IJMMS 2004:61, 3285–3290
PII. S0161171204311063
http://ijmms.hindawi.com

A FORMULA FOR THE INNER SPECTRAL RADIUS

S. MAHMOUD MANJEGANI

Received 3 November 2003

This note presents an asymptotic formula for the minimum of the moduli of the elements in the spectrum of a bounded linear operator acting on Banach space X. This minimum moduli is called the inner spectral radius, and the formula established herein is an analogue of Gelfand's spectral radius formula.

2000 Mathematics Subject Classification: 47A10.

1. Introduction. Let X be a Banach space and let $B(X)$ denote the Banach algebra of all bounded linear operators on X, and let T be an element of $B(X)$. We denote by $m(T)$ the "minimum moduli" of T and define it by

$$m(T) = \inf\{\|Tx\|,\ x \in X,\ \|x\| = 1\}. \tag{1.1}$$

In what follows, $i(T)$ and $i_{ap}(T)$ will denote, respectively, inner spectral radius and inner approximate spectral radius of T. We define $i(T)$ and $i_{ap}(T)$ by

$$\begin{aligned} i(T) &= \inf\{|\lambda| : \lambda \in \sigma(T)\}, \\ i_{ap}(T) &= \inf\{|\lambda| : \lambda \in \sigma_{ap}(T)\}. \end{aligned} \tag{1.2}$$

Makai and Zemanek [3] proved that

$$i_{ap}(T) = \lim_{n\to\infty} [m(T^n)]^{1/n}. \tag{1.3}$$

In this note, we prove the same formula for $i(T)$. The main results established herein are the following theorem and corollary.

Theorem 1.1. *Let $T \in B(X)$. Then, $r_i(T) = i(T)$ if and only if $r_i(T) \le r_i(T^*)$.*

Corollary 1.2. *It is not necessary that $r_i(T) = r_i(T^*)$ for any $T \in B(X)$.*

2. Basic concepts. Throughout, X will denote a Banach space, $B(X)$ is the Banach algebra of all bounded linear operators on X. T will denote an element of $B(X)$. We denote T^* as the transpose of T (T^* is an element of B(X^*), where X^* is dual space of X) and define

$$(T^*g)(x) = g(T(x)), \quad x \in X,\ g \in X^*. \tag{2.1}$$

If X is a Hilbert space, then T^* is the adjoint of T and $T^* \in B(X)$. We denote by $\sigma(T)$, $\sigma_{ap}(T)$, $\sigma_p(T)$, and $\sigma_c(T)$, respectively, spectrum, approximate point spectrum, point spectrum, and compression spectrum of T and define

$$\begin{aligned} \sigma(T) &= \{\lambda : (T-\lambda I) \text{ is not invertible, } \lambda \in \mathbb{C}\}, \\ \sigma_{ap}(T) &= \{\lambda : (T-\lambda I) \text{ is not bounded below, } \lambda \in \mathbb{C}\}, \\ \sigma_p(T) &= \{\lambda : \ker(T-\lambda I) \neq 0,\ \lambda \in \mathbb{C}\}, \\ \sigma_c(T) &= \{\lambda : \operatorname{ran}(T-\lambda I) \text{ is not dense in } X,\ \lambda \in \mathbb{C}\}. \end{aligned} \tag{2.2}$$

The spectral radius of T is denoted by $r(T)$ and defined by

$$r(T) = \sup\{|\lambda| : \lambda \in \sigma(T)\}. \tag{2.3}$$

We recall the following statements. One can see their proof in [1].

(1) $\|T\| = \|T^*\|$.
(2) $r(T) = \lim_{n\to\infty} \|T^n\|^{1/n}$ (Gelfand's formula).
(3) $r(T) = r(T^*)$.
(4) $\sigma(T) = \sigma(T^*)$, if X is a Hilbert space, then $\sigma(T^*) = \overline{\sigma(T)}$, where $\overline{\sigma(T)} = \{\overline{\lambda}, \lambda \in \sigma(T)\}$.

For operator $T \in B(X)$, define

$$m(T) = \inf\{\|Tx\|,\ x \in X,\ \|x\| = 1\}. \tag{2.4}$$

$m(T)$ is called the minimum moduli of T. Note that by definition of $m(T)$, we have $\|Tx\| \geq m(T)\|x\|$. It is clear that; if T is an invertible element in $B(X)$, then $m(T) = \|T^{-1}\|^{-1}$.

Definition 2.1. The inner spectral radius and inner approximate spectral radius of T are denoted, respectively, by $i(T)$ and $i_{ap}(T)$ and defined by (1.2).

Proposition 2.2. *If $|\lambda| < m(T)$, then $(T-\lambda I)$ is bounded below.*

Proof. We have

$$\|(T-\lambda I)x\| \geq \|Tx\| - \|\lambda x\| \geq (m(T) - |\lambda|)\|x\|. \tag{2.5}$$

The assumption implies that $m(T) - |\lambda| > 0$ and hence $(T-\lambda I)$ is bounded below by the definition. □

Proposition 2.3. *For every operator $T \in B(X)$,*

$$\lim_{n\to\infty} [m(T^n)]^{1/n} = \sup [m(T^n)]^{1/n}. \tag{2.6}$$

Proof. For every operator T and S in $B(X)$, we have

$$m(TS) \geq m(T)m(S), \tag{2.7}$$

by definition of the minimum moduli. Therefore, for every positive integers i and j,

$$m(T^{i+j}) \geq m(T^i)m(T^j). \tag{2.8}$$

This is the crucial inequality. Let k be fixed. For every integer number n, we have $n = kq + r$, $0 \leq r < k$, where $q = q(n)$ and $r = r(n)$ are functions of n. Note that $\lim_{n\to\infty} q(n)/n = 1/k$. Thus, by (2.8) we have

$$m(T^n) \geq m(T^k)^q m(T)^r, \quad \text{for each positive integer } n. \tag{2.9}$$

Hence,

$$\liminf_{n\to\infty} [m(T^n)]^n \geq m(T^k)^{1/k}. \tag{2.10}$$

Since this equation holds for all k, we have

$$\liminf_{n\to\infty} [m(T^n)]^n \geq \sup [m(T^n)]^{1/n} \geq \limsup_{n\to\infty} [m(T^n)]^n \tag{2.11}$$

and the result follows. □

Assume that $r_i(T) = \lim_{n\to\infty}[m(T^n)]^{1/n}$. By Gelfand's formula, it is clear that if T is invertible, then $r_i(T) = [r(T^{-1})]^{-1}$.

Corollary 2.4. *Let $T \in B(X)$. Then, $0 \in \sigma_{ap}(T)$ if and only if $m(T) = 0$.*

Proof. The result follows from the facts that $0 \in \sigma_{ap}(T)$ if and only if T is not bounded below and $\|Tx\| \geq m(T)\|x\|$ for each $x \in X$. □

Proposition 2.5. *Let $T \in B(X)$. If $\lambda \in \sigma_{ap}(T)$, then $|\lambda| \geq r_i(T)$.*

Proof. Suppose $\lambda \in \sigma_{ap}(T)$. Assume, contrary to what we wish to prove, that $|\lambda| < r_i(T)$. Thus, $|\lambda|^n < m(T^n)$ for some integer n by the definition of $r_i(T)$. By Proposition 2.2, $(T^n - \lambda^n I)$ is bounded below. We have

$$T^n - \lambda^n = (T^{n-1} + T^{n-2}\lambda + \cdots + \lambda^n)(T - \lambda). \tag{2.12}$$

Hence, $(T - \lambda)$ is bounded below and so $\lambda \notin \sigma_{ap}(T)$, which is contradictory to our assumption. □

Corollary 2.6. *For each $T \in B(X)$,*

$$\sigma_{ap}(T) \subseteq \{\lambda : r_i(T) \le |\lambda| \le r(T)\}. \tag{2.13}$$

Makai and Zemanek in [3] proved that $i_{ap}(T) = r_i(T)$ for every $T \in B(X)$. In the next section, we will prove that $i(T) = r_i(T)$ if and only if $r_i(T) \le r_i(T^*)$.

3. Inner spectral radius.

The purpose of this section is to prove the main result.

We know that $\partial\sigma_{ap}(T) \subseteq \sigma(T)$ and $r_i(T) = i_{ap}(T)$ and so $r_i(T) \in \sigma(T)$. Therefore, for every $T \in B(X)$, we have

$$i(T) \le r_i(T). \tag{3.1}$$

Fact 3.1. If X is a finite-dimensional space, then $\sigma_{ap}(T) = \sigma(T)$ for each $T \in B(X)$ and hence $r_i(T) = i(T)$.

Fact 3.2. If T is a compact operator acting on Banach space X, then $r_i(T) = i(T)$.

We begin with some general lemmas that we need in the proof of the main theorem.

Lemma 3.3. *Let $T \in B(X)$. Then, $\sigma_c(T) = \sigma_p(T^*)$. (If X is a Hilbert space, then $\sigma_c(T) = \overline{\sigma_p(T^*)}$).*

Proof. First, we show that $\sigma_c(T) \subseteq \sigma_p(T^*)$. Suppose λ is an element in $\sigma_c(T)$. Consider M the closure of $\operatorname{ran}(T-\lambda I)$. By definition of $\sigma_c(T)$, $M \ne X$. If x_0 is a nonzero element in $X - M$, then by the Hahn-Banach theorem there is $f_0 \in X^*$ such that $f_0(M) = 0$ and $f_0(x_0) = 1$. We have $((T^* - \lambda I)f_0)(x) = f_0((T-\lambda I)x) = 0$ for every $x \in X$ and hence $f_0 \in \ker(T^* - \lambda I)$, that is $\lambda \in \sigma_p(T^*)$.

Now, we prove $\sigma_p(T^*) \subseteq \sigma_c(T)$. Suppose $\lambda \in \sigma_p(T^*)$, thus, there is a nonzero functional g in X^* such that $(T^* - \lambda I)g = 0$ and so, $g((T-\lambda I)x) = 0$ for each $x \in X$ by (2.1). Hence, $g(t) = 0$ for any t in closure $\operatorname{ran}(T-\lambda I)$.

But $g \ne 0$ on X, and hence there is $x_0 \in X - M$ such that $g(x_0) \ne 0$. Therefore, $M \ne X$, that is, $\lambda \in \sigma_c(T)$.

If X is a Hilbert space, then we know that $\ker(T) = (\operatorname{ran} T^*)^{\perp}$ and $\operatorname{closure}(\operatorname{ran} T^*) = (\ker T)^{\perp}$ in [1, Theorem II.2.19]. Thus, by the definition of $\sigma_p(T)$ and $\sigma_c(T)$, we get the following result. □

Lemma 3.4. *Let $T \in B(X)$. Then, $\sigma(T) = \sigma_{ap}(T) \bigcup \sigma_c(T)$.*

Proof. It follows from [1, Proposition VII.6.4] and the definition of $\sigma_{ap}(T)$ and $\sigma_c(T)$. □

Lemma 3.5. *Let $T \in B(X)$. If $\sigma(T) \subseteq \{\lambda : r_i(T) \le |\lambda| \le r(T)\}$, then $r_i(T) = i(T)$.*

Proof. By assumption, we have $r_i(T) \le i(T)$ and the result follows the fact that $i(T) \le r_i(T)$. □

Theorem 3.6. *Let $T \in B(X)$. Then, $r_i(T) = i(T)$ if and only if $r_i(T) \le r_i(T^*)$.*

Proof. First, suppose that $r_i(T) \le r_i(T^*)$. By Lemmas 3.3 and 3.4, $\sigma(T) = \sigma_{ap}(T) \bigcup \sigma_p(T^*)$ (if X is a Hilbert space, then $\sigma(T) = \sigma_{ap}(T) \bigcup \overline{\sigma_p(T^*)}$). We have

$$\sigma(T) \subseteq \{\lambda : r_i(T) \le |\lambda| \le r(T)\}. \tag{3.2}$$

Hence, by Lemma 3.5, $r_i(T) = i(T)$.

Conversely, suppose that $r_i(T) = i(T)$. We have $\sigma(T) = \sigma(T^*)$ (if X is a Hilbert space, then $\sigma(T^*) = \overline{\sigma(T)}$). Thus, $i(T) = i(T^*)$ by definition of $i(T)$, and, therefore,

$$r_i(T) = i(T) = i(T^*) \le r_i(T^*). \tag{3.3}$$

□

Example 3.7. Let X be a Hilbert space and $N \in B(X)$ a normal operator. Then,

$$i(N) = i(N^*) = r_i(N) = r_i(N^*). \tag{3.4}$$

Since N is normal, $\|Nx\| = \|N^*x\|$ for every x in X, and, therefore, $m(N) = m(N^*)$. Similarly, we have $m(N^n) = m(N^{*n})$ for each n, and so, $i(N) = i(N^*) = r_i(N) = r_i(N^*)$.

If X is a Hilbert space and $N \in B(X)$ is a normal operator, then $r(N) = \|N\|$. In the next proposition, we prove that $r_i(N) = m(N)$ for the normal operator N in $B(X)$.

Recall that for each operator $T \in B(X)$ the numerical range of T is defined and denoted as follows:

$$W(T) = \{\lambda \in \mathbb{C} : \lambda = \langle Tx, x\rangle,\ x \in X \text{ with } \|x\| = 1\}. \tag{3.5}$$

The following interesting theorem was proved in [2, Theorem 27.9].

Theorem 3.8. *If T is a selfadjoint operator in $B(X)$, M_1 and M_2 denote, respectively, the infimum and the supremum of the numerical range of T, then M_1 and M_2 are approximate eigenvalues of T, and the spectrum of T is contained in the interval $[M_1, M_2]$.*

By this theorem, for each positive operator $T \in B(X)$ we have

$$i(T) = r_i(T) = \inf\{\langle Tx, x\rangle,\ x \in X \text{ with } \|x\| = 1\}. \tag{3.6}$$

Proposition 3.9. *If N is normal operator acting on Hilbert space X, then $i(N) = r_i(N) = m(N)$.*

Proof. As shown in Example 3.7, we have $m(N) = m(N^*)$. Now, we prove that $m(NN^*) = m(N)^2$. Since NN^* is positive, by (3.6) and Proposition 2.3, we have

$$\begin{aligned} m(NN^*) \le r_i(NN^*) &= \inf\{\langle NN^*x, x\rangle,\ x \in X \text{ with } \|x\| = 1\} \\ &= \inf\{\|Nx\|^2,\ x \in X \text{ with } \|x\| = 1\} = m(N)^2. \end{aligned} \tag{3.7}$$

By (2.8), we get

$$m(NN^*) \ge m(N)m(N^*) = m(N)^2. \tag{3.8}$$

Hence,

$$m(NN^*) = m(N)^2. \tag{3.9}$$

By induction, we show that if $j = 2^n$, $n = 0,1,2,\ldots$, then $m(N^j) = m(N)^j$. This is clearly true for $n = 0$. Assume it to be true for some n, then for all $x \in \mathfrak{H}$, we have

$$\|N^{2^{n+1}}(x)\| = \|N^{2^n}(N^{2^n}(x))\| = \|(N^{2^n})^*(N^{2^n}(x))\|, \tag{3.10}$$

because N^{2^n} is normal. This shows that $m(N^{2^{n+1}}) = m((N^{2^n})^* N^{2^n})$, which is equal to $m(N^{2^n})^2$. Thus, $m(N^{2^{n+1}}) = (m(N)^{2^n})^2 = m(N)^{2^{n+1}}$. Therefore,

$$r_i(N) = \lim_{n\to\infty}[m(N^n)]^{1/n} = \lim_{n\to\infty}[m(N^{2^n})]^{1/2^n} = m(N). \tag{3.11}$$

□

Example 3.10. Suppose U is a unilateral weighted shift with weights $(1,2,1,\ldots)$ acting on separable Hilbert space $\mathfrak{H}$. William Ridge [4] proved that $\sigma_{ap}(U) = \{\lambda : |\lambda| = \sqrt{2}\}$, $\sigma(U) = \{\lambda : |\lambda| \leq \sqrt{2}\}$, and $\sigma_{ap}(U^*) = \sigma(U^*) = \sigma(U)$. Hence, $r_i(U) = r(U) = \sqrt{2}$, $r_i(U^*) = i(U^*) = 0$, and $i(U) = 0$. Therefore, we have $i(U) \neq r_i(U)$ and $r_i(U^*) < r_i(U)$.

We know that $r(T) = r(T^*)$ for any $T \in B(X)$. But in the above example $r_i(U^*) < r_i(U)$ so, we can write the next corollary.

Corollary 3.11. *It is not necessary that $r_i(T) = r_i(T^*)$ for any $T \in B(X)$.*

Acknowledgments. The author wishes to thank Professors Douglas R. Farenick and Juliana Erlijman for their advice and helpful comments regarding the results herein. This work is supported by the Isfahan University of Technology, Iran, and the University of Regina, Canada.

References

[1] J. B. Conway, *A Course in Functional Analysis*, 2nd ed., Graduate Texts in Mathematics, vol. 96, Springer-Verlag, New York, 1990.
[2] B. V. Limaye, *Functional Analysis*, Wiley Eastern, New Delhi, 1981.
[3] E. Makai, Jr. and J. Zemánek, *The surjectivity radius, packing numbers and boundedness below of linear operators*, Integral Equations Operator Theory **6** (1983), no. 3, 372–384.
[4] W. C. Ridge, *Approximate point spectrum of a weighted shift*, Trans. Amer. Math. Soc. **147** (1970), 349–356.

S. Mahmoud Manjegani: School of Mathematical Science, Isfahan University of Technology, Isfahan 84156, Iran
E-mail address: manjgani@cc.iut.ac.ir
Current address: Department of Mathematics and Statistics, University of Regina, Regina, Saskatchewan, Canada S4S 0A2
E-mail address: manjegan@math.uregina.ca

Journal of Applied Mathematics and Decision Sciences

Special Issue on Intelligent Computational Methods for Financial Engineering

Call for Papers

As a multidisciplinary field, financial engineering is becoming increasingly important in today's economic and financial world, especially in areas such as portfolio management, asset valuation and prediction, fraud detection, and credit risk management. For example, in a credit risk context, the recently approved Basel II guidelines advise financial institutions to build comprehensible credit risk models in order to optimize their capital allocation policy. Computational methods are being intensively studied and applied to improve the quality of the financial decisions that need to be made. Until now, computational methods and models are central to the analysis of economic and financial decisions.

However, more and more researchers have found that the financial environment is not ruled by mathematical distributions or statistical models. In such situations, some attempts have also been made to develop financial engineering models using intelligent computing approaches. For example, an artificial neural network (ANN) is a nonparametric estimation technique which does not make any distributional assumptions regarding the underlying asset. Instead, ANN approach develops a model using sets of unknown parameters and lets the optimization routine seek the best fitting parameters to obtain the desired results. The main aim of this special issue is not to merely illustrate the superior performance of a new intelligent computational method, but also to demonstrate how it can be used effectively in a financial engineering environment to improve and facilitate financial decision making. In this sense, the submissions should especially address how the results of estimated computational models (e.g., ANN, support vector machines, evolutionary algorithm, and fuzzy models) can be used to develop intelligent, easy-to-use, and/or comprehensible computational systems (e.g., decision support systems, agent-based system, and web-based systems)

This special issue will include (but not be limited to) the following topics:

- **Computational methods**: artificial intelligence, neural networks, evolutionary algorithms, fuzzy inference, hybrid learning, ensemble learning, cooperative learning, multiagent learning
- **Application fields**: asset valuation and prediction, asset allocation and portfolio selection, bankruptcy prediction, fraud detection, credit risk management
- **Implementation aspects**: decision support systems, expert systems, information systems, intelligent agents, web service, monitoring, deployment, implementation

Authors should follow the Journal of Applied Mathematics and Decision Sciences manuscript format described at the journal site http://www.hindawi.com/journals/jamds/. Prospective authors should submit an electronic copy of their complete manuscript through the journal Manuscript Tracking System at http://mts.hindawi.com/, according to the following timetable:

Manuscript Due	December 1, 2008
First Round of Reviews	March 1, 2009
Publication Date	June 1, 2009

Guest Editors

Lean Yu, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, China; Department of Management Sciences, City University of Hong Kong, Tat Chee Avenue, Kowloon, Hong Kong; yulean@amss.ac.cn

Shouyang Wang, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, China; sywang@amss.ac.cn

K. K. Lai, Department of Management Sciences, City University of Hong Kong, Tat Chee Avenue, Kowloon, Hong Kong; mskklai@cityu.edu.hk

Hindawi Publishing Corporation
http://www.hindawi.com