

# ON THE GENUS OF FREE LOOP FIBRATIONS OVER $F_0$ -SPACES

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We give a lower bound of the genus of the fibration of free loops on an elliptic space whose rational cohomology is concentrated in even degrees.

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**1. Introduction.** In this note, all spaces are supposed to be connected and having the rational homotopy type of a CW complex of finite type. The LS category,  $\text{cat}(X)$ , of a space  $X$  is the least integer  $n$  such that  $X$  can be covered by  $n+1$  open subsets, each contractible in  $X$ . The genus,  $\text{genus}(\eta)$  or  $\text{genus}(p)$ , of a fibration  $\eta : F \rightarrow E \xrightarrow{p} B$  is the least integer  $n$  such that  $B$  can be covered by  $n+1$  open subsets, over each of which  $p$  is a trivial fibration, in the sense of fiber homotopy type. The sectional category,  $\text{secat}(\eta)$ , is the least integer  $n$  such that  $B$  can be covered by  $n+1$  open subsets, over each of which  $p$  has a section. Let

$$\mathcal{L}_X : \Omega X \longrightarrow LX \longrightarrow X \quad (1.1)$$

be the fibration of free loops on a 2-connected space  $X$  and let  $\mathcal{P}_X : \Omega X \rightarrow PX \rightarrow X$  be the path fibration. It is known that  $\mathcal{L}_X$  is an interesting object in topology and geometry [1, 9]. We know that  $\text{cat}(X) = \text{secat}(\mathcal{P}_X) = \text{genus}(\mathcal{P}_X)$  (see [4, page 599]). On the other hand, since  $\mathcal{L}_X$  has a section,  $\text{secat}(\mathcal{L}_X) = 0$ . But it seems hard to know  $\text{genus}(\mathcal{L}_X)$  in general. In this note, we consider a certain case for  $X$  by using the argument of the Sullivan minimal model in [4].

A simply connected space is said to be elliptic if the dimensions of rational cohomology and homotopy are finite. An elliptic space  $X$  is said to be an  $F_0$ -space if the rational cohomology is concentrated in even degrees. Then there is an isomorphism  $H^*(X; \mathbb{Q}) \cong \mathbb{Q}[x_1, \dots, x_n]/(f_1, \dots, f_n)$  with a regular sequence  $f_1, \dots, f_n$ . For example, the homogeneous space  $G/H$  where  $G$  and  $H$  have same rank is an  $F_0$ -space. Note that there is a conjecture of Halperin for an  $F_0$ -space (see [3, page 516], [7]).

**THEOREM 1.1.** *Let  $X$  be a 2-connected  $F_0$ -space of  $n$  variables. Then  $\text{genus}(\mathcal{L}_X) \geq n$ .*

In the following, Section 2 is a preliminary in Sullivan minimal models and we prove the theorem in Section 3. Refer to [3] for the rational model theory.

**2. Sullivan model of classifying map.** Let  $M(X) = (\Lambda V, d)$  be the Sullivan minimal model [3, Section 12] of a 2-connected space  $X$ , in which  $V = \bigoplus_{i \geq 2} V^i$  as a graded vector space. Let  $\overline{V}^i = V^{i+1}$  and let  $\beta : \Lambda \overline{V} \otimes \Lambda V \rightarrow \Lambda \overline{V} \otimes \Lambda V$  be the derivation  $(\beta(xy) = \beta(x)y + (-1)^{\deg x} x\beta(y))$  of degree  $-1$  with the properties  $\beta(v) = \overline{v}$  and  $\beta(\overline{v}) = 0$ .

Then  $M(\Omega X) = (\Lambda \bar{V}, 0)$  and  $M(LX) \cong (\Lambda \bar{V} \otimes \Lambda V, \delta)$  with  $\delta v = dv$  and  $\delta \bar{v} = -\beta dv = \sum_j \pm_j \partial dv / \partial v_j \cdot \bar{v}_j$  for a basis  $v_j$  of  $V$  [9].

Let  $Y$  be a simply connected space and let  $\text{Der}_i M(Y)$  be the set of derivations of  $M(Y)$  decreasing the degree by  $i > 0$ . We denote  $\bigoplus_{i>0} \text{Der}_i M(Y)$  by  $\text{Der} M(Y)$ . The Lie bracket is defined by  $[\sigma, \tau] = \sigma \circ \tau - (-1)^{\deg \sigma \deg \tau} \tau \circ \sigma$ . The boundary operator  $\partial : \text{Der}_* M(Y) \rightarrow \text{Der}_{*-1} M(Y)$  is defined by  $\partial(\sigma) = [d, \sigma]$ . Let  $B \text{aut } Y$  be the Dold-Lashof classifying space [2] for fibrations with fiber  $Y$  and  $\tilde{B} \text{aut } Y$  the universal covering. The differential graded Lie algebra  $L = (\text{Der} M(Y), \partial)$  is a model for  $\tilde{B} \text{aut } Y$  (see [8, page 313]).

Any fibration with fiber  $Y$  over a simply connected space  $B$  is the pullback of the universal fibration by a classifying map  $h : B \rightarrow \tilde{B} \text{aut } Y$ . Let  $Y \rightarrow E \rightarrow B$  be a fibration whose model [3, Section 15] is

$$M(B) = (\Lambda W, d) \longrightarrow (\Lambda W \otimes \Lambda V, D) \longrightarrow (\Lambda V, \bar{D}) = M(Y). \quad (2.1)$$

Take a basis  $a_i$  of  $(\Lambda W)^+$ , then there are derivations  $\theta_i$  of  $\Lambda V$  such that for each  $z \in V$ , we have  $D(z) = \bar{D}(z) + \sum_i a_i \theta_i(z)$ . A differential graded algebra model for  $\tilde{B} \text{aut } Y$  is given by the cochain algebra  $C^*(L)$  [3, 23(a)] on  $L$ , and a model for the classifying map of the fibration  $h$  is given by

$$h^* : C^*(L) = \text{Hom}(\text{Der}_{*-1} M(Y), \mathbb{Q}) \longrightarrow \Lambda W, \quad h^*(\psi) = \sum_i a_i \psi(\theta_i) \quad (2.2)$$

(see [6, Section 9]). Put the derivation which sends a generator  $p$  to an element  $q$  and other generators to zero as  $(p, q)$  and the dual element with the degree shifted by  $+1$  as  $s(p, q)^*$ .

**LEMMA 2.1.** *The fibration  $\mathcal{L}_X$  is the pullback of the universal fibration by a classifying map  $h : X \rightarrow \tilde{B} \text{aut } \Omega X$ , where the model is given by  $h^*(s(\bar{v}_i, \bar{v}_j)^*) = \pm_{i,j} \partial dv_i / \partial v_j$  for  $v_i, v_j \in V$  and  $h^*(\text{other}) = 0$ .*

**3. Proof.** The category,  $\text{cat}(f)$ , of a map  $f : X \rightarrow Y$  is the least integer  $n$  such that  $X$  can be covered by  $n+1$  open subsets  $U_i$ , for which the restriction of  $f$  to each  $U_i$  is null-homotopic. Note that  $\text{cat}(f) \leq \text{cat}(X)$ . Recall that if  $\eta : F \rightarrow E \rightarrow B$  is a simply connected fibration, then  $\text{genus}(\eta) = \text{cat}(h)$  for the classifying map of  $\eta$ ,  $h : B \rightarrow \tilde{B} \text{aut } F$  [5].

**PROOF OF THEOREM 1.1.** Let  $M(X) = (\Lambda(x_1, \dots, x_n, y_1, \dots, y_n), d)$  with  $\deg x_i$  even,  $\deg y_i$  odd,  $d(x_i) = 0$ , and  $d(y_i) = f_i \neq 0 \in \Lambda(x_1, \dots, x_n)$  for  $i = 1, \dots, n$ . Then  $M(\Omega X) = (\Lambda(\bar{x}_1, \dots, \bar{x}_n, \bar{y}_1, \dots, \bar{y}_n), 0)$  with  $\deg \bar{v} = \deg v - 1$  for any element  $v$ . The minimal model of the space  $LX$  of free loops on  $X$  is given by

$$M(LX) = (\Lambda(x_1, \dots, x_n, y_1, \dots, y_n, \bar{x}_1, \dots, \bar{x}_n, \bar{y}_1, \dots, \bar{y}_n), \delta), \quad (3.1)$$

where  $\delta x_i = \delta \bar{x}_i = 0$ ,  $\delta y_i = dy_i = f_i$ ,  $\delta \bar{y}_i = -\sum_{j=1}^n \partial f_i / \partial x_j \cdot \bar{x}_j$ . Then we see from Lemma 2.1 that

$$h^*(s(\bar{y}_i, \bar{x}_j)^*) = -\frac{\partial f_i}{\partial x_j} \quad \text{for } 1 \leq i, j \leq n, \quad h^*(\text{other}) = 0. \quad (3.2)$$

Let  $J$  be the determinant of the matrix whose  $(i, j)$ -component is  $s(\overline{y}_i, \overline{x}_j)^*$ . Then  $(-1)^n h^*(J)$  is the Jacobian  $|\partial f_i / \partial x_j|_{1 \leq i, j \leq n}$  of  $f_1, \dots, f_n$  and it is a cocycle which is not cohomologous to zero in  $M(X)$  [7, Theorem B]. Therefore, as in [4, page 598(2)],

$$\text{genus}(\mathcal{L}_X) = \text{cat}(h) \geq \text{nil}(\text{Im} \tilde{H}(h^*)) \geq n, \quad (3.3)$$

where  $\text{nil}R$  is the least integer  $n$  such that  $R^{n+1} = 0$  for a ring  $R$  and  $\tilde{H}(h^*)$  is the induced morphism in reduced cohomology.  $\square$

**COROLLARY 3.1.** *If  $X$  is an  $F_0$ -space of  $n$  variables with  $\text{cat}(X) = n$ , then  $\text{genus}(\mathcal{L}_X) = n$ .*

**EXAMPLE 3.2.** Let  $X = S^{2n} \vee S^{2n} \cup e^{4n} \not\cong_0 S^{2n} \vee S^{2n} \vee S^{4n}$ .  $X$  is an  $F_0$ -space where  $H^*(X; \mathbb{Q}) \cong \mathbb{Q}[x_1, x_2] / (x_1^2 + ax_2^2, x_1x_2)$  with some  $a \neq 0 \in \mathbb{Q}$  and  $\deg x_i = 2n$ . Then from Theorem 1.1 and [3, Lemma 27.3],  $2 \leq \text{genus}(\mathcal{L}_X) \leq \text{cat}(X) \leq \text{cat}(S^{2n} \vee S^{2n}) + 1 = 2$ , that is,  $\text{genus}(\mathcal{L}_X) = \text{cat}(X) = 2$ .

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## REFERENCES

- [1] M. Chas and D. Sullivan, *String topology*, preprint, 1999, <http://arxiv.org/abs/math.GT/9911159v1>.
- [2] A. Dold and R. Lashof, *Principal quasi-fibrations and fibre homotopy equivalence of bundles*, Illinois J. Math. **3** (1959), 285–305.
- [3] Y. Félix, S. Halperin, and J.-C. Thomas, *Rational Homotopy Theory*, Graduate Texts in Mathematics, vol. 205, Springer-Verlag, New York, 2001.
- [4] J.-B. Gatsinzi, *On the genus of elliptic fibrations*, Proc. Amer. Math. Soc. **132** (2004), no. 2, 597–606.
- [5] I. M. James, *On category, in the sense of Lusternik-Schnirelmann*, Topology **17** (1978), no. 4, 331–348.
- [6] M. Schlessinger and J. Stasheff, *Deformation theory and rational homotopy type*, to appear in Publ. Math. Inst. Hautes Études Sci.
- [7] H. Shiga and M. Tezuka, *Rational fibrations, homogeneous spaces with positive Euler characteristics and Jacobians*, Ann. Inst. Fourier (Grenoble) **37** (1987), no. 1, 81–106.
- [8] D. Sullivan, *Infinitesimal computations in topology*, Publ. Math. Inst. Hautes Études Sci. (1977), no. 47, 269–331.
- [9] M. Vigué-Poirrier and D. Sullivan, *The homology theory of the closed geodesic problem*, J. Differential Geom. **11** (1976), no. 4, 633–644.

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