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# ON THE CARLEMAN CLASSES OF VECTORS OF A SCALAR TYPE SPECTRAL OPERATOR

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The Carleman classes of a scalar type spectral operator in a reflexive Banach space are characterized in terms of the operator's resolution of the identity. A theorem of the Paley-Wiener type is considered as an application.

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**1. Introduction.** As was shown in [8] (see also [9, 10]), under certain conditions, the *Carleman classes* of vectors of a *normal operator* in a complex Hilbert space can be characterized in terms of the operator's *spectral measure* (the *resolution of the identity*).

The purpose of the present paper is to generalize this characterization to the case of a *scalar type spectral operator* in a complex *reflexive* Banach space.

## 2. Preliminaries

**2.1. The Carleman classes of vectors.** Let $A$ be a linear operator in a Banach space $X$ with norm $\|\cdot\|$, $\{m_n\}_{n=0}^{\infty}$ a sequence of positive numbers, and

$$C^{\infty}(A) \stackrel{\text{def}}{=} \bigcap_{n=0}^{\infty} D(A^n) \tag{2.1}$$

($D(\cdot)$ is the *domain* of an operator).

The sets

$$\begin{aligned} C_{\{m_n\}}(A) &\stackrel{\text{def}}{=} \{f \in C^{\infty}(A) \mid \exists \alpha > 0,\ \exists c > 0 : \|A^n f\| \le c\alpha^n m_n,\ n = 0,1,2,\dots\}, \\ C_{(m_n)}(A) &\stackrel{\text{def}}{=} \{f \in C^{\infty}(A) \mid \forall \alpha > 0\ \exists c > 0 : \|A^n f\| \le c\alpha^n m_n,\ n = 0,1,2,\dots\} \end{aligned} \tag{2.2}$$

are called the *Carleman classes* of vectors of the operator $A$ corresponding to the sequence $\{m_n\}_{n=0}^{\infty}$ of *Roumie's* and *Beurling's types*, respectively.

Obviously, the inclusion

$$C_{(m_n)}(A) \subseteq C_{\{m_n\}}(A) \tag{2.3}$$

holds.

For $m_n := [n!]^{\beta}$ (or, due to *Stirling's formula*, for $m_n := n^{\beta n}$), $n = 0,1,2,\dots$ $(0 \le \beta < \infty)$, we obtain the well-known $\beta$th-order *Gevrey classes* of vectors, $\mathscr{E}^{\{\beta\}}(A)$ and $\mathscr{E}^{(\beta)}(A)$, respectively. In particular, $\mathscr{E}^{\{1\}}(A)$ are the *analytic* and $\mathscr{E}^{(1)}(A)$ are the *entire* vectors of the operator $A$ [7, 17].

The sequence $\{m_n\}_{n=0}^{\infty}$ will be subject to the following condition.

(WGR) For any $\alpha > 0$, there exist such a $C = C(\alpha) > 0$ that

$$C\alpha^n \le m_n, \quad n = 0, 1, 2, \dots. \tag{2.4}$$

Note that the name WGR originates from the words *"weak growth."*

Under this condition, the numerical function

$$T(\lambda) := m_0 \sum_{n=0}^{\infty} \frac{\lambda^n}{m_n}, \quad 0 \le \lambda < \infty, \ (0^0 := 1), \tag{2.5}$$

first introduced by Mandelbrojt [15], is well defined.

This function is *nonnegative*, *continuous*, and *increasing*.

As established in [8] (see also [9, 10]), for a *normal operator* $A$ with a *spectral measure* $E_A(\cdot)$ in a complex Hilbert space $H$ with inner product $(\cdot, \cdot)$ and the sequence $\{m_n\}_{n=0}^{\infty}$ satisfying the condition (WGR),

$$\begin{aligned} C_{\{m_n\}}(A) &= \bigcup_{t>0} D(T(t|A|)), \\ C_{(m_n)}(A) &= \bigcap_{t>0} D(T(t|A|)), \end{aligned} \tag{2.6}$$

the normal operators $T(t|A|)$ $(0 < t < \infty)$ being defined in the sense of the spectral *operational calculus* for a normal operator:

$$\begin{aligned} T(t|A|) &:= \int_{\sigma(A)} T(t|\lambda|)\, dE_A, \\ D(T(t|A|)) &:= \left\{ f \in H \mid \int_{\sigma(A)} T^2(t|\lambda|)(dE_A(\lambda)f, f) < \infty \right\}, \end{aligned} \tag{2.7}$$

where the function $T(\cdot)$ can be replaced by any *nonnegative*, *continuous*, and *increasing* function $L(\cdot)$ defined on $[0, \infty)$ such that

$$c_1 L(\gamma_1 \lambda) \le T(\lambda) \le c_2 L(\gamma_2 \lambda), \quad \lambda > R, \tag{2.8}$$

with some positive $\gamma_1$, $\gamma_2$, $c_1$, $c_2$, and a nonnegative $R$.

In particular, $T(\cdot)$ in (2.6) is replaceable by

$$S(\lambda) := m_0 \sup_{n \ge 0} \frac{\lambda^n}{m_n}, \quad 0 \le \lambda < \infty, \tag{2.9}$$

or

$$P(\lambda) := m_0 \left[ \sum_{n=0}^{\infty} \frac{\lambda^{2n}}{m_n^2} \right]^{1/2}, \quad 0 \le \lambda < \infty, \tag{2.10}$$

(see [10]).

**2.2. Carleman ultradifferentiability.** Let $I$ be an interval of the real axis, $\mathbb{C}^\infty(I)$ the set of all complex-valued functions strongly infinite differentiable on $I$, and $\{m_n\}_{n=0}^\infty$ a sequence of positive numbers.

$$C_{\{m_n\}}(I) \stackrel{\text{def}}{=} \begin{cases} \Big\{ f(\cdot) \in C^\infty(I) \mid \forall [a,b] \subseteq I, \ \exists \alpha > 0, \ \exists c > 0 : \\ \max_{a \le x \le b} \| f^{(n)}(x) \| \le c \alpha^n m_n, \ n = 0,1,2,\ldots \Big\}, \\ \Big\{ f(\cdot) \in C^\infty(I) \mid \forall [a,b] \subseteq I, \ \forall \alpha > 0, \ \exists c > 0 : \\ \max_{a \le x \le b} \| f^{(n)}(x) \| \le c \alpha^n m_n, \ n = 0,1,2,\ldots \Big\} \end{cases} \tag{2.11}$$

are the *Carleman classes of ultradifferentiable functions* of *Roumie's* and *Beurling's types*, respectively, [1, 12, 13, 14].

In particular, for $m_n := [n!]^\beta$ (or, due to *Stirling's formula*, for $m_n := n^{\beta n}$), $n = 0,1,2,\ldots$ $(0 \le \beta < \infty)$, these are the well-known $\beta$th-order *Gevrey classes*, $\mathscr{E}^{\{\beta\}}(I)$ and $\mathscr{E}^{(\beta)}(I)$, respectively, [6, 12, 13, 14].

Observe that $\mathscr{E}^{\{1\}}(I)$ is the class of the *real analytic* on $I$ functions and $\mathscr{E}^{(1)}(I)$ is the class of *entire* functions, that is, the restrictions to $I$ of *analytic* and *entire* functions, correspondingly, [15].

Note that condition (WGR), in particular, implies that $\lim_{n\to\infty} m_n = \infty$. Since, as is easily seen, the *Carleman classes* of vectors and functions coincide for the sequence $\{m_n\}_{n=1}^\infty$ and the sequence $\{dm_n\}_{n=1}^\infty$ for any $d > 0$, without loss of generality, we can regard that

$$\inf_{n \ge 0} m_n \ge 1. \tag{2.12}$$

**2.3. Scalar type spectral operators.** Henceforth, unless specified otherwise, $A$ is a *scalar type spectral operator* in a complex Banach space $X$ with norm $\|\cdot\|$ and $E_A(\cdot)$ is its *spectral measure* (the *resolution of the identity*), the operator's spectrum $\sigma(A)$ being the *support* for the latter [2, 5].

Note that, in a Hilbert space, the *scalar type spectral operators* are those similar to the *normal* ones [21].

For such operators, there has been developed an *operational calculus* for Borel measurable functions on $\mathbb{C}$ (on $\sigma(A)$) [2, 5], $F(\cdot)$ being such a function; a new *scalar type spectral operator*

$$F(A) = \int_{\mathbb{C}} F(\lambda)\, dE_A(\lambda) = \int_{\sigma(A)} F(\lambda)\, dE_A(\lambda) \tag{2.13}$$

is defined as follows:

$$F(A)f := \lim_{n\to\infty} F_n(A)f, \quad f \in D(F(A)),$$
$$D(F(A)) := \left\{ f \in X \mid \lim_{n\to\infty} F_n(A)f \text{ exists} \right\} \tag{2.14}$$

($D(\cdot)$ is the *domain* of an operator), where

$$F_n(\cdot) := F(\cdot)\chi_{\{\lambda\in\sigma(A)\,|\,|F(\lambda)|\le n\}}(\cdot), \quad n = 1,2,\ldots, \tag{2.15}$$

($\chi_\alpha(\cdot)$ is the *characteristic function* of a set $\alpha$), and

$$F_n(A) := \int_{\sigma(A)} F_n(\lambda)\, dE_A(\lambda), \quad n = 1,2,\ldots, \tag{2.16}$$

being the integrals of *bounded* Borel measurable functions on $\sigma(A)$, are *bounded scalar type spectral operators* on $X$ defined in the same manner as for *normal operators* (see, e.g., [4, 19]).

The properties of the *spectral measure*, $E_A(\cdot)$, and the *operational calculus* underlying the entire subsequent argument are exhaustively delineated in [2, 5]. We just observe here that, due to its *strong countable additivity*, the spectral measure $E_A(\cdot)$ is *bounded* [3], that is, there is an $M > 0$ such that, for any Borel set $\delta$,

$$\|E_A(\delta)\| \le M. \tag{2.17}$$

Observe that, in (2.17), the notation $\|\cdot\|$ was used to designate the norm in the space of bounded linear operators on $X$. We will adhere to this rather common economy of symbols in what follows adopting the same notation for the norm in the dual space $X^*$ as well.

Due to (2.17), for any $f \in X$ and $g^* \in X^*$ ($X^*$ is the *dual space*), the total variation $v(f,g^*,\cdot)$ of the complex-valued measure $\langle E_A(\cdot)f, g^*\rangle$ ($\langle\cdot,\cdot\rangle$ is the *pairing* between the space $X$ and its dual, $X^*$) is *bounded*. Indeed, $\delta$ being an arbitrary Borel subset of $\sigma(A)$, [3],

$$\begin{aligned} &v(f,g^*,\sigma(A)) \\ &\quad\le 4 \sup_{\delta\subseteq\sigma(A)} |\langle E_A(\delta)f, g^*\rangle| \le 4 \sup_{\delta\subseteq\sigma(A)} \|E_A(\delta)\|\|f\|\|g^*\| \quad \text{(by (2.17))} \\ &\quad\le 4M\|f\|\|g^*\|. \end{aligned} \tag{2.18}$$

For the reader's convenience, we reformulate here [16, Proposition 3.1], heavily relied upon in what follows, which allows to characterize the domains of the Borel measurable functions of a scalar type spectral operator in terms of positive measures (see [16] for a complete proof).

On account of compactness, the terms *spectral measure* and *operational calculus* for scalar type spectral operators, frequently referred to, will be abbreviated to *s.m.* and *o.c.*, respectively.

**Proposition 2.1.** *Let $A$ be a scalar type spectral operator in a complex Banach space $X$ and $F(\cdot)$ a complex-valued Borel measurable function on $\mathbb{C}$ (on $\sigma(A)$). Then $f \in D(F(A))$ if and only if*

(i) *for any $g^* \in X^*$,*

$$\int_{\sigma(A)} |F(\lambda)|\, dv(f,g^*,\lambda) < \infty, \tag{2.19}$$

(ii)

$$\sup_{\{g^*\in X^* \mid \|g^*\|=1\}} \int_{\{\lambda\in\sigma(A) \mid |F(\lambda)|>n\}} |F(\lambda)|\, dv(f,g^*,\lambda) \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{2.20}$$

Observe that, for $F(\cdot)$ being an arbitrary Borel measurable function on $\mathbb{C}$ (on $\sigma(A)$), for any $f \in D(F(A))$, $g^* \in X^*$, and arbitrary Borel sets $\delta \subseteq \sigma$,

$$\begin{aligned}
&\int_{\sigma} |F(\lambda)|\, dv(f,g^*,\lambda) \quad \text{(see [3])}\\
&\quad \le 4 \sup_{\delta\subseteq\sigma} \left| \int_{\delta} F(\lambda)\, d\langle E_A(\lambda)f,g^*\rangle \right|\\
&\quad = 4 \sup_{\delta\subseteq\sigma} \left| \int_{\sigma} \chi_\delta(\lambda)F(\lambda)\, d\langle E_A(\lambda)f,g^*\rangle \right| \qquad \text{(by the properties of the } o.c.\text{)}\\
&\quad = 4 \sup_{\delta\subseteq\sigma} \left| \left\langle \int_{\sigma} \chi_\delta(\lambda)F(\lambda)\, dE_A(\lambda)f,g^* \right\rangle \right| \qquad \text{(by the properties of the } o.c.\text{)}\\
&\quad = 4 \sup_{\delta\subseteq\sigma} \left| \langle E_A(\delta)E_A(\sigma)F(A)f,g^*\rangle \right|\\
&\quad \le 4 \sup_{\delta\subseteq\sigma} \|E_A(\delta)E_A(\sigma)F(A)f\| \|g^*\|\\
&\quad \le 4 \sup_{\delta\subseteq\sigma} \|E_A(\delta)\| \|E_A(\sigma)F(A)f\| \|g^*\| \qquad \text{(by (2.17))}\\
&\quad \le 4M \|E_A(\sigma)F(A)f\| \|g^*\| \le 4M \|E_A(\sigma)\| \|F(A)f\| \|g^*\|.
\end{aligned} \tag{2.21}$$

In particular,

$$\begin{aligned}
&\int_{\sigma(A)} |F(\lambda)|\, dv(f,g^*,\lambda) \quad \text{(by (2.21))}\\
&\quad \le 4M \|E_A(\sigma(A))\| \|F(A)f\| \|g^*\|\\
&\qquad\qquad\qquad \text{(since } E_A(\sigma(A)) = I \text{ (} I \text{ is the } \textit{identity operator} \text{ in } X\text{))}\\
&\quad \le 4M \|F(A)f\| \|g^*\|.
\end{aligned} \tag{2.22}$$

## 3. The Carleman classes of a scalar type spectral operator

**Theorem 3.1.** *Let $A$ be a scalar type spectral operator in a complex reflexive Banach space $X$. If a sequence of positive numbers $\{m_n\}_{n=0}^{\infty}$ satisfies condition* (WGR)*, equalities (2.6) hold, the scalar type spectral operators $T(t|A|)$ $(0 < t < \infty)$ defined in the sense of the operational calculus for a scalar type spectral operator and the function $T(\cdot)$ being replaceable by any nonnegative, continuous, and increasing function $L(\cdot)$ defined on* $[0,\infty)$ *such that*

$$c_1L(\gamma_1\lambda) \le T(\lambda) \le c_2L(\gamma_2\lambda), \quad \lambda > R, \tag{3.1}$$

*with some positive $\gamma_1$, $\gamma_2$, $c_1$, $c_2$, and a nonnegative $R$.*

**Proof.** First, we prove the replaceability of $T(\cdot)$ in (2.6) by a *nonnegative*, *continuous*, and *increasing* function satisfying (3.1) with some positive $\gamma_1$, $\gamma_2$, $c_1$, $c_2$, and a nonnegative $R \ge 0$.

Let

$$f \in \bigcup_{t>0} T(t|A|) \quad \left(\bigcap_{t>0} T(t|A|)\right). \tag{3.2}$$

Then, for some (any) $0 < t < \infty$, $f \in D(T(t|A|))$, which, according to Proposition 2.1, implies, in particular, that, for any $g^* \in X^*$,

$$\int_{\sigma(A)} T(t|\lambda|)\, dv(f,g^*,\lambda) < \infty. \tag{3.3}$$

For any $g^* \in X^*$,

$$\int_{\sigma(A)} L(\gamma_1 t|\lambda|)\, dv(f,g^*,\lambda) < \infty. \tag{3.4}$$

Indeed,

$$\begin{aligned}
&\int_{\sigma(A)} L(\gamma_1 t|\lambda|)\, dv(f,g^*,\lambda) \\
&\quad = \int_{\{\lambda\in\sigma(A)|t|\lambda|\le R\}} L(\gamma_1 t|\lambda|)\, dv(f,g^*,\lambda) + \int_{\{\lambda\in\sigma(A)|t|\lambda|>R\}} L(\gamma_1 t|\lambda|)\, dv(f,g^*,\lambda) \\
&\quad \le L(\gamma_1 R) v(f,g^*,\sigma(A)) + \int_{\{\lambda\in\sigma(A)|t|\lambda|>R\}} L(\gamma_1 t|\lambda|)\, dv(f,g^*,\lambda) \quad \text{(by (2.18))} \\
&\quad \le L(\gamma_1 R)4M\|f\|\|g^*\| + \int_{\{\lambda\in\sigma(A)|t|\lambda|>R\}} L(\gamma_1 t|\lambda|)\, dv(f,g^*,\lambda) \quad \text{(by (3.1))} \\
&\quad \le L(\gamma_1 R)4M\|f\|\|g^*\| + \frac{1}{c_1}\int_{\{\lambda\in\sigma(A)|t|\lambda|>R\}} F(t|\lambda|)\, dv(f,g^*,\lambda) \\
&\quad \le L(\gamma_1 R)4M\|f\|\|g^*\| + \frac{1}{c_1}\int_{\sigma(A)} F(t|\lambda|)\, dv(f,g^*,\lambda) \quad \text{(by (3.3))} \\
&\quad < \infty.
\end{aligned} \tag{3.5}$$

Further,

$$\sup_{\{g^*\in X^*\,|\,\|g^*\|=1\}} \int_{\{\lambda\in\sigma(A)\,|\,t|\lambda|\le R,\ L(\gamma_1 t|\lambda|)>n\}} L(\gamma_1 t|\lambda|)\,dv(f,g^*,\lambda) = 0 \tag{3.6}$$

for all sufficiently large natural $n$'s since, when $t|\lambda| \le R$, $L(\gamma_1 t|\lambda|) \le L(\gamma_1 R)$.

On the other hand,

$$\begin{aligned} &\int_{\{\lambda\in\sigma(A)\,|\,t|\lambda|>R,\ L(\gamma_1 t|\lambda|)>n\}} L(\gamma_1 t|\lambda|)\,dv(f,g^*,\lambda) \quad \text{(by (3.1))} \\ &\quad\le \frac{1}{c_1}\int_{\{\lambda\in\sigma(A)\,|\,t|\lambda|>R,\ T(t|\lambda|)>c_1 n\}} T(t|\lambda|)\,dv(f,g^*,\lambda) \quad \text{(by (2.21))} \\ &\quad\le \frac{1}{c_1}\big\|E_A(\{\lambda\in\sigma(A)\mid T(t|\lambda|)>c_1 n\})T(t|A|)f\big\|\,\|g^*\| \\ &\qquad\qquad\qquad\text{(by the continuity of the } s.m.) \\ &\quad\longrightarrow 0 \quad \text{as } n\longrightarrow\infty. \end{aligned} \tag{3.7}$$

Therefore, by Proposition 2.1, $f\in D(L(\gamma_1 t|A|))$.

Thus, we have proved the inclusions

$$\begin{aligned} \bigcup_{t>0} D(T(t|A|)) &\subseteq \bigcup_{t>0} D(L(t|A|)), \\ \bigcap_{t>0} D(T(t|A|)) &\subseteq \bigcap_{t>0} D(L(t|A|)). \end{aligned} \tag{3.8}$$

Similarly, one can derive from (3.1) the inverse inclusions:

$$\begin{aligned} \bigcup_{t>0} D(T(t|A|)) &\supseteq \bigcup_{t>0} D(L(t|A|)), \\ \bigcap_{t>0} D(T(t|A|)) &\supseteq \bigcap_{t>0} D(L(t|A|)). \end{aligned} \tag{3.9}$$

Thus,

$$\begin{aligned} \bigcup_{t>0} D(T(t|A|)) &= \bigcup_{t>0} D(L(t|A|)), \\ \bigcap_{t>0} D(T(t|A|)) &= \bigcap_{t>0} D(L(t|A|)). \end{aligned} \tag{3.10}$$

Let $f \in C_{\{m_n\}}(A)$ $(C_{(m_n)}(A))$. Then $f \in C^\infty(A)$ and, for a certain (an arbitrary) $\alpha > 0$, there is a $c > 0$ such that

$$\|A^n f\| \le c\alpha^n m_n, \quad n = 0,1,2,\dots. \tag{3.11}$$

For any $g^* \in X^*$,

$$\begin{aligned}
\int_{\sigma(A)} T\left(\frac{1}{2\alpha}|\lambda|\right) dv(f,g^*,\lambda) &= \int_{\sigma(A)} \sum_{n=0}^{\infty} \frac{|\lambda|^n}{2^n\alpha^n m_n} dv(f,g^*,\lambda) \\
&\qquad \text{(by the \textit{monotone convergence theorem})} \\
&= \sum_{n=0}^{\infty} \int_{\sigma(A)} \frac{|\lambda|^n}{2^n\alpha^n m_n} dv(f,g^*,\lambda) \\
&= \sum_{n=0}^{\infty} \frac{1}{2^n\alpha^n m_n} \int_{\sigma(A)} |\lambda|^n dv(f,g^*,\lambda) \quad \text{(by (2.22))} \\
&\le \sum_{n=0}^{\infty} \frac{1}{2^n\alpha^n m_n} 4M\|A^n f\|\|g^*\| \quad \text{(by (3.11))} \\
&\le 4Mc \sum_{n=0}^{\infty} \frac{1}{2^n}\|g^*\| = 8Mc\|g^*\| < \infty.
\end{aligned} \tag{3.12}$$

Let

$$\Delta_n := \{\lambda \in \sigma(A) \mid |\lambda| \le n\}, \quad n = 0,1,2,\dots. \tag{3.13}$$

By the properties of the *o.c.*, $T((1/2\alpha)|A|)E_A(\Delta_n)$, $n = 0,1,2,\dots$, is a bounded operator on $X$ and

$$\begin{aligned}
\left\|T\left(\frac{1}{2\alpha}|A|\right)E_A(\Delta_n)\right\| &\le 4M \sum_{k=0}^{\infty} \frac{n^k}{2^k\alpha^k m_k} \\
&\left(\text{by condition (WGR), there is a } C = C(\alpha,n) > 0 : \frac{n^k}{\alpha^k m_k} \le C,\ k = 0,1,\dots\right) \\
&\le 4MC \sum_{k=0}^{\infty} \frac{1}{2^k} = 8MC.
\end{aligned} \tag{3.14}$$

For any $1 \le m < n$,

$$\begin{aligned}
&\left|\left\langle T\left(\frac{1}{2\alpha}|A|\right)E_A(\Delta_n)f - T\left(\frac{1}{2\alpha}|A|\right)E_A(\Delta_m)f, g^*\right\rangle\right| \\
&\qquad\qquad\qquad\qquad \text{(by the properties of the } o.c.\text{)} \\
&\left|\left\langle \int_{\{\lambda\in\sigma(A)\,|\,m<|\lambda|\le n\}} T\left(\frac{1}{2\alpha}|\lambda|\right) dE_A(\lambda)f, g^*\right\rangle\right| \\
&\qquad\qquad\qquad\qquad \text{(by the properties of the } o.c.\text{)} \\
&\quad = \left|\int_{\{\lambda\in\sigma(A)\,|\,m<|\lambda|\le n\}} T\left(\frac{1}{2\alpha}|\lambda|\right) d\langle E_A(\lambda)f, g^*\rangle\right| \\
&\quad \le \int_{\{\lambda\in\sigma(A)\,|\,m<|\lambda|\}} T\left(\frac{1}{2\alpha}|\lambda|\right) dv(f, g^*, \lambda) \quad \text{(by (3.12))} \\
&\quad \longrightarrow 0 \quad \text{as } m \longrightarrow \infty.
\end{aligned} \tag{3.15}$$

Since a *reflexive* Banach space is *weakly complete* (see, e.g., [3]), we infer that the sequence $\{T((1/2\alpha)|A|)E_A(\Delta_n)f\}_{n=1}^{\infty}$ *weakly converges* in $X$. This, considering the fact that, by the continuity of the $s.m.$,

$$E_A(\Delta_n)f \longrightarrow f \quad \text{as } n \longrightarrow \infty \tag{3.16}$$

and the *closedness* of the operator $T((1/2\alpha)|A|)$, implies

$$f \in D\left(T\left(\frac{1}{2\alpha}|A|\right)\right). \tag{3.17}$$

Therefore,

$$f \in \bigcup_{t>0} D(T(t|A|)) \quad \left(\bigcap_{t>0} D(T(t|A|)), \text{ resp.}\right), \tag{3.18}$$

which proves the inclusions

$$\begin{aligned}
C_{\{m_n\}}(A) &\subseteq \bigcup_{t>0} D(T(t|A|)), \\
C_{(m_n)}(A) &\subseteq \bigcap_{t>0} D(T(t|A|)).
\end{aligned} \tag{3.19}$$

Now, we are to prove the inverse inclusions.

Let

$$f \in \bigcup_{t>0} D(T(t|A|)) \quad \left(\bigcap_{t>0} D(T(t|A|))\right). \tag{3.20}$$

Then, for a certain (any) $t > 0$, $f \in D(T(t|A|))$.

We infer from the latter that $f \in C^\infty(A)$.

Indeed, for an arbitrary $N = 0, 1, 2, \dots$ and any $g^* \in X^*$,

$$\begin{aligned}\int_{\sigma(A)} \frac{t^N}{m_N} |\lambda|^N \, dv(f, g^*, \lambda) &\le \int_{\sigma(A)} \sum_{k=0}^{\infty} \frac{[t|\lambda|]^k}{m_k} \, dv(f, g^*, \lambda) \\ &= \int_{\sigma(A)} T(t|\lambda|) \, dv(f, g^*, \lambda) \\ &\qquad\qquad \text{(by Proposition 2.1)}, \\ &< \infty.\end{aligned} \tag{3.21}$$

Further, for any $N = 0, 1, 2, \dots$,

$$\begin{aligned}&\sup_{\{g^* \in X^* \mid \|g^*\|=1\}} \int_{\{\lambda \in \sigma(A) \mid (t^N/m_N)|\lambda|^N > n\}} \frac{t^N}{m_N} |\lambda|^N \, dv(f, g^*, \lambda) \\ &\quad \le \sup_{\{g^* \in X^* \mid \|g^*\|=1\}} \int_{\{\lambda \in \sigma(A) \mid T(t|\lambda|) > n\}} T(t|\lambda|) \, dv(f, g^*, \lambda) \quad \text{(by Proposition 2.1)}, \\ &\quad \longrightarrow 0 \quad \text{as } n \longrightarrow \infty.\end{aligned} \tag{3.22}$$

By Proposition 2.1, (3.21) and (3.22) imply that

$$f \in C^\infty(A). \tag{3.23}$$

Further, by (2.22),

$$\begin{aligned}&\sup_{\{g^* \in X^* \mid \|g^*\|=1\}} \int_{\sigma(A)} T(t|\lambda|) \, dv(f, g^*, \lambda) \quad \text{(by (2.22))} \\ &\le 4M \| T(t|A|) f \| < \infty.\end{aligned} \tag{3.24}$$

By (2.22),

$$\begin{aligned}0 < c := &\sup_{\{g^* \in X^* \mid \|g^*\|=1\}} \int_{\sigma(A)} T(t|\lambda|) \, dv(f, g^*, \lambda) + 1 \\ &\le 4M \| T(t|A|) f \| < \infty.\end{aligned} \tag{3.25}$$

Whence, for any $n = 0, 1, 2, \ldots$,

$$\begin{aligned}
c &\geq \sup_{\{g^* \in X^* \mid \|g^*\|=1\}} \int_{\sigma(A)} \frac{t^n}{m_n} |\lambda|^n \, dv(f, g^*, \lambda) \\
&\geq \frac{t^n}{m_n} \sup_{\{g^* \in X^* \mid \|g^*\|=1\}} \left| \int_{\sigma(A)} \lambda^n \, d\langle E_A(\lambda) f, g^* \rangle \right| \\
&\qquad \text{(by the properties of the } o.c.\text{)} \\
&\geq \frac{t^n}{m_n} \sup_{\{g^* \in X^* \mid \|g^*\|=1\}} \left| \left\langle \int_{\sigma(A)} \lambda^n \, dE_A(\lambda) f, g^* \right\rangle \right| \\
&\qquad \text{(by the properties of the } o.c.\text{)} \\
&= \frac{t^n}{m_n} \sup_{\{g^* \in X^* \mid \|g^*\|=1\}} |\langle A^n f, g^* \rangle| \\
&\qquad \text{(as follows from the } \textit{Hahn-Banach theorem)} \\
&= \frac{t^n}{m_n} \|A^n f\|.
\end{aligned} \tag{3.26}$$

Thus, for some (any) $t > 0$,

$$\|A^n f\| \leq c \left( \frac{1}{t} \right)^n m_n, \quad n = 0, 1, 2, \ldots. \tag{3.27}$$

Hence,

$$f \in C_{\{m_n\}}(A) \quad (C_{(m_n)}(A), \text{ resp.}), \tag{3.28}$$

which proves the inverse inclusions

$$\begin{aligned}
C_{\{m_n\}}(A) &\supseteq \bigcup_{t>0} D(T(t|A|)), \\
C_{(m_n)}(A) &\supseteq \bigcap_{t>0} D(T(t|A|)).
\end{aligned} \tag{3.29}$$

From (3.19) and (3.29), we infer equalities (2.6). □

**Remark 3.2.** Observe that the assumption of the *reflexivity* of the space $X$ was utilized for proving the inclusions

$$\begin{aligned}
C_{\{m_n\}}(A) &\subseteq \bigcup_{t>0} D(T(t|A|)), \\
C_{(m_n)}(A) &\subseteq \bigcap_{t>0} D(T(t|A|))
\end{aligned} \tag{3.30}$$

only.

The inverse inclusions

$$\begin{aligned}
C_{\{m_n\}}(A) &\supseteq \bigcup_{t>0} D(T(t|A|)), \\
C_{(m_n)}(A) &\supseteq \bigcap_{t>0} D(T(t|A|))
\end{aligned} \tag{3.31}$$

hold regardless whether $X$ is reflexive or not.

**4. The Gevrey classes of a scalar type spectral operator.** Let $0 < \beta < \infty$. As is easily seen, the sequence $m_n = [n!]^\beta$, $n = 0,1,2,\dots$, satisfies condition (WGR) and, thus, the function

$$T(\lambda) := \sum_{n=0}^{\infty} \frac{\lambda^n}{[n!]^\beta}, \quad 0 \le \lambda < \infty, \tag{4.1}$$

is well defined.

According to *Stirling's formula*,

$$n^{\beta n} \sim (2\pi n)^{-\beta/2} e^{\beta n} [n!]^\beta \quad \text{as } n \longrightarrow \infty. \tag{4.2}$$

Hence, there is such a $C = C(\beta) \ge 1$ such that

$$[n!]^\beta \le n^{\beta n} \le C(2\pi n)^{-\beta/2} e^{\beta n} [n!]^\beta \le C e^{\beta n} [n!]^\beta, \quad n = 0,1,2,\dots. \tag{4.3}$$

Taking this into account, we infer

$$\begin{aligned} \sup_{n\ge 0} \frac{\lambda^n}{n^{\beta n}} &\le \sum_{n=0}^{\infty} \frac{\lambda^n}{n^{\beta n}} \le T(\lambda) \le C \sum_{n=0}^{\infty} \frac{(e^\beta\lambda)^n}{n^{\beta n}} = C \sum_{n=0}^{\infty} \frac{1}{2^n} \frac{(2e^\beta\lambda)^n}{n^{\beta n}} \\ &\le C \sup_{n\ge 0} \frac{(2e^\beta\lambda)^n}{n^{\beta n}} \sum_{n=0}^{\infty} \frac{1}{2^n} = 2C \sup_{n\ge 0} \frac{(2e^\beta\lambda)^n}{n^{\beta n}}, \quad 0 \le \lambda < \infty. \end{aligned} \tag{4.4}$$

Now, we consider the family of functions

$$\rho_\lambda(x) := \frac{\lambda^x}{x^{\beta x}}, \quad 0 \le x < \infty,\ 1 \le \lambda < \infty\ (0^0 := 1). \tag{4.5}$$

It is easy to make sure that the function $\rho_\lambda(\cdot)$ attains its maximum value on $[0,\infty)$ at the point $x_\lambda = e^{-1}\lambda^{1/\beta}$.

Therefore,

$$\sup_{n\ge 0} \frac{\lambda^n}{n^{\beta n}} \le \sup_{x\ge 0} \frac{\lambda^x}{x^{\beta x}} = \rho_\lambda(x_\lambda) = e^{\beta e^{-1}\lambda^{1/\beta}}. \tag{4.6}$$

For $\lambda \ge e^\beta$, let $N$ be the *integer part* of $x_\lambda = e^{-1}\lambda^{1/\beta}$.

Hence, $N \ge 1$ and

$$\begin{aligned} \sup_{n\ge 0} \frac{\lambda^n}{n^{\beta n}} &\ge \frac{\lambda^N}{N^{\beta N}} = \exp\left(N\ln\lambda - \beta N \ln N\right) \\ &\ge \exp\left((x_\lambda - 1)\ln\lambda - \beta x_\lambda \ln x_\lambda\right) = \frac{1}{\lambda} e^{\beta e^{-1}\lambda^{1/\beta}}, \quad \lambda \ge e^\beta. \end{aligned} \tag{4.7}$$

Obviously, for all sufficiently large positive $\lambda$'s,

$$e^{-(\beta e^{-1}/2)\lambda^{1/\beta}} \le \frac{1}{\lambda}. \tag{4.8}$$

Based on (4.4), (4.6), (4.7), and (4.8), for all sufficiently large positive $\lambda$'s,

$$\begin{aligned} e^{(\beta^\beta(e^{-\beta}/2^\beta)\lambda)^{1/\beta}} &\le T(\lambda) \le 2C\sup_{n\ge 0}\frac{(2e^\beta\lambda)^n}{n^{\beta n}} \le 2C\sup_{x\ge 0}\rho_{2e^\beta\lambda}(x) \\ &= 2Ce^{\beta e^{-1}(2e^\beta\lambda)^{1/\beta}} \le e^{(4\beta^\beta\lambda)^{1/\beta}}. \end{aligned} \tag{4.9}$$

Thus, by Theorem 3.1, in the considered case, the function $T(\lambda)$ can be replaced by $e^{\lambda^{1/\beta}}$ $(0 \le \lambda < \infty)$ and we arrive at the following.

**Corollary 4.1.** *Let $A$ be a scalar type spectral operator in a complex reflexive Banach space and $0 < \beta < \infty$. Then*

$$\begin{aligned} \mathscr{E}^{\{\beta\}}(A) &= \bigcup_{t>0} D\left(e^{t|A|^{1/\beta}}\right), \\ \mathscr{E}^{(\beta)}(A) &= \bigcap_{t>0} D\left(e^{t|A|^{1/\beta}}\right). \end{aligned} \tag{4.10}$$

In particular, for $\beta = 1$, Corollary 4.1 gives the description of the *analytic* and *entire* vectors of the scalar type spectral operator $A$.

Corollary 4.1 generalizes the corresponding result of [8] (see also [9, 10]) for a *normal operator* in a complex Hilbert space.

Observe that the inclusions

$$\begin{aligned} \mathscr{E}^{\{\beta\}}(A) &\supseteq \bigcup_{t>0} D\left(e^{t|A|^{1/\beta}}\right), \\ \mathscr{E}^{(\beta)}(A) &\supseteq \bigcap_{t>0} D\left(e^{t|A|^{1/\beta}}\right). \end{aligned} \tag{4.11}$$

are valid without the assumption of the *reflexivity* of $X$ (see Remark 3.2).

**5. A theorem of the Paley-Wiener type.** Consider the self-adjoint differential operator $A = i(d/dx)$ ($i$ is the *imaginary unit*) in the complex Hilbert space $L^2(-\infty,\infty)$. With the unitary equivalence of this operator and the operator of multiplication by the independent variable $x$ in view, by Theorem 3.1 as well as by [9, 10], we arrive at the following theorem of the Paley-Wiener type [18, 22].

**Theorem 5.1.** *Let $\{m_n\}_{n=0}^{\infty}$ be a sequence of positive numbers satisfying condition* (WGR)*, then*

$$f \in C_{\{m_n\}}(A)\ \left(C_{(m_n)}(A)\right) \Longleftrightarrow \int_{-\infty}^{\infty} |\hat{f}(x)|^2 T^2(t|x|)\,dx < \infty \tag{5.1}$$

*($\hat{f}$ is the Fourier transform of $f$) for some (any) $0 < t < \infty$, the function $T(\cdot)$ being replaceable by any nonnegative, continuous, and increasing function $L(\cdot)$ defined on $[0,\infty)$ and satisfying (3.1) with some positive $\gamma_1$, $\gamma_2$, $c_1$, $c_2$, and a nonnegative $R$.*

The only natural question to be answered now is how the abstract smoothness relative to the differential operator $A$ in $L^2(-\infty,\infty)$ reveals itself as the smoothness in the ordinary sense.

For any $f \in W_2^n(I)$, where $I$ is an interval of the real axis and $W_2^n(I) = H^n(I)$ is the $n$th-order *Sobolev space* [20], let $f(\cdot)$ be the representative of the equivalence class $f$ continuously differentiable $n-1$ times and such that $f^{(n-1)}(\cdot)$ is *absolutely continuous* on $I$.

For

$$f \in W_2^\infty(-\infty,\infty) := \bigcap_{n=0}^{\infty} W_2^n(-\infty,\infty), \tag{5.2}$$

let $f(\cdot)$ be the infinite-differentiable representative of the equivalence class $f$ such that

$$\int_{-\infty}^{\infty} |f^{(n)}(t)|^2\,dt < \infty, \quad n = 0,1,2,\dots. \tag{5.3}$$

Let

$$\begin{aligned} \hat{C}_{\{m_n\}}(-\infty,\infty) &\overset{\text{def}}{=} \Big\{ f \in W_2^\infty(-\infty,\infty) \mid \forall [a,b] \subseteq (-\infty,\infty)\ \exists \alpha > 0, \\ &\qquad \exists c > 0 : \max_{a \le t \le b} \|f^{(n)}(t)\| \le c\alpha^n m_n,\ n = 0,1,2,\dots \Big\}, \\ \hat{C}_{(m_n)}(-\infty,\infty) &\overset{\text{def}}{=} \Big\{ f \in W_2^\infty(-\infty,\infty) \mid \forall [a,b] \subseteq (-\infty,\infty),\ \forall \alpha > 0 \\ &\qquad \exists c > 0 : \max_{a \le t \le b} \|f^{(n)}(t)\| \le c\alpha^n m_n,\ n = 0,1,2,\dots \Big\}. \end{aligned} \tag{5.4}$$

We will impose upon the sequence $\{m_n\}_{n=0}^\infty$ an additional condition.

(DI) There are an $L > 0$ and a $\gamma > 1$ such that

$$m_{n+1} \le L\gamma^n m_n, \quad n = 0,1,2,\dots.$$

Note that the name (DI) originates from the words *"differentiation invariant"* since, as is easily verifiable, under this condition, the Carleman classes $C_{\{m_n\}}(-\infty,\infty)$ and $C_{(m_n)}(-\infty,\infty)$ along with a function $f(\cdot)$ contain its first derivative, $f'(\cdot)$.

Observe that, for $0 \le \beta < \infty$, the Gevrey sequence $m_n = [n!]^\beta$, $n = 0,1,2,\dots$, meets condition (DI) with any $\gamma > 1$. Indeed, in this case, $m_{n+1}/m_n = (n+1)^\beta$, $n = 0,1,2,\dots$.

**Lemma 5.2.** *Let a sequence of positive numbers* $\{m_n\}_{n=0}^\infty$ *satisfy condition* (DI). *Then*

$$\begin{aligned} C_{\{m_n\}}(A) &\subseteq \hat{C}_{\{m_n\}}(-\infty,\infty), \\ C_{(m_n)}(A) &\subseteq \hat{C}_{(m_n)}(-\infty,\infty). \end{aligned} \tag{5.5}$$

**Proof.** Let $f \in C_{\{m_n\}}(A)$ ($C_{(m_n)}(A)$), Then

$$f \in W_2^\infty(-\infty,\infty), \tag{5.6}$$

and for some (any) $\alpha > 0$, there is a $c > 0$ such that

$$\|f\|_{L^2(-\infty,\infty)} = \left[ \int_{-\infty}^{\infty} |f^{(n)}(x)|^2\,dx \right]^{1/2} \le c\alpha^n m_n, \quad n = 0,1,2,\dots. \tag{5.7}$$

We fix a finite segment $[a,b]$ of the real axis. Then, according to the *Sobolev embedding theorems* [20] (see also [22, 23]), the space $W_2^1(a,b)$ is *continuously embedded* into $C[a,b]$, that is, for some $M>0$ and any $f\in W_2^1(a,b)$,

$$\max_{a\le t\le b}|f(x)|\le M\|f\|_{W_2^1(a,b)}\le M\Big[\|f\|_{L^2(a,b)}+\|f'\|_{L^2(a,b)}\Big]. \tag{5.8}$$

Since $f\in C_{\{m_n\}}(A)$ ($C_{(m_n)}(A)$). Then, obviously, $f^{(n)}\in W_2^1(a,b)$ for any $n=0,1,2,\ldots$. Therefore, for an arbitrary $n=0,1,2,\ldots$,

$$\begin{aligned}\max_{a\le t\le b}\left|f^{(n)}(x)\right| &\le M\|f\|_{W_2^1(a,b)}\le M\Big[\|f^{(n)}\|_{L^2(a,b)}+\|f^{(n+1)}\|_{L^2(a,b)}\Big]\\ &\le M\Big[\|f^{(n)}\|_{L^2(-\infty,\infty)}+\|f^{(n+1)}\|_{L^2(-\infty,\infty)}\Big]\\ &\le M\left[c\alpha^n m_n+c\alpha^{n+1}m_{n+1}\right]\quad \text{(by (DI))}\\ &\le M\left[c\alpha^n m_n+c\alpha^{n+1}L\gamma^n m_n\right]=Mc\left[1+L\alpha\gamma^n\right]\alpha^n m_n\\ &\text{(considering that } \gamma>1\text{, there is a } c_1>0 \text{ such that } \gamma>1,\ c_1>0)\\ &\le c_1(\gamma\alpha)^n m_n,\quad n=0,1,2,\ldots.\end{aligned} \tag{5.9}$$

□

Based on this Lemma, we obtain the following proposition.

**Proposition 5.3.** *Let $\{m_n\}_{n=0}^{\infty}$ be a sequence of positive numbers satisfying* (WGR) *and* (DI). *If $f\in L^2(-\infty,\infty)$ is such that, for some (any) $0<t<\infty$,*

$$\int_{-\infty}^{\infty}\left|\hat f(x)\right|^2T^2(t|x|)\,dx<\infty, \tag{5.10}$$

*there is a representative $f(\cdot)$ of the equivalence class $f$ such that $f(\cdot)\in C^\infty(-\infty,\infty)$,*

$$\begin{gathered}\int_{-\infty}^{\infty}\left|f^{(n)}(x)\right|^2dx<\infty,\quad n=0,1,2,\ldots,\\ f(\cdot)\in C_{\{m_n\}}(-\infty,\infty)\ \left(C_{(m_n)}(-\infty,\infty)\right),\end{gathered} \tag{5.11}$$

*the function $T(\cdot)$ being replaceable by any nonnegative, continuous, and increasing function $L(\cdot)$ defined on $[0,\infty)$ and satisfying (3.1) with some positive $\gamma_1$, $\gamma_2$, $c_1$, $c_2$, and a nonnegative $R$.*

**Corollary 5.4.** *Let $0<\beta<\infty$. If $f\in L^2(-\infty,\infty)$ is such that, for some (any) $0<t<\infty$,*

$$\int_{-\infty}^{\infty}\left|\hat f(x)\right|^2e^{2t|x|^{1/\beta}}\,dx<\infty, \tag{5.12}$$

*there is a representative $f(\cdot)$ of the equivalence class $f$ such that $f(\cdot)\in C^\infty(-\infty,\infty)$,*

$$\begin{gathered}\int_{\infty}^{\infty}\left|f^{(n)}(x)\right|^2dx<\infty,\quad n=0,1,2,\ldots,\\ f(\cdot)\in \mathscr{E}^{\{\beta\}}(-\infty,\infty)\ \left(\mathscr{E}^{(\beta)}(-\infty,\infty)\right).\end{gathered} \tag{5.13}$$

In particular, for $\beta=1$, we obtain sufficient conditions for the *real analyticity* and *entireness.*

**6. Remarks.** It is to be noted that, in [10] (see also [8, 9]), not only were equalities (2.6) for a *normal operator* in a complex Hilbert space proved to hold in the set-theoretical sense but also in the topological sense, the sets $C_{\{m_n\}}(A)$ and $C_{(m_n)}(A)$ considered as the *inductive* and, respectively, *projective* limits of the Banach spaces

$$C_{\alpha[m_n]}(A) := \left\{ f \in C^{\infty}(A) \mid \exists c > 0 : \|A^n f\| \le c\alpha^n m_n,\ n = 0, 1, \dots \right\}, \tag{6.1}$$

$0 < \alpha < \infty$, with the norms

$$\|f\|_{C_{\alpha[m_n]}(A)} := \sup_{n \ge 0} \frac{\|A^n f\|}{\alpha^n m_n} \tag{6.2}$$

and the sets $\bigcup_{t>0} D(T(t|A|))$ and $\bigcap_{t>0} D(T(t|A|))$ as the *inductive* and, respectively, *projective* limits of the Hilbert spaces

$$H_{t[T]}(A) := D(T(t|A|)), \quad 0 < t < \infty, \tag{6.3}$$

with inner products

$$(f, g)_{H_{t[T]}(A)} := (T(t|A|)f, T(t|A|)g), \quad 0 < t < \infty. \tag{6.4}$$

Observe also that, in [11] (see also [10]), similar results were obtained for the *generator of a bounded analytic semigroup* in a Banach space.

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