus function.

**PROOF.** Let  $x = \{x_k\} \in \Gamma_M$ , but then  $\{M(|x_k|^{1/k}/\rho)\} \in \Gamma$ , and hence

$$\sup_{k \geq n+1} M\left(\frac{|x_k|^{1/k}}{\rho}\right) \rightarrow 0 \quad \text{as } n \rightarrow \infty. \quad (2.4)$$

By using (2.4),  $d(x, x^{[n]}) = \sup_{k \geq n+1} M(|x_k|^{1/k}/\rho) \rightarrow 0$  as  $n \rightarrow \infty$ , which implies that  $x^{[n]} \rightarrow x$  as  $n \rightarrow \infty$ , implying that  $\Gamma_M$  has AK. This completes the proof.  $\square$

**PROPOSITION 2.5.**  $\Gamma_M$  is solid.

**PROOF.** Let  $|x_k| \leq |y_k|$  and let  $y = (y_k) \in \Gamma_M$ .  $M(|x_k|^{1/k}/\rho) \leq M(|y_k|^{1/k}/\rho)$ , because  $M$  is nondecreasing. But  $M(|y_k|^{1/k}/\rho) \in \Gamma$  because  $y \in \Gamma_M$ . That is,  $M(|y_k|^{1/k}/\rho) \rightarrow 0$  as  $k \rightarrow \infty$  and  $M(|x_k|^{1/k}/\rho) \rightarrow 0$  as  $k \rightarrow \infty$ . Therefore  $x = \{x_k\} \in \Gamma_M$ . This completes the proof.  $\square$

**PROPOSITION 2.6.** Let  $M$  be an Orlicz function which satisfies the  $\Delta_2$ -condition. Then  $\Gamma \subset \Gamma_M$ .

**PROOF.** Let

$$x \in \Gamma. \quad (2.5)$$

Then  $|x_k|^{1/k} \leq \varepsilon$  for sufficiently large  $k$  and every  $\varepsilon > 0$ . But then by taking  $\rho \geq 1/2$ ,

$$\begin{aligned} M\left(\frac{|x_k|^{1/k}}{\rho}\right) &\leq M\left(\frac{\varepsilon}{\rho}\right) \quad (\text{because } M \text{ is nondecreasing}) \\ &\leq M(2\varepsilon) \\ \Rightarrow M\left(\frac{|x_k|^{1/k}}{\rho}\right) &\leq KM(\varepsilon) \quad (\text{by the } \Delta_2\text{-condition, for some } K > 0) \\ &\leq \varepsilon \quad \left(\text{by defining } M(\varepsilon) < \frac{\varepsilon}{K}\right) \\ \Rightarrow M\left(\frac{|x_k|^{1/k}}{\rho}\right) &\rightarrow 0 \quad \text{as } k \rightarrow \infty. \end{aligned} \quad (2.6)$$

Hence  $x \in \Gamma_M$ .

From (2.5) and since

$$x \in \Gamma_M, \quad (2.7)$$

we get

$$\Gamma \subset \Gamma_M. \quad (2.8)$$

This completes the proof.  $\square$

**PROPOSITION 2.7.** If  $M$  is a modulus function, then  $\Gamma_M$  is a linear set over the set of complex numbers  $\mathbb{C}$ .

**PROOF.** Let  $x, y \in \Gamma_M$  and  $\alpha, \beta \in \mathbb{C}$ . In order to prove the result, we need to find some  $\rho_3$  such that

$$M\left(\frac{|\alpha x_k + \beta y_k|^{1/k}}{\rho_3}\right) \rightarrow 0 \quad \text{as } k \rightarrow \infty. \quad (2.9)$$

Since  $x, y \in \Gamma_M$ , there exist some positive  $\rho_1$  and  $\rho_2$  such that

$$\begin{aligned} M\left(\frac{|x_k|^{1/k}}{\rho_1}\right) &\rightarrow 0 \quad \text{as } k \rightarrow \infty, \\ M\left(\frac{|y_k|^{1/k}}{\rho_2}\right) &\rightarrow 0 \quad \text{as } k \rightarrow \infty. \end{aligned} \quad (2.10)$$

Since  $M$  is a nondecreasing modulus function, we have

$$\begin{aligned} M\left(\frac{|\alpha x_k + \beta y_k|^{1/k}}{\rho_3}\right) &\leq M\left(\frac{|\alpha x_k|^{1/k}}{\rho_3} + \frac{|\beta y_k|^{1/k}}{\rho_3}\right) \\ &\leq M\left(\frac{|\alpha|^{1/k} |x_k|^{1/k}}{\rho_3} + \frac{|\beta|^{1/k} |y_k|^{1/k}}{\rho_3}\right) \\ &\leq M\left(\frac{|\alpha| |x_k|^{1/k}}{\rho_3} + \frac{|\beta| |y_k|^{1/k}}{\rho_3}\right). \end{aligned} \quad (2.11)$$

Take  $\rho_3$  such that

$$\frac{1}{\rho_3} = \min\left\{\frac{1}{|\alpha|} \frac{1}{\rho_1}, \frac{1}{|\beta|} \frac{1}{\rho_2}\right\}. \quad (2.12)$$

Then

$$\begin{aligned} M\left(\frac{|\alpha x_k + \beta y_k|^{1/k}}{\rho_3}\right) &\leq M\left(\frac{|x_k|^{1/k}}{\rho_1} + \frac{|y_k|^{1/k}}{\rho_2}\right) \\ &\leq M\left(\frac{|x_k|^{1/k}}{\rho_1}\right) + M\left(\frac{|y_k|^{1/k}}{\rho_2}\right) \\ &\rightarrow 0 \quad (\text{by (2.10)}). \end{aligned} \quad (2.13)$$

Hence

$$M\left(\frac{|\alpha x_k + \beta y_k|^{1/k}}{\rho_3}\right) \rightarrow 0 \quad \text{as } k \rightarrow \infty. \quad (2.14)$$

So  $(\alpha x + \beta y) \in \Gamma_M$ . Therefore  $\Gamma_M$  is linear. This completes the proof.  $\square$

**DEFINITION 2.8.** Let  $p = (p_k)$  be any sequence of positive real numbers. Then

$$\Gamma_M(p) = \left\{ x = \{x_k\} : \left( M \left( \frac{|x_k|^{1/k}}{\rho} \right) \right)^{p_k} \rightarrow 0 \text{ as } k \rightarrow \infty \right\}. \quad (2.15)$$

Suppose that  $p_k$  is a constant for all  $k$ , then  $\Gamma_M(p) = \Gamma_M$ .

**PROPOSITION 2.9.** Let  $0 \leq p_k \leq q_k$  and let  $\{q_k/p_k\}$  be bounded. Then  $\Gamma_M(q) \subset \Gamma_M(p)$ .

**PROOF.** Let

$$x \in \Gamma_M(q), \quad (2.16)$$

$$\left( M \left( \frac{|x_k|^{1/k}}{\rho} \right) \right)^{q_k} \rightarrow 0 \quad \text{as } k \rightarrow \infty. \quad (2.17)$$

Let  $t_k = (M(|x_k|^{1/k}/\rho))^{q_k}$  and  $\lambda_k = p_k/q_k$ . Since  $p_k \leq q_k$ , we have  $0 \leq \lambda_k \leq 1$ .

Take  $0 < \lambda < \lambda_k$ . Define

$$\begin{aligned} u_k &= \begin{cases} t_k & (t_k \geq 1) \\ 0 & (t_k < 1), \end{cases} \\ v_k &= \begin{cases} 0 & (t_k \geq 1) \\ t_k & (t_k < 1), \end{cases} \\ t_k &= u_k + v_k, \quad t_k^{\lambda_k} = u_k^{\lambda_k} + v_k^{\lambda_k}. \end{aligned} \quad (2.18)$$

Now it follows that

$$u_k^{\lambda_k} \leq u_k \leq t_k, \quad v_k^{\lambda_k} \leq v_k^{\lambda}. \quad (2.19)$$

Since  $t_k^{\lambda_k} = u_k^{\lambda_k} + v_k^{\lambda_k}$ , then  $t_k^{\lambda_k} \leq t_k + v_k^{\lambda}$ .

$$\begin{aligned} \left( M \left( \frac{|x_k|^{1/k}}{\rho} \right)^{q_k} \right)^{\lambda_k} &\leq \left( M \left( \frac{|x_k|^{1/k}}{\rho} \right) \right)^{q_k} \\ &\Rightarrow \left( M \left( \frac{|x_k|^{1/k}}{\rho} \right)^{q_k} \right)^{p_k/q_k} \leq \left( M \left( \frac{|x_k|^{1/k}}{\rho} \right) \right)^{q_k} \\ &\Rightarrow \left( M \left( \frac{|x_k|^{1/k}}{\rho} \right) \right)^{p_k} \leq \left( M \left( \frac{|x_k|^{1/k}}{\rho} \right) \right)^{q_k}. \end{aligned} \quad (2.20)$$

But

$$\left( M \left( \frac{|x_k|^{1/k}}{\rho} \right) \right)^{q_k} \rightarrow 0 \quad (\text{by (2.17)}). \quad (2.21)$$

Hence  $(M(|x_k|^{1/k}/\rho))^{p_k} \rightarrow 0$  as  $k \rightarrow \infty$ . Hence

$$x \in \Gamma_M(p). \quad (2.22)$$

From (2.16) and (2.22), we get

$$\Gamma_M(q) \subset \Gamma_M(p). \quad (2.23)$$

This completes the proof.  $\square$

**PROPOSITION 2.10.** (a) Let  $0 < \inf p_k \leq p_k \leq 1$ . Then  $\Gamma_M(p) \subset \Gamma_M$ .

(b) Let  $1 \leq p_k \leq \sup p_k < \infty$ . Then  $\Gamma_M \subset \Gamma_M(p)$ .

**PROOF.** (a) Let  $x \in \Gamma_M(p)$ ,

$$\left( M \left( \frac{|x_k|^{1/k}}{\rho} \right) \right)^{p_k} \rightarrow 0 \quad \text{as } k \rightarrow \infty. \quad (2.24)$$

Since  $0 < \inf p_k \leq p_k \leq 1$ ,

$$\left( M \left( \frac{|x_k|^{1/k}}{\rho} \right) \right) \leq \left( M \left( \frac{|x_k|^{1/k}}{\rho} \right) \right)^{p_k}, \quad (2.25)$$

From (2.24) and (2.25) it follows that

$$x \in \Gamma_M. \quad (2.26)$$

Thus

$$\Gamma_M(p) \subset \Gamma_M. \quad (2.27)$$

We have thus proven (a).

(b) Let  $p_k \geq 1$  for each  $k$  and  $\sup p_k < \infty$  and let  $x \in \Gamma_M$ .

$$M \left( \frac{|x_k|^{1/k}}{\rho} \right) \rightarrow 0 \quad \text{as } k \rightarrow \infty. \quad (2.28)$$

Since  $1 \leq p_k \leq \sup p_k < \infty$ , we have

$$\begin{aligned} \left( M \left( \frac{|x_k|^{1/k}}{\rho} \right) \right)^{p_k} &\leq \left( M \left( \frac{|x_k|^{1/k}}{\rho} \right) \right), \\ \left( M \left( \frac{|x_k|^{1/k}}{\rho} \right) \right)^{p_k} &\rightarrow 0 \quad \text{as } k \rightarrow \infty \text{ (by using (2.28))}. \end{aligned} \quad (2.29)$$

Therefore  $x \in \Gamma_M(p)$ . This completes the proof.  $\square$

**PROPOSITION 2.11.** *Let  $0 < p_k \leq q_k < \infty$  for each  $k$ . Then  $\Gamma_M(p) \subseteq \Gamma_M(q)$ .*

**PROOF.** Let  $x \in \Gamma_M(p)$

$$\left( M \left( \frac{|x_k|^{1/k}}{\rho} \right) \right)^{p_k} \rightarrow 0 \quad \text{as } k \rightarrow \infty. \quad (2.30)$$

This implies that  $(M(|x_k|^{1/k}/\rho)) \leq 1$  for sufficiently large  $k$ . Since  $M$  is nondecreasing, we get

$$\begin{aligned} \left( M \left( \frac{|x_k|^{1/k}}{\rho} \right) \right)^{q_k} &\leq \left( M \left( \frac{|x_k|^{1/k}}{\rho} \right) \right)^{p_k} \\ &\Rightarrow \left( M \left( \frac{|x_k|^{1/k}}{\rho} \right) \right)^{q_k} \rightarrow 0 \quad \text{as } k \rightarrow \infty \text{ (by using (2.30))}. \end{aligned} \quad (2.31)$$

Since  $x \in \Gamma_M(q)$ , hence  $\Gamma_M(p) \subseteq \Gamma_M(q)$ . This completes the proof.  $\square$

**PROPOSITION 2.12.**  *$\Gamma_M(p)$  is  $r$ -convex for all  $r$ , where  $0 \leq r \leq \inf p_k$ . Moreover, if  $p_k = p \leq 1$  for all  $k$ , then they are  $p$ -convex.*

**PROOF.** We will prove the theorem for  $\Gamma_M(p)$ . Let  $x \in \Gamma_M(p)$  and  $r \in (0, \lim_{n \rightarrow \infty} \inf p_n)$ . Then, there exists  $k_0$  such that  $r \leq p_k$  for all  $k > k_0$ .

Now, define

$$g^*(x) = \inf \left\{ \rho : M \left( \frac{|x_k - y_k|^{1/k}}{\rho} \right)^r + M \left( \frac{|x_k - y_k|^{1/k}}{\rho} \right)^{p_n} \right\}. \quad (2.32)$$

Since  $r \leq p_k \leq 1$  for all  $k > k_0$ ,  $g^*$  is subadditive. Further, for  $0 \leq |\lambda| \leq 1$ ,

$$|\lambda|^{p_k} \leq |\lambda|^r \quad \forall k > k_0. \quad (2.33)$$

Therefore, for each  $\lambda$ , we have

$$g^*(\lambda x) \leq |\lambda|^r \cdot g^*(x). \quad (2.34)$$

Now, for  $0 < \delta < 1$ ,

$$U = \{x : g^*(x) \leq \delta\}, \quad (2.35)$$

which is an absolutely  $r$ -convex set, for

$$|\lambda|^r + |\mu|^r \leq 1, \quad x, y \in U. \quad (2.36)$$

Now

$$\begin{aligned}
g^*(\lambda x + \mu y) &\leq g^*(\lambda x) + g^*(\mu y) \\
&\leq |\lambda|^r g^*(x) + |\mu|^r g^*(y) \\
&\leq |\lambda|^r \delta + |\mu|^r \delta \quad (\text{using (2.34) and (2.35)}) \\
&\leq (|\lambda|^r + |\mu|^r) \delta \\
&\leq 1 \cdot \delta \quad (\text{by using (2.36)}) \\
&\leq \delta.
\end{aligned} \tag{2.37}$$

If  $p_k = p \leq 1$  for all  $k$ , then for  $0 < r < 1$ ,  $U = \{x : g^*(x) \leq \delta\}$  is an absolutely  $p$ -convex set. This can be obtained by a similar analysis and therefore we omit the details. This completes the proof.  $\square$

**PROPOSITION 2.13.**  $(\Gamma_M)^\beta = \wedge$ .

**PROOF**

**STEP 1.**  $\Gamma \subset \Gamma_M$  by [Proposition 2.3](#); this implies that  $(\Gamma_M)^\beta \subset \Gamma^\beta = \wedge$ . Therefore,

$$(\Gamma_M)^\beta \subset \wedge. \tag{2.38}$$

**STEP 2.** Let  $y \in \wedge$ . Then  $|y_k| < M^k$  for all  $k$  and for some constant  $M > 0$ .

Let  $x \in \Gamma_M$ . Then  $M(|x_k|^{1/k}/\rho) \rightarrow 0$  as  $k \rightarrow \infty$ . Hence  $M(|x_k|^{1/k}/\rho) < \varepsilon$  for given  $\varepsilon > 0$  for sufficiently large  $k$ .

Take  $\varepsilon = 1/2M$  so that  $M(|x_k|/\rho) < 1/(2M)^k$ .

But then  $M(|x_k y_k|/\rho) \leq 1/2^k$  so that  $\sum_{k=1}^{\infty} M(|x_k y_k|/\rho)$  converges. Therefore  $\sum_{k=1}^{\infty} M(x_k y_k/\rho)$  converges. Hence  $\sum_{k=1}^{\infty} x_k y_k$  converges so that  $y \in (\Gamma_M)^\beta$ . Thus

$$\wedge \subset (\Gamma_M)^\beta. \tag{2.39}$$

**STEP 3.** From [\(2.38\)](#) and [\(2.39\)](#), we obtain

$$(\Gamma_M)^\beta = \wedge. \tag{2.40}$$

This completes the proof.  $\square$

**PROPOSITION 2.14.**  $(\Gamma_M)^\mu = \wedge$  for  $\mu = \alpha, \beta, \gamma, f$ .

**PROOF**

**STEP 1.**  $\Gamma_M$  has AK by [Proposition 2.4](#). Hence, by [Lemma 2.2\(i\)](#), we get  $(\Gamma_M)^\beta = (\Gamma_M)^f$ . But  $(\Gamma_M)^\beta = \wedge$ . Hence

$$(\Gamma_M)^f = \wedge. \tag{2.41}$$

**STEP 2.** Since AK implies AD, hence by [Lemma 2.2\(iii\)](#) we get  $(\Gamma_M)^\beta = (\Gamma_M)^\gamma$ . Therefore

$$(\Gamma_M)^\gamma = \wedge. \tag{2.42}$$

**STEP 3.**  $\Gamma_M$  is normal by [Proposition 2.5](#). Hence, by [2, Proposition 2.7], we get

$$(\Gamma_M)^\alpha = (\Gamma_M)^\gamma = \wedge. \quad (2.43)$$

From [\(2.41\)](#), [\(2.42\)](#), and [\(2.43\)](#), we have

$$(\Gamma_M)^\alpha = (\Gamma_M)^\beta = (\Gamma_M)^\gamma = (\Gamma_M)^f = \wedge. \quad (2.44)$$

□

**PROPOSITION 2.15.** *The dual space of  $\Gamma_M$  is  $\wedge$ . In other words,  $\Gamma_M^* = \wedge$ .*

**PROOF.** We recall that  $\delta^k$  has 1 in the  $k$ th place and zeros elsewhere, with

$$\begin{aligned} x = \delta^k, \quad \left\{ M\left(\frac{|x_k|^{1/k}}{\rho}\right)\right\} &= \left\{ \frac{M(0)^1}{\rho}, \frac{M(0)^{1/2}}{\rho}, \dots, \frac{M(1)^{1/k}}{\rho}, \frac{M(0)^{1/(k+1)}}{\rho}, \dots \right\} \\ &= \left\{ 0, 0, \dots, \frac{M(1)^{1/k}}{\rho}, 0, \dots \right\} \end{aligned} \quad (2.45)$$

which is a null sequence. Hence  $\delta^k \in \Gamma_M$ .  $f(x) = \sum_{k=1}^{\infty} x_k y_k$  with  $x \in \Gamma_M$  and  $f \in \Gamma_M^*$ , where  $\Gamma_M^*$  is the dual space of  $\Gamma_M$ . Take  $x = \delta^k \in \Gamma_M$ . Then

$$|y_k| \leq \|f\| d(\delta^k, 0) < \infty \quad \forall k. \quad (2.46)$$

Thus  $(y_k)$  is a bounded sequence and hence an analytic sequence. In other words,  $y \in \wedge$ . Therefore  $\Gamma_M^* = \wedge$ . This completes the proof. □

**LEMMA 2.16** [6, Theorem 8.6.1].  *$Y \supset X \Leftrightarrow Y^f \subset X^f$ , where  $X$  is an AD-space and  $Y$  an FK-space.*

**PROPOSITION 2.17.** *Let  $Y$  be any FK-space  $\supset \Phi$ . Then  $Y \supset \Gamma_M$  if and only if the sequence  $\delta^{(k)}$  is weakly analytic.*

**PROOF.** The following implications establish the result: since  $\Gamma_M$  has AD and by [Lemma 2.16](#),

$$\begin{aligned} Y \supset \Gamma_M &\Leftrightarrow Y^f \subset (\Gamma_M)^f \\ &\Leftrightarrow Y^f \subset \wedge \quad (\text{since } (\Gamma_M)^f = \wedge) \\ &\Leftrightarrow \text{for each } f \in Y', \text{ the topological dual of } Y \cdot f(\delta^{(k)}) \in \wedge \\ &\Leftrightarrow f(\delta^{(k)}) \text{ is analytic} \\ &\Leftrightarrow \delta^{(k)} \text{ is weakly analytic,} \end{aligned} \quad (2.47)$$

this completes the proof. □

## REFERENCES

- [1] H. I. Brown, *The summability field of a perfect  $l\text{-}l$  method of summation*, J. Anal. Math. **20** (1967), 281–287.
- [2] P. K. Kamthan and M. Gupta, *Sequence Spaces and Series*, Lecture Notes in Pure and Applied Mathematics, vol. 65, Marcel Dekker, New York, 1981.
- [3] J. Lindenstrauss and L. Tzafriri, *On Orlicz sequence spaces*, Israel J. Math. **10** (1971), 379–390.
- [4] I. J. Maddox, *Sequence spaces defined by a modulus*, Math. Proc. Cambridge Philos. Soc. **100** (1986), no. 1, 161–166.
- [5] W. H. Ruckle, *FK spaces in which the sequence of coordinate vectors is bounded*, Canad. J. Math. **25** (1973), 973–978.
- [6] A. Wilansky, *Summability through Functional Analysis*, North-Holland Mathematics Studies, vol. 85, North-Holland Publishing, Amsterdam, 1984.

K. Chandrasekhara Rao: Srinivasa Ramanujan Centre, Shanmugha Arts, Science, Technology, and Research Academy (ASTRA), Kumbakonam 612 001, India

*E-mail address:* [kcrao2008@yahoo.co.in](mailto:kcrao2008@yahoo.co.in)

N. Subramanian: Department of Mathematics, Shanmugha Arts, Science, Technology, and Research Academy (ASTRA), Thanjavur 613 402, India

*E-mail address:* [nsmaths@yahoo.com](mailto:nsmaths@yahoo.com)

## Special Issue on Modeling Experimental Nonlinear Dynamics and Chaotic Scenarios

### Call for Papers

Thinking about nonlinearity in engineering areas, up to the 70s, was focused on intentionally built nonlinear parts in order to improve the operational characteristics of a device or system. Keying, saturation, hysteretic phenomena, and dead zones were added to existing devices increasing their behavior diversity and precision. In this context, an intrinsic nonlinearity was treated just as a linear approximation, around equilibrium points.

Inspired on the rediscovering of the richness of nonlinear and chaotic phenomena, engineers started using analytical tools from "Qualitative Theory of Differential Equations," allowing more precise analysis and synthesis, in order to produce new vital products and services. Bifurcation theory, dynamical systems and chaos started to be part of the mandatory set of tools for design engineers.

This proposed special edition of the *Mathematical Problems in Engineering* aims to provide a picture of the importance of the bifurcation theory, relating it with nonlinear and chaotic dynamics for natural and engineered systems. Ideas of how this dynamics can be captured through precisely tailored real and numerical experiments and understanding by the combination of specific tools that associate dynamical system theory and geometric tools in a very clever, sophisticated, and at the same time simple and unique analytical environment are the subject of this issue, allowing new methods to design high-precision devices and equipment.

Authors should follow the Mathematical Problems in Engineering manuscript format described at <http://www.hindawi.com/journals/mpe/>. Prospective authors should submit an electronic copy of their complete manuscript through the journal Manuscript Tracking System at <http://mts.hindawi.com/> according to the following timetable:

Manuscript Due	December 1, 2008
First Round of Reviews	March 1, 2009
Publication Date	June 1, 2009

### Guest Editors

**José Roberto Castilho Piqueira**, Telecommunication and Control Engineering Department, Polytechnic School, The University of São Paulo, 05508-970 São Paulo, Brazil; [piqueira@lac.usp.br](mailto:piqueira@lac.usp.br)

**Elbert E. Neher Macau**, Laboratório Associado de Matemática Aplicada e Computação (LAC), Instituto Nacional de Pesquisas Espaciais (INPE), São José dos Campos, 12227-010 São Paulo, Brazil ; [elbert@lac.inpe.br](mailto:elbert@lac.inpe.br)

**Celso Grebogi**, Center for Applied Dynamics Research, King's College, University of Aberdeen, Aberdeen AB24 3UE, UK; [grebogi@abdn.ac.uk](mailto:grebogi@abdn.ac.uk)