

ON THE GENUS OF FREE LOOP FIBRATIONS OVER F_0 -SPACES

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We give a lower bound of the genus of the fibration of free loops on an elliptic space whose rational cohomology is concentrated in even degrees.

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1. Introduction. In this note, all spaces are supposed to be connected and having the rational homotopy type of a CW complex of finite type. The LS category, $\text{cat}(X)$, of a space X is the least integer n such that X can be covered by $n+1$ open subsets, each contractible in X . The genus, $\text{genus}(\eta)$ or $\text{genus}(p)$, of a fibration $\eta: F \rightarrow E \xrightarrow{p} B$ is the least integer n such that B can be covered by $n+1$ open subsets, over each of which p is a trivial fibration, in the sense of fiber homotopy type. The sectional category, $\text{secat}(\eta)$, is the least integer n such that B can be covered by $n+1$ open subsets, over each of which p has a section. Let

$$\mathcal{L}_X: \Omega X \rightarrow LX \rightarrow X \quad (1.1)$$

be the fibration of free loops on a 2-connected space X and let $\mathcal{P}_X: \Omega X \rightarrow PX \rightarrow X$ be the path fibration. It is known that \mathcal{L}_X is an interesting object in topology and geometry [1, 9]. We know that $\text{cat}(X) = \text{secat}(\mathcal{P}_X) = \text{genus}(\mathcal{P}_X)$ (see [4, page 599]). On the other hand, since \mathcal{L}_X has a section, $\text{secat}(\mathcal{L}_X) = 0$. But it seems hard to know $\text{genus}(\mathcal{L}_X)$ in general. In this note, we consider a certain case for X by using the argument of the Sullivan minimal model in [4].

A simply connected space is said to be elliptic if the dimensions of rational cohomology and homotopy are finite. An elliptic space X is said to be an F_0 -space if the rational cohomology is concentrated in even degrees. Then there is an isomorphism $H^*(X; \mathbb{Q}) \cong \mathbb{Q}[x_1, \dots, x_n]/(f_1, \dots, f_n)$ with a regular sequence f_1, \dots, f_n . For example, the homogeneous space G/H where G and H have same rank is an F_0 -space. Note that there is a conjecture of Halperin for an F_0 -space (see [3, page 516], [7]).

THEOREM 1.1. *Let X be a 2-connected F_0 -space of n variables. Then $\text{genus}(\mathcal{L}_X) \geq n$.*

In the following, Section 2 is a preliminary in Sullivan minimal models and we prove the theorem in Section 3. Refer to [3] for the rational model theory.

2. Sullivan model of classifying map. Let $M(X) = (\Lambda V, d)$ be the Sullivan minimal model [3, Section 12] of a 2-connected space X , in which $V = \bigoplus_{i>2} V^i$ as a graded vector space. Let $\bar{V}^i = V^{i+1}$ and let $\beta: \Lambda \bar{V} \otimes \Lambda V \rightarrow \Lambda \bar{V} \otimes \Lambda V$ be the derivation $(\beta(xy) = \beta(x)y + (-1)^{\deg x} x\beta(y))$ of degree -1 with the properties $\beta(v) = \bar{v}$ and $\beta(\bar{v}) = 0$.

Then $M(\Omega X) = (\Lambda \bar{V}, 0)$ and $M(LX) \cong (\Lambda \bar{V} \otimes \Lambda V, \delta)$ with $\delta v = dv$ and $\delta \bar{v} = -\beta dv = \sum_j \pm_j \partial dv / \partial v_j \cdot \bar{v}_j$ for a basis v_j of V [9].

Let Y be a simply connected space and let $\text{Der}_i M(Y)$ be the set of derivations of $M(Y)$ decreasing the degree by $i > 0$. We denote $\bigoplus_{i>0} \text{Der}_i M(Y)$ by $\text{Der} M(Y)$. The Lie bracket is defined by $[\sigma, \tau] = \sigma \circ \tau - (-1)^{\deg \sigma \deg \tau} \tau \circ \sigma$. The boundary operator $\partial : \text{Der}_* M(Y) \rightarrow \text{Der}_{*-1} M(Y)$ is defined by $\partial(\sigma) = [d, \sigma]$. Let $B \text{aut } Y$ be the Dold-Lashof classifying space [2] for fibrations with fiber Y and $\tilde{B} \text{aut } Y$ the universal covering. The differential graded Lie algebra $L = (\text{Der} M(Y), \partial)$ is a model for $\tilde{B} \text{aut } Y$ (see [8, page 313]).

Any fibration with fiber Y over a simply connected space B is the pullback of the universal fibration by a classifying map $h : B \rightarrow \tilde{B} \text{aut } Y$. Let $Y \rightarrow E \rightarrow B$ be a fibration whose model [3, Section 15] is

$$M(B) = (\Lambda W, d) \longrightarrow (\Lambda W \otimes \Lambda V, D) \longrightarrow (\Lambda V, \bar{D}) = M(Y). \quad (2.1)$$

Take a basis a_i of $(\Lambda W)^+$, then there are derivations θ_i of ΛV such that for each $z \in V$, we have $D(z) = \bar{D}(z) + \sum_i a_i \theta_i(z)$. A differential graded algebra model for $\tilde{B} \text{aut } Y$ is given by the cochain algebra $C^*(L)$ [3, 23(a)] on L , and a model for the classifying map of the fibration h is given by

$$h^* : C^*(L) = \text{Hom}(\text{Der}_{*-1} M(Y), \mathbb{Q}) \longrightarrow \Lambda W, \quad h^*(\psi) = \sum_i a_i \psi(\theta_i) \quad (2.2)$$

(see [6, Section 9]). Put the derivation which sends a generator p to an element q and other generators to zero as (p, q) and the dual element with the degree shifted by $+1$ as $s(p, q)^*$.

LEMMA 2.1. *The fibration \mathcal{L}_X is the pullback of the universal fibration by a classifying map $h : X \rightarrow \tilde{B} \text{aut } \Omega X$, where the model is given by $h^*(s(\bar{v}_i, \bar{v}_j)^*) = \pm_{i,j} \partial dv_i / \partial v_j$ for $v_i, v_j \in V$ and $h^*(\text{other}) = 0$.*

3. Proof. The category, $\text{cat}(f)$, of a map $f : X \rightarrow Y$ is the least integer n such that X can be covered by $n+1$ open subsets U_i , for which the restriction of f to each U_i is null-homotopic. Note that $\text{cat}(f) \leq \text{cat}(X)$. Recall that if $\eta : F \rightarrow E \rightarrow B$ is a simply connected fibration, then $\text{genus}(\eta) = \text{cat}(h)$ for the classifying map of η , $h : B \rightarrow \tilde{B} \text{aut } F$ [5].

PROOF OF THEOREM 1.1. Let $M(X) = (\Lambda(x_1, \dots, x_n, y_1, \dots, y_n), d)$ with $\deg x_i$ even, $\deg y_i$ odd, $d(x_i) = 0$, and $d(y_i) = f_i \neq 0 \in \Lambda(x_1, \dots, x_n)$ for $i = 1, \dots, n$. Then $M(\Omega X) = (\Lambda(\bar{x}_1, \dots, \bar{x}_n, \bar{y}_1, \dots, \bar{y}_n), 0)$ with $\deg \bar{v} = \deg v - 1$ for any element v . The minimal model of the space LX of free loops on X is given by

$$M(LX) = (\Lambda(x_1, \dots, x_n, y_1, \dots, y_n, \bar{x}_1, \dots, \bar{x}_n, \bar{y}_1, \dots, \bar{y}_n), \delta), \quad (3.1)$$

where $\delta x_i = \delta \bar{x}_i = 0$, $\delta y_i = dy_i = f_i$, $\delta \bar{y}_i = -\sum_{j=1}^n \partial f_i / \partial x_j \cdot \bar{x}_j$. Then we see from Lemma 2.1 that

$$h^*(s(\bar{y}_i, \bar{x}_j)^*) = -\frac{\partial f_i}{\partial x_j} \quad \text{for } 1 \leq i, j \leq n, \quad h^*(\text{other}) = 0. \quad (3.2)$$

Let J be the determinant of the matrix whose (i,j) -component is $s(\bar{y}_i, \bar{x}_j)^*$. Then $(-1)^n h^*(J)$ is the Jacobian $|(\partial f_i / \partial x_j)_{1 \leq i, j \leq n}|$ of f_1, \dots, f_n and it is a cocycle which is not cohomologous to zero in $M(X)$ [7, Theorem B]. Therefore, as in [4, page 598(2)],

$$\text{genus}(\mathcal{L}_X) = \text{cat}(h) \geq \text{nil}(\text{Im} \tilde{H}(h^*)) \geq n, \quad (3.3)$$

where $\text{nil}R$ is the least integer n such that $R^{n+1} = 0$ for a ring R and $\tilde{H}(h^*)$ is the induced morphism in reduced cohomology. \square

COROLLARY 3.1. *If X is an F_0 -space of n variables with $\text{cat}(X) = n$, then $\text{genus}(\mathcal{L}_X) = n$.*

EXAMPLE 3.2. Let $X = S^{2n} \vee S^{2n} \cup e^{4n} \neq_0 S^{2n} \vee S^{2n} \vee S^{4n}$. X is an F_0 -space where $H^*(X; \mathbb{Q}) \cong \mathbb{Q}[x_1, x_2]/(x_1^2 + ax_2^2, x_1x_2)$ with some $a \neq 0 \in \mathbb{Q}$ and $\deg x_i = 2n$. Then from Theorem 1.1 and [3, Lemma 27.3], $2 \leq \text{genus}(\mathcal{L}_X) \leq \text{cat}(X) \leq \text{cat}(S^{2n} \vee S^{2n}) + 1 = 2$, that is, $\text{genus}(\mathcal{L}_X) = \text{cat}(X) = 2$.

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