

## UNITS IN FAMILIES OF TOTALLY COMPLEX ALGEBRAIC NUMBER FIELDS

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Multidimensional continued fraction algorithms associated with  $GL_n(\mathbb{Z}_K)$ , where  $\mathbb{Z}_K$  is the ring of integers of an imaginary quadratic field  $K$ , are introduced and applied to find systems of fundamental units in families of totally complex algebraic number fields of degrees four, six, and eight.

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**1. Introduction.** Let  $F$  be an algebraic number field of degree  $n$ . There exist exactly  $n$  field embeddings of  $F$  in  $\mathbb{C}$ . Let  $s$  be the number of embeddings of  $F$  whose images lie in  $\mathbb{R}$ , and let  $2t$  be the number of nonreal complex embeddings, so that  $n = s + 2t$ . The pair  $(s, t)$  is said to be the *signature* of  $F$ . Let  $\mathbb{Z}_F$  be the ring of integers of the field  $F$ . A *unit* in  $F$  is an invertible element of  $\mathbb{Z}_F$ . The set of units in  $F$  forms a multiplicative group which will be denoted by  $\mathbb{Z}_F^\times$ . In 1840, P. G. Lejeune-Dirichlet determined the structure of the group  $\mathbb{Z}_F^\times$ . He showed that  $\mathbb{Z}_F^\times$  is a finitely generated Abelian group of rank  $r = s + t - 1$ , that is,  $\mathbb{Z}_F^\times$  is isomorphic to  $\mu_F \times \mathbb{Z}^r$ , where  $\mu_F$  is a finite cyclic group.  $\mu_F$  is called the *torsion* subgroup of  $\mathbb{Z}_F^\times$ . Thus, there exist units  $\epsilon_1, \dots, \epsilon_r$  such that every element of  $\mathbb{Z}_F^\times$  can be written in a unique way as  $\zeta \epsilon_1^{n_1} \cdots \epsilon_r^{n_r}$ , where  $n_i \in \mathbb{Z}$  and  $\zeta$  is a root of unity in  $F$ . Such a set  $\{\epsilon_1, \dots, \epsilon_r\}$  is called a *system of fundamental units* of  $F$ . Finding a system of fundamental units of  $F$  is one of the main computational problems of algebraic number theory (see, e.g., [4, page 217]). Much work has been done to solve this problem for certain classes of algebraic number fields (see, e.g., [11]). In the case of the real quadratic fields, the continued fraction algorithm provides a very efficient method for solving this problem (see, e.g., [11, page 119]). This approach goes back to L. Euler, who applied continued fractions to solve Pell's equation  $x^2 - dy^2 = \pm 1$ . (If a square-free positive integer  $d \equiv 2$  or  $3 \pmod{4}$  and  $x, y$  is an integral solution of this equation, then  $x + \sqrt{d}y$  is a unit in the real quadratic field  $\mathbb{Q}(\sqrt{d})$ . Moreover, any unit in  $\mathbb{Q}(\sqrt{d})$  can be obtained this way.) Many attempts have been made to develop a similar algorithm that would find a system of fundamental units in other algebraic number fields. In the case of a cubic field, one of the most successful such algorithms was introduced by Voronoi [16]. A review of the multidimensional continued fraction algorithms and their properties that were known by 1980 can be found in [1].

Let  $d > 0$  be a square-free integer. Let  $\mathbb{Z}_K$  be the ring of integers of the field  $K = \mathbb{Q}(\sqrt{-d})$ . The group of units  $\mu_K$  of  $K$  is a finite cyclic group of order 6 if  $d = 3$ , 4 if  $d = 1$ , and  $\mu_K = \{\pm 1\}$  otherwise. Let  $\omega = (1 + \sqrt{-d})/2$  if  $d \equiv 3 \pmod{4}$  and  $\omega = \sqrt{-d}$

otherwise. Then  $\{1, \omega\}$  is a  $\mathbb{Z}$ -basis of  $\mathbb{Z}_K$ . Let  $F/K$  be a relative extension of relative degree  $n$ , so that the signature of  $F$  is  $(0, n)$ .

In [24], the (multidimensional continued fraction) Algorithm II associated with the discrete group  $\mathrm{GL}_n(\mathbb{Z})/\{\pm 1\}$  acting on the symmetric space  $\mathcal{P}_n = \mathrm{SL}_n(\mathbb{R})/\mathrm{SO}_n(\mathbb{R})$  was introduced and applied to the problem of finding a system of fundamental units in an algebraic number field. In the present paper, an analog of Algorithm II associated with the group  $\Gamma = \mathrm{GL}_n(\mathbb{Z}_K)/\mu_K$  acting on the Hermitian symmetric space  $\mathcal{H} = \mathrm{SL}_n(\mathbb{C})/\mathrm{SU}(n)$  is applied to the problem of finding a system of fundamental units in the relative extension  $F/K$  and in the field  $F$ . The space  $\mathcal{H}$  can be identified with the set of positive definite Hermitian forms in  $n$  complex variables with the leading coefficient one. Denote by  $\hat{X}$  the positive definite quadratic form in  $2n$  real variables associated with a Hermitian form  $X \in \mathcal{H}$ . The set  $\{\hat{X} : X \in \mathcal{H}\}$  is a totally geodesic submanifold of  $\mathcal{P}_{2n}$  of dimension  $n^2 - 1$  (see, e.g., [2, Chapter II.10]).

Assume that  $g \in \mathrm{GL}_n(\mathbb{C})$ . Let  $ga_i = \lambda_i a_i$ ,  $i = 1, \dots, n$ , so that  $a_i$  is an eigenvector of  $g$  corresponding to its eigenvalue  $\lambda_i$ . For simplicity, assume that all the eigenvalues of  $g$  are distinct. Let  $P = (a_1, \dots, a_n)$  be the matrix with columns  $a_1, \dots, a_n$ . The set of points in  $\mathcal{H}$  fixed by  $g$  will be called the *axis*  $L_P$  of  $g$ . The axis  $L_P$  of  $g$  depends only on eigenvectors of  $g$ , that is, on  $P$ , but not on its eigenvalues (see Section 3).  $L_P$  is a totally geodesic submanifold of  $\mathcal{H}$  of dimension  $n - 1$ .

In Section 2, the notion of the height of a point in  $\mathcal{H}$  is introduced. Let  $w = (1, 0, \dots, 0)^T$  and  $W = ww^T$ . In what follows, the point  $W$  which belongs to the boundary of  $\mathcal{H}$  is analogous to the point  $\infty$  in the upper half-space model  $H^{n+1} = \{(z, t) : z \in \mathbb{R}^n, t > 0\}$  of the  $(n + 1)$ -dimensional hyperbolic space (see [21, 22]). The set  $K_n = K(w)$  in  $\mathcal{H}$  is defined so that, for every point  $X \in \mathcal{H}$ , the points in the  $\Gamma$ -orbit of  $X$  with the largest height belong to  $K(w)$ . The images  $K_n[g]$  of  $K_n$ ,  $g \in \Gamma$ , under the action of  $\Gamma$  form the *K-tessellation* of  $\mathcal{H}$ . The *K-tessellation* of  $\mathcal{H}$  is  $\Gamma$ -invariant.

If  $L_P \cap K_n[g] \neq \emptyset$ ,  $g \in \Gamma$ , then the vector  $u = g^{-1}w \in \mathbb{Z}_K^n$  is called a *convergent* of  $L_P$ . In Section 3, it is shown that if  $u$  is a convergent of  $L_P$ , then  $|\langle a_1, u \rangle \cdots \langle a_n, u \rangle / \det P|$ , where  $\langle \cdot, \cdot \rangle$  denotes the complex dot product in  $\mathbb{C}^n$ , is small (Theorem 3.3). Algorithm II, which is introduced in [24], can be applied in  $\mathcal{H}$  to find the sets  $R(g^{-1}w) = L_P \cap K_n[g] \neq \emptyset$ , which form a tessellation of  $L_P$ , and the set of convergents of  $L_P$ .

It is proved in Section 4 that a system of fundamental units in the relative extension  $F/K$  is a system of fundamental units in the field  $F$  provided  $\mathbb{Z}_{F/K}$  is a free  $\mathbb{Z}_K$ -module.

The upper half-space  $H^3 = \{(z, t) : z \in \mathbb{C}, t > 0\}$  with the metric  $ds^2 = t^{-2}(|dz|^2 + dt^2)$  can be used as a model of the three-dimensional hyperbolic space.  $\mathrm{SL}_2(\mathbb{C})$  is the group of orientation-preserving isometries of  $H^3$ . In Section 5, for  $n = 2$ , a bijection  $\psi$  of  $\mathcal{H}$  and  $H^3$  is introduced, so that  $\psi$  is also a bijection between the *K-tessellations* of  $\mathcal{H}$  and  $H^3$ . Thus, Algorithm I from [21] in  $H^3$  coincides with Algorithm II from [24] in  $\mathcal{H}$  in this case. In Examples 5.3, 5.4, and 5.6, Algorithm I is applied to find fundamental units in some families of number fields with signature  $(0, 2)$ .

If  $g \in \Gamma = \mathrm{GL}_n(\mathbb{Z}_K)/\mu_K$ , then there are only finitely many sets  $R(u)$  which are not congruent modulo the action of  $\Gamma$ . The union of noncongruent sets  $R(u)$  forms a fundamental domain of  $\Gamma_L$  in  $L_P$ . Assume that the characteristic polynomial  $p(x)$  of  $g$  is irreducible over  $K$ . Let  $p(\epsilon) = 0$ . In Section 6, the problem of finding a system of

fundamental units in  $F/K$  is solved for some families of fields  $F = \mathbb{Q}(\epsilon)$  with signature  $(0, n)$ ,  $n \leq 4$ , by reducing it, as explained in [Section 4](#), to the problem of finding a set of generators of  $\Gamma_L$ . Here, the families of fields with signature  $(0, n)$  are obtained from some families of fields with signature  $(n, 0)$  by complexification, that is, by replacing a real parameter  $t \in \mathbb{Z}$  by a nonreal complex parameter  $m \in \mathbb{Z}_K$ .

In [Example 5.3](#) (and, for  $\delta = -1$ , in [Example 6.1](#)), the following result is obtained.

**THEOREM 1.1.** *Let  $d$  be a square-free positive integer and let  $K = \mathbb{Q}(\sqrt{-d})$ . Let  $\{1, \omega\}$  be the standard  $\mathbb{Z}$ -basis of  $\mathbb{Z}_K$ . Let  $p(x) = x^2 - mx + \delta$ , where nonreal  $m \in \mathbb{Z}_K$ ,  $|m| \geq 4$ , and  $\delta \in \mathbb{Z}_K^\times$ . Assume that either  $m^2 - 4\delta$  or  $m^2/4 - \delta$  is a square-free ideal in  $\mathbb{Z}_K$ . Let  $p(\epsilon) = 0$  and  $F = \mathbb{Q}(\epsilon)$ .*

*Then  $\{1, \omega, \epsilon^{-1}, \epsilon^{-1}\omega\}$  is a  $\mathbb{Z}$ -basis of  $\mathbb{Z}_F$  and  $\mathbb{Z}_F^\times/\mu_F = \langle \epsilon \rangle$ .*

Similar results (Theorems [5.5](#) and [5.7](#)) are proved in [Examples 5.4](#) and [5.6](#), where  $\Gamma = B_d/\mu_K$  and  $B_d$  is the extended Bianchi group (see [[19](#), [20](#)]). Complexification of the family of simplest cubic fields of Shanks [[15](#)] leads to the following result obtained in [Example 6.2](#).

**THEOREM 1.2.** *Let  $d$  be a square-free positive integer and let  $K = \mathbb{Q}(\sqrt{-d})$ . Let  $f(x) = x^3 - mx^2 - (m+3)x - 1$ , where nonreal  $m \in \mathbb{Z}_K$ ,  $|m| \geq \sqrt{20} + 3$ . Assume that  $m^2 + 3m + 9$  is a square-free ideal in  $\mathbb{Z}_K$ . Let  $f(\epsilon) = 0$  and  $F = \mathbb{Q}(\epsilon)$ .*

*Then  $\{1, \epsilon, \epsilon^2\}$  is a  $\mathbb{Z}_K$ -basis of  $\mathbb{Z}_{F/K}$  and  $\mathbb{Z}_F^\times/\mu_F = \langle \epsilon, \epsilon + 1 \rangle$ .*

In [Example 6.4](#), the fundamental domain of  $\Gamma_L$  in  $L_P$  is found for the family of the simplest quartic fields of Gras [[8](#)]. By complexification of this family, in [Example 6.5](#), we prove the following.

**THEOREM 1.3.** *Let  $d$  be a square-free positive integer and let  $K = \mathbb{Q}(\sqrt{-d})$ . Let  $f(x) = x^4 - 2mx^3 - 6x^2 + 2mx + 1$ , where nonreal  $m \in \mathbb{Z}_K$ ,  $\gcd(m, 2) = 1$ , and  $|m| \geq \sqrt{84}$ . Assume that  $m^2 + 4$  is a square-free ideal in  $\mathbb{Z}_K$ . Let  $f(\epsilon) = 0$  and  $F = \mathbb{Q}(\epsilon)$ .*

*Then  $\{1, \epsilon, (\epsilon^2 - 1)/2, \epsilon(\epsilon^2 - 1)/2\}$  is a  $\mathbb{Z}_K$ -basis of  $\mathbb{Z}_{F/K}$ , and  $\mathbb{Z}_F^\times/\mu_F = \langle \epsilon, (\epsilon - 1)/(\epsilon + 1), (\epsilon - \epsilon^{-1})/2 \rangle$ .*

Note that the families of algebraic number fields  $F$  considered in the theorems above are parameterized by complex parameters  $m = a + \omega b \in \mathbb{Z}_K$ ,  $a, b \in \mathbb{Z}$ , or by three real parameters  $a$ ,  $b$ , and  $d$ .

In [[23](#)], Algorithm II is used to find a system of fundamental units in a two-parameter family of complex cubic fields. In [[24](#)], it is used to find a system of fundamental units in some families of algebraic number fields  $F$  of degree less than or equal to 4, which have at least one real embedding. Thus, the present paper, where Algorithm II is applied only to the totally complex algebraic number fields, can be considered as a complement of [[24](#)].

**2. Fundamental domains and  $K$ -tessellation.** Almost all the definitions in this section and in [Section 3](#) are similar to the corresponding definitions from [[24](#), Sections 2 and 3]. We reproduce them here for completeness.

Let  $n \geq 2$  be a positive integer. Let  $V_n$  be the vector space of Hermitian  $n \times n$  matrices. A complex matrix  $X \in V_n$  if and only if  $X = X^* = \overline{X}^T$ . The real dimension of  $V_n$  is  $N = n^2$ .

The action of  $g \in G = \mathrm{GL}(n, \mathbb{C})$  on  $X \in V_n$  is given by

$$X \mapsto X[g] = g^* X g. \quad (2.1)$$

For a subset  $S$  of  $V_n$ , denote  $S[g] = \{X[g] \in V_n : X \in S\}$ .

The one-dimensional subspaces of  $V_n$  form the real projective space  $V$  of dimension  $N-1$ , so that for any fixed nonzero  $X \in V_n$ , all the vectors  $kX \in V_n$ ,  $0 \neq k \in \mathbb{R}$ , represent one point in  $V$ . Denote by  $\mathcal{H} \subset V$  the set of (positive) definite elements of  $V$  and by  $\mathcal{B}$  the boundary of  $\mathcal{H}$  ( $\mathcal{B}$  can be identified with nonnegative elements of  $V$  of rank less than  $n$ ). The group  $G$  preserves both  $\mathcal{H}$  and  $\mathcal{B}$  as does its arithmetic subgroup  $\mathrm{GL}(n, \mathbb{Z}_K)$ .

The space  $V_n$  (and  $V$ ) can be also identified with the set of Hermitian forms  $A[x] = x^* A x$ ,  $A \in V_n$ ,  $x \in \mathbb{C}^n$ . With each point  $a = (a_1, \dots, a_n)^T \in \mathbb{C}^n$ , we associate the matrix  $A = a a^* \in \mathcal{B}$  and the Hermitian form

$$A[x] = |\langle a, x \rangle|^2 = |\bar{a}_1 x_1 + \dots + \bar{a}_n x_n|^2. \quad (2.2)$$

Here,  $\langle a, x \rangle = \overline{\langle x, a \rangle} = a^* x$ . For  $g \in G$ , we have  $\langle ga, x \rangle = a^* g^* x = \langle a, g^* x \rangle$ .

Let  $w = (1, 0, \dots, 0)^T$  and  $W = w w^*$ . Then  $\langle w, x \rangle^2 = x_1^2$  and  $W[g] = U = u u^*$ , where  $u = g^* w$ .

Denote by  $G_\infty$  and  $\Gamma_\infty$  the stabilizers of  $w$  in  $G$  and  $\Gamma = \mathrm{GL}(n, \mathbb{Z}_F)/\mu_F$ , respectively. Then

$$G_\infty = \{g \in G : gw = w\} = \{g \in G : g_1 = w\}, \quad (2.3)$$

where  $g_1$  is the first column of  $g$ . Thus,  $g \in G_\infty$  if and only if  $W[g^*] = W$ .

We will say that  $A \in V$  is *extremal* if  $|A[x]| \geq |A[w]| = |a_{11}|^2$  for any  $x \in \mathbb{Z}_K^n$ ,  $x \neq (0, \dots, 0)$ . Let  $\mathcal{A}_n = \{X \in V : X[w] \neq 0\}$ . It is clear that  $\mathcal{H} \subset \mathcal{A}_n$ . For  $X \in \mathcal{A}_n$ , we will say that  $\mathrm{ht}(X) = |\det(X)|^{1/n}/|X[w]|$  is the *height* of  $X$  and, for a subset  $S$  of  $V$ , we define the *height* of  $S$  as  $\mathrm{ht}(S) = \max \mathrm{ht}(X)$ ,  $X \in S$ .

The elements of  $\mathcal{A}_n$  will be normalized so that  $X[w] = 1$ . For a fixed  $g \in \Gamma$ , the set  $\{X \in \mathcal{A}_n : |X[gw]| < 1\}$  is called the *g-strip*. It is clear that the *gh-strip* coincides with the *g-strip* for any  $h \in \Gamma_\infty$ . Since  $X[gw] \in \mathbb{R}$  for any  $g \in G$ , the boundary of the *g-strip* consists of two planes  $X[gw] = \pm 1$ . The plane

$$L^+(gw) = L^+(g) = \{X \in \mathcal{A}_n : X[gw] = 1\} \quad (2.4)$$

is the boundary of the *g-strip*, which cuts  $\mathcal{H}$ . Let  $\mathcal{R}_w$  be the set of all extremal points of  $V$ . Denote

$$K_n = K(w) = \mathcal{H} \cap \mathcal{R}_w. \quad (2.5)$$

Note that  $K(w) \subset \mathcal{A}_n$  is bounded by the planes  $L^+(g)$ . If  $h \in \Gamma_\infty$ , then  $X[hw] = X[w]$  and, therefore,  $\mathrm{ht}(X[h]) = \mathrm{ht}(X)$ . Thus,

$$K_n[h] = K_n, \quad h \in \Gamma_\infty. \quad (2.6)$$

By (2.6),  $K_n[hg] = K_n[g]$  for any  $g \in \Gamma$  and  $h \in \Gamma_\infty$ . Thus, the sets  $K_n[g]$  are parameterized by the classes  $\Gamma_\infty \backslash \Gamma$  or by primitive vectors  $u = g^{-1} h^{-1} w = g^{-1} w$ , so that  $\pm u$

represent the same  $K_n[g]$ . The sets  $K_n[g]$ ,  $g \in \Gamma_\infty \setminus \Gamma$ , form a tessellation of  $\mathcal{H}$  which will be called the  $K$ -tessellation. It is clear that the  $K$ -tessellation of  $\mathcal{H}$  is  $\Gamma$ -invariant.

Let  $w_{2n} = (1, 0, \dots, 0)^T \in \mathcal{P}_{2n}$ . Denote by  $K(w_{2n})$  the set of  $w_{2n}$ -extremal points in  $\mathcal{P}_{2n}$ . Denote  $\mathcal{P}_H = \{\hat{X} : X \in \mathcal{H}\}$  and  $K_H = K(w_{2n}) \cap \mathcal{P}_H$ . Let  $X \in \mathcal{A}_n$ . A Hermitian matrix can be reduced to a diagonal form by a unitary transformation. Hence  $\det(\hat{X}) = \det^2(X)$  and, therefore,  $\text{ht}(\hat{X}) = (\text{ht}(X))^2$ , where  $\text{ht}(\hat{X})$  is the height of  $\hat{X} \in \mathcal{P}_{2n}$  (see [24]). It follows that  $X \rightarrow \hat{X}$  is a bijection between the  $K$ -tessellations of  $\mathcal{H}$  and  $\mathcal{P}_H$ . In Sections 5 and 6, to show that  $X \in \mathcal{H}$  is extremal, we will show that  $\hat{X}$  is Minkowski-reduced (see, e.g., [6, pages 396–397]).

**3. Axes of elements of  $G$ .** Let  $g \in G$ . Let  $ga_i = \lambda_i a_i$ ,  $i = 1, \dots, n$ , where, for simplicity, we assume that  $\lambda_i \neq \lambda_j$  if  $i \neq j$ . Here,  $a_i$  is an eigenvector of  $g$  corresponding to its eigenvalue  $\lambda_i$ . Assume that  $\langle a_i, w \rangle \neq 0$ ,  $i = 1, \dots, n$ . Then we can choose  $a_i$  so that

$$\langle a_i, w \rangle = 1, \quad i = 1, \dots, n. \quad (3.1)$$

$g \in \Gamma$  is said to be  $K$ -irreducible if its characteristic polynomial is irreducible over the field  $K$ . If  $g \in \Gamma$  is  $K$ -irreducible, then all its eigenvalues are distinct. Let  $\lambda_k \neq \pm 1$ . Let  $P = (a_1, \dots, a_n)$  be the matrix with columns  $a_1, \dots, a_n$ , and let  $H = \text{diag}(\lambda_1, \dots, \lambda_n)$ . Then  $g = PHP^{-1}$ .

The totally geodesic submanifold  $L_P$  of  $\mathcal{H}$  fixed by  $g = PHP^{-1}$  will be called the *axis* of  $g$ . The dimension of  $L_P$  is  $n - 1$ . A point  $q \in L_P$  can be represented as

$$q = \sum_{k=1}^n \mu_k A_k, \quad A_k = a_k a_k^*, \quad \mu_k \geq 0, \quad \sum_{k=1}^n \mu_k = 1. \quad (3.2)$$

It can also be identified with the set of Hermitian forms in  $\mathcal{A}_n$

$$q[x] = \sum_{k=1}^n \mu_k A_k[x] = \sum_{k=1}^n \mu_k |\langle x, a_k \rangle|^2, \quad \mu_k \geq 0, \quad \sum_{k=1}^n \mu_k = 1. \quad (3.3)$$

Hence

$$\det q = \mu_1 \cdots \mu_n |\det P|^2. \quad (3.4)$$

It follows from (3.3) that  $L_P$  is the axis of  $h \in G$  if and only if  $a_i$ ,  $i = 1, \dots, n$ , are eigenvectors of  $h$ . Hence, the axis of  $g$  depends only on its set of eigenvectors, that is, on  $P$ , but not on the eigenvalues of  $g$ .

Thus,  $L_P$  is the simplex with vertices  $A_k$ ,  $k = 1, \dots, n$ . All the faces of  $L_P$  belong to  $\mathcal{B}$ . Note that  $L_P[g^*] = L_P$ .

Denote  $K_n(g^{-1}w) = K_n[g]$  and

$$R(g^{-1}w) = K_n[g] \cap L_P \neq \emptyset, \quad g \in \Gamma_\infty \setminus \Gamma. \quad (3.5)$$

The sets  $R(u)$ ,  $u = g^{-1}w$ , form a tessellation of  $L_P$  which is invariant modulo the action of  $\Gamma$  since the  $K$ -tessellation of  $\mathcal{H}$  is  $\Gamma$ -invariant. We say that this tessellation is *periodic* if there are only a finite number of noncongruent sets  $R(u)$  modulo the action

of  $\text{Stab}(L_P, \Gamma)$ . In that case, the union of all noncongruent sets  $R(u)$  is a fundamental domain of  $\text{Stab}(L_P, \Gamma)$ . The number of noncongruent sets  $R(u)$  in the tessellation of  $L_P$  will be called the *period length*.

Let  $N_P(x) = \langle x, a_1 \rangle \cdots \langle x, a_n \rangle$ , where  $\langle x, a_k \rangle = x^* a_k$ . Define

$$v(L_P) = \inf \left| \frac{N_P(gw)}{\det P} \right|, \quad (3.6)$$

where the infimum is taken over all  $g \in \Gamma$ . It is clear that  $v(L_P) = v(L_{MP}[h])$  for any  $h \in \Gamma$  and  $M = \text{diag}(\mu_1, \dots, \mu_n)$ , where  $\mu_1, \dots, \mu_n \in \mathbb{C}$  and  $\mu_1 \cdots \mu_n \neq 0$ . The projective invariant  $v(L_P)$  is well known in the geometry of numbers (see, e.g., [3] or [9]).

Let  $n = 2$ . The approximation constants  $\sup v(L_P)$  are known for  $d = 1, 2, 3, 5, 6, 7, 11, 15, 19$  (see, e.g., [20, 22], where in the cases of  $d = 5, 6$ , and  $15$ ,  $\Gamma$  is the extended Bianchi group). For  $d = 1, 3$ , and  $11$ , more information is available (see [12, 13, 14, 17, 18]). Thus, when  $d = 1$ , it is proved in [17, 18] that if  $v(L_P) > 1/2$ , then  $v(L_P) = (4 - |m|^{-4})^{-1/2}$  or  $14.76^{-1/4}$ , where  $(m, m')$  is a solution of the Diophantine equation  $(m\overline{m'})^2 + (m'\overline{m})^2 = |m|^2 + |m'|^2$  in nonzero  $m, m' \in \mathbb{Z}_K$ , the ring of integers of the Gaussian field  $K = \mathbb{Q}(i)$ .

A point  $q_m \in L_P$  is said to be the *summit* of  $L_P$  if  $|\det(q_m)| = \max |\det(q)|$ , the maximum being taken over all  $q \in L_P$ . It is clear that if  $R = L_P \cap K_n(w) \neq \emptyset$ , then  $q_m \in R$ . The following two lemmas are analogous to [24, Lemmas 5 and 6].

**LEMMA 3.1.** *Let  $L_P$  be the totally geodesic manifold fixed by  $g \in G$  and defined by (3.3), where  $ga_i = \lambda_i a_i$ . Let  $P = (a_1, \dots, a_n)$  be the matrix with columns  $a_1, \dots, a_n$ . Then*

$$q_m = \frac{1}{n} \sum_{k=1}^n A_k \quad (3.7)$$

is the summit of  $L_P$ ,

$$\begin{aligned} \text{ht}(L_P) &= \frac{1}{n} \left| \frac{\det P}{N_P(w)} \right|^{2/n}, \\ v(L_P) &= \inf (n \text{ht}(L_P[g]))^{-n/2}, \quad g \in \Gamma. \end{aligned} \quad (3.8)$$

**LEMMA 3.2.** *Let  $L_P$  be the totally geodesic manifold fixed by  $g \in G$  and defined by (3.3), where  $ga_i = \lambda_i a_i$ . Then*

$$v(L_P) = \inf (n \text{ht}(L_P[g_j]))^{-n/2}, \quad L_P \cap K_n(g_j w) \neq \emptyset, \quad g_j \in \Gamma. \quad (3.9)$$

Assume that  $L_P \cap K_n(gw) \neq \emptyset$ , where  $g \in \Gamma$ . Denote

$$h_n = \inf (\text{ht}(X)), \quad X \in K_n. \quad (3.10)$$

Since  $L_P[g] \cap K_n(w) \neq \emptyset$ , by Lemma 3.1,

$$\text{ht}(L_P[g]) = \text{ht}(L_{g^*P}) = \frac{1}{n} \left| \frac{\det P}{N_{g^*P}(w)} \right|^{2/n} > h_n. \quad (3.11)$$

But  $N_{g^*P}(x) = \langle x, g^*a_1 \rangle \cdots \langle x, g^*a_n \rangle = \langle gx, a_1 \rangle \cdots \langle gx, a_n \rangle$ . Hence  $N_{g^*P}(w) = N_P(gw)$ .

A vector  $gw \in \mathbb{Z}_K^n$ , such that  $L_P \cap K_n(gw) \neq \emptyset$ , will be called a *convergent* of  $L_P$ . We have proved the following.

**THEOREM 3.3.** *If a vector  $u$  is a convergent of  $L_P$  (i.e., if  $L_P \cap K_n(u) \neq \emptyset$ ), then*

$$|N_P(u)| < C_n^{n/2} |\det P|, \quad (3.12)$$

where  $C_n = 1/(nh_n)$ . Hence, if  $L_P$  cuts infinitely many sets  $K_n(u)$ , then this inequality has infinitely many solutions in  $u \in \mathbb{Z}_K^n$ .

A component of the boundary of a set  $R(u)$  of codimension one will be called a *face* of  $R(u)$ .

Algorithm II from [24] can be applied in  $\mathcal{H}$ . In this case, a simplex  $L \subset \mathcal{H}$  has vertices at  $A_i \in \mathcal{B}$ , where  $A_i = a_i a_i^*$ , for  $i = 1, \dots, n$ , and  $\det(a_1, \dots, a_n) \neq 0$ .

The following result shows that any matrix  $g$  in the stabilizer of the simplex  $L$  is uniquely determined by the first row of  $g$ .

**PROPOSITION 3.4.** *Let  $f(x)$  be an irreducible polynomial over  $\mathbb{Z}_K$  of degree  $n$  with coefficients in  $\mathbb{Z}_K$ . Let  $E^*$  be the companion matrix of  $f(x)$ . Let  $L_P$  be the axis of  $E$ . Let  $g = (g_{ij}) \in G_L$ , the torsion-free subgroup of the stabilizer of  $L_P$  in  $G$ . Then*

$$g = g_{11}I + g_{12}E + \cdots + g_{1n}E^{n-1}. \quad (3.13)$$

**PROOF.** The first row of  $E^{i-1}$  is the standard unit vector  $e_i = (0, \dots, 0, 1, 0, \dots, 0)$ .  $\square$

**4. Fundamental units in  $\mathbb{Z}_F$  and  $\mathbb{Z}_{F/K}$ .** Let  $\mathbb{Z}_F$  be the ring of integers of the field  $F$ . Assume that  $\mathbb{Z}_F$  is a free  $\mathbb{Z}_K$ -module. Let  $\{1, \alpha_2, \dots, \alpha_n\}$  be a  $\mathbb{Z}_K$ -basis of  $\mathbb{Z}_F$ . Then the  $\mathbb{Z}$ -basis of  $\mathbb{Z}_F$  is  $\{1, \omega, \alpha_2, \omega\alpha_2, \dots, \alpha_n, \omega\alpha_n\}$ . Let  $a_1 = (1, \alpha_2, \dots, \alpha_n)^T$ . Let  $\gamma \in \mathbb{Z}_F$ . Then  $\gamma\alpha_j = \sum m_{jk}\alpha_k$  or  $\gamma a_1 = M_\gamma a_1$ , where  $\alpha_1 = 1$ ,  $m_{jk} \in \mathbb{Z}_K$ , and  $M_\gamma = (m_{jk})$  is a square matrix of order  $n$ . Let  $\sigma_i$  be the  $n$  distinct embeddings of  $F/K$  in  $\mathbb{C}$ . Let  $a_k = \sigma_k(a_1)$  and  $\gamma_k = \sigma_k(\gamma)$ , where  $\gamma_1 = \gamma$ . Then  $\gamma_k a_k = M_\gamma a_k$  for  $k = 1, \dots, n$ . Thus,  $a_k$  is an eigenvector of  $M_\gamma$  corresponding to its eigenvalue  $\gamma_k$ . It is clear that the map  $\gamma \mapsto M_\gamma$  is an isomorphism of the ring of integers  $\mathbb{Z}_{F/K}$  and the commutative ring of  $\mathbb{Z}_K$ -integral square matrices of order  $n$  with the common axis  $L_P$ . The relative norm of  $\gamma$  equals  $\det(M_\gamma)$  so that  $\gamma$  is a unit in  $\mathbb{Z}_{F/K}$  if and only if  $M_\gamma \in \text{GL}_n(\mathbb{Z}_K)$ . The torsion-free subgroup  $\Gamma_L$  of the stabilizer of  $L_P$  is isomorphic to  $\mathbb{Z}_{F/K}^\times / \mu_{F/K}$ . Thus, the problem of finding a system of fundamental units of  $F/K$  is equivalent to the problem of finding a set of generators of  $\Gamma_L$ . The analog of the (multidimensional continued fraction) Algorithm II introduced in [24] can be used to solve the latter problem. In Section 6, a set of generators of  $\Gamma_L$ , and, therefore, a system of fundamental units, is found in some families of relative extensions  $F/K$  of relative degree  $n \leq 4$  and in the fields  $F$ .

Let  $\hat{a}_1 = (1, \omega, \alpha_2, \omega\alpha_2, \dots, \alpha_n, \omega\alpha_n)$ . Let  $m_{jk} = b_{jk} + \omega c_{jk}$ , where  $b_{jk}, c_{jk} \in \mathbb{Z}$ . Then  $y\alpha_j = \sum b_{jk}\alpha_k + \sum c_{jk}\omega\alpha_k$ . Let  $d_1 = (d+1)/4$ . If  $d \equiv 3 \pmod{4}$ , then  $y\omega\alpha_j = \sum (-d_1 c_{jk})\alpha_k + \sum (b_{jk} + c_{jk})\omega\alpha_k$ , and  $y\omega\alpha_j = \sum (-dc_{jk})\alpha_k + \sum b_{jk}\omega\alpha_k$  otherwise. Denote  $\hat{M}_y = (\hat{m}_{jk})$ , where

$$\hat{m}_{jk} = \begin{cases} \begin{bmatrix} b_{jk} & c_{jk} \\ -d_1 c_{jk} & b_{jk} + c_{jk} \end{bmatrix} & \text{if } d \equiv 3 \pmod{4}, \\ \begin{bmatrix} b_{jk} & c_{jk} \\ -dc_{jk} & b_{jk} \end{bmatrix} & \text{otherwise.} \end{cases} \quad (4.1)$$

Then  $y_k \hat{a}_k = \widehat{M}_y \hat{a}_k$ , where  $\hat{a}_k = \sigma_k(\hat{a}_1)$ , for  $k = 1, \dots, n$ , and  $(\widehat{a_k})$  is also an eigenvector of  $\widehat{M}_y$  corresponding to its eigenvalue  $\bar{y}_k$ . Let  $\hat{L}_P$  be the axis of  $\widehat{M}_y$  in  $\mathcal{P}_{2n}$ . It is clear that the map  $y \mapsto \widehat{M}_y$  is an isomorphism of the ring of integers  $\mathbb{Z}_F$  and the commutative ring of  $\mathbb{Z}$ -integral square matrices of order  $2n$  with the common axis  $\hat{L}_P$ . The norm of  $y$  equals  $\det(\widehat{M}_y)$  so that  $y$  is a unit in  $\mathbb{Z}_F$  if and only if  $\widehat{M}_y \in \text{GL}_{2n}(\mathbb{Z})$ . The torsion-free subgroup  $\hat{\Gamma}_L$  of the stabilizer of  $\hat{L}_P$  is isomorphic to  $\mathbb{Z}_F^\times / \mu_F$ . Thus, the problem of finding a system of fundamental units of  $F$  is equivalent to the problem of finding a set of generators of  $\hat{\Gamma}_L$ . Note that  $\det(\widehat{M}_y) = |\det(M_y)|^2$  since  $N(y) = |N_{F/K}(y)|^2$ .

We have proved the following.

**LEMMA 4.1.** *Let  $d > 0$  be a square-free integer. Let  $\mathbb{Z}_K$  be the ring of integers of the field  $K = \mathbb{Q}(\sqrt{-d})$ . Let  $F$  be an extension of  $K$ . Let  $\mathbb{Z}_F$  be the ring of integers of the field  $F$ . Assume that  $\mathbb{Z}_F$  is a free  $\mathbb{Z}_K$ -module. Then a system of fundamental units of the relative extension  $F/K$  is a system of fundamental units of  $F$ .*

**5.  $2 \times 2$  Hermitian matrices.** In this section, we consider a model of the three-dimensional hyperbolic space which is similar to the Klein model of the hyperbolic plane used in Example 1 from [23] or [24].

When  $n = 2$ , the space  $V$  consists of all Hermitian  $2 \times 2$  matrices

$$A = \begin{bmatrix} x_1 & x_2 + ix_3 \\ x_2 - ix_3 & x_4 \end{bmatrix}, \quad (5.1)$$

where  $(x_1, x_2, x_3, x_4) \in \mathbb{R}^4$ . The formula

$$\rho(g)A = g^* A g = A[g], \quad (5.2)$$

where  $g \in \text{PSL}(2, \mathbb{C})$ ,  $A \in V$ , defines a representation  $\rho$  of the group  $\text{PSL}(2, \mathbb{C})$  in the space  $V$ . All the transformations  $\rho(g)$  as well as the complex conjugation  $A \mapsto \bar{A}$  preserve the form  $\Delta(A) = \det(A) = x_1 x_4 - x_2^2 - x_3^2$ . The space  $\mathcal{H}$  of (positive) definite matrices in  $V$ , considered with the action of the group  $\rho(\text{PSL}(2, \mathbb{C}))$  extended by the complex conjugation, is isomorphic to the three-dimensional hyperbolic space.



The action of  $g = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \in \mathrm{SL}(2, \mathbb{C})$  on  $(z, t) \in H^3$  is given by

$$g(z, t) = \left( \frac{(\alpha z + \beta)(\overline{\gamma z + \delta}) + \alpha \overline{\gamma} t^2}{|\gamma z + \delta|^2 + |\gamma|^2 t^2}, \frac{t}{|\gamma z + \delta|^2 + |\gamma|^2 t^2} \right) \quad (5.3)$$

(see, e.g., [7, page 569]). Thus, the height of  $g(z, t)$  is  $t(|\gamma z + \delta|^2 + |\gamma|^2 t^2)^{-1}$ .

**LEMMA 5.1** [7, page 409]. Define  $\psi : \mathcal{H} \rightarrow H^3$  by

$$\psi(A) = \left( \frac{x_2 + ix_3}{x_1}, \frac{\sqrt{\Delta(A)}}{|x_1|} \right), \quad A = \begin{bmatrix} x_1 & x_2 + ix_3 \\ x_2 - ix_3 & x_4 \end{bmatrix} \in \mathcal{H}. \quad (5.4)$$

Then  $\psi(A[g]) = g\psi(A)$ . Hence  $\psi$  induces a bijection of  $\mathcal{H}$  and  $H^3$ , which commutes with the action of  $\mathrm{PSL}(2, \mathbb{C})$ .

The height of  $\psi(A)$  is  $\sqrt{\Delta(A)}/|x_1|$ .

Let  $g \in \Gamma = \mathrm{PGL}(2, \mathbb{Z}_K)$ . Denote

$$K(\infty) = \{(z, t) \in H^3 : |g_{21}z + g_{22}|^2 + |g_{12}|^2 t^2 \geq 1, g = (g_{ij}) \in \Gamma\}. \quad (5.5)$$

We have the following.

**THEOREM 5.2.** Let  $\psi$  be the bijection of  $\mathcal{H}$  and  $H^3$  defined in Lemma 5.1. Then  $\psi(K_2) = K(\infty)$  and, therefore,  $\psi$  is a bijection between the  $K$ -tessellations of  $\mathcal{H}$  and  $H^3$ . Hence  $\psi$  is a bijection between the tessellations of the axis of  $g \in \mathrm{SL}(2, \mathbb{C})$  in  $H^3$  and the axis of  $g$  in  $\mathcal{H}$ . Thus, Algorithm I from [22] in  $H^3$  coincides with Algorithm II from [24] in  $\mathcal{H}$  in this case.

**PROOF.** The height of  $\psi(A) \in H^3$  equals  $\mathrm{ht}(A)$  in  $\mathcal{H}$ . Hence  $\psi(K_2) = K(\infty)$ .  $\square$

Let  $F$  be a field with signature  $(0, 2)$ , which has an imaginary quadratic subfield  $K$ , so that  $\mathbb{Z}_F$ , the ring of integers of  $F$ , is a free  $\mathbb{Z}_K$ -module. Lemma 4.1 and Theorem 5.2 imply that to find a fundamental unit in  $F$ , one can apply either Algorithm I in  $H^3$  or Algorithm II in  $\mathcal{H}$ . But, in general, it is easier to apply Algorithm II in  $\mathcal{H}$  than Algorithm I in  $H^3$ , since to find the point of intersection of the axis  $L_P$  of  $g \in \Gamma$  with the boundary of  $K(w)$  in  $\mathcal{H}$ , one has to solve a system of linear equations. On the other hand, to solve this problem in  $H^3$ , we have to find the point of intersection of a semicircle with a hemisphere. However, the application of Algorithm I in  $H^3$  in Examples 5.3, 5.4, and 5.6 is quite simple. In the next section, we apply Algorithm II in  $\mathcal{H}$  to find a system of fundamental units in some families of fields with signature  $(0, n)$ ,  $n \leq 4$ . The period length in any of Examples 5.3, 5.4, 5.6, 6.1, 6.2, 6.4, 6.5 is one.

The discriminant of  $F$  is  $d_K^2 |d_{F/K}|^2$ , where  $d_K$  is the discriminant of  $K$  and  $d_{F/K}$  is the discriminant of the extension  $F/K$  (see, e.g., [5, page 209]). In all the examples below, we assume that  $\mathbb{Z}_F$  has a free basis over  $\mathbb{Z}_K$ . In the case when  $F/K$  is a quadratic extension, (i.e.,  $F = K(\sqrt{\Delta})$ ), as in Examples 5.3, 5.4, and 5.6, such a basis exists if and only if  $\mathfrak{D}_{F/K}/\sqrt{\Delta}$  is a principal ideal (of  $\mathbb{Z}_F$ ) generated by an element of  $K$  (see, e.g., [5, page 222]). Here,  $\mathfrak{D}_{F/K}$  is the relative different.

**EXAMPLE 5.3.** Let

$$U = \begin{bmatrix} 1 & m \\ 0 & 1 \end{bmatrix}, \quad S = \begin{bmatrix} 0 & -\delta \\ 1 & 0 \end{bmatrix}, \quad W = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \quad (5.6)$$

where  $m = a + \omega b \in \mathbb{Z}_K$ ,  $\delta \in \mathbb{Z}_K^\times$ . Then reflection  $S$  fixes the unit hemisphere  $\phi_1$  with equation  $|z|^2 + t^2 = 1$  in the hyperbolic space  $H^3$ . The axes of reflections  $U' = UW$  and  $S' = WS$  are perpendicular to the axis  $A_g$  of  $g = US = U'S'$  in  $H^3$ . Let  $M$  and  $M_1$  be the points of intersection of  $A_g$  with axes of  $S'$  and  $U'$ , respectively. Let  $R_0$  be the arc  $MM_1$  on  $A_g$ . Since the axis of  $U'$  is the vertical line in  $H^3$  through the point  $m/2 \in \mathbb{C}$ ,  $R_0 \subset A_g \cap K(\infty)$  if and only if  $M \in K(\infty)$ . For  $|m|$  fixed, it can be easily seen that the height of  $M$  is smallest when  $A_g$  and the axis of  $S'$  lie in the same vertical plane in  $H^3$ . It is clear that the part of the unit hemisphere  $\phi_1$  which lies above  $|z| \leq 1/2$  belongs to  $K(\infty)$  for any  $d$ . It follows that  $M \in K(\infty)$  and  $R_0 \subset A_g \cap K(\infty)$  if  $|m| \geq 4$ . Thus,  $g$  is a generator of the torsion-free subgroup of the stabilizer of  $A_g$  in  $\text{PGL}(2, \mathbb{Z}_F)$  and, by [Theorem 5.2](#), of  $\Gamma_L$ , provided  $|m| \geq 4$ . The characteristic polynomial of  $g$  is  $p(x) = x^2 - mx + \delta$  with discriminant  $d(p) = m^2 - 4\delta$ . Let  $p(\epsilon) = 0$ . Let  $F = \mathbb{Q}(\epsilon)$ . If either the ideal  $d(p) = m^2 - 4\delta$  or  $d(p)/4$  is square-free in  $\mathbb{Z}_K$ , then  $\{1, \epsilon\}$  is a  $\mathbb{Z}_K$ -basis of  $\mathbb{Z}_{F/K}$ , and  $\{1, \omega, \epsilon^{-1}, \epsilon^{-1}\omega\}$  is a  $\mathbb{Z}$ -basis of  $\mathbb{Z}_F$ . By [Lemma 4.1](#),  $\mathbb{Z}_F^\times/\mu_F = \langle \epsilon \rangle$ . Note that  $\hat{a} = (1, \omega, \epsilon^{-1}, \epsilon^{-1}\omega)^T$  is an eigenvector of  $\hat{g}$  corresponding to its eigenvalue  $\epsilon$ . We have proved [Theorem 1.1](#).

**EXAMPLE 5.4.** Let  $d = 5$  or  $6$ . Let  $B_d$  be the extended Bianchi group (see [\[19\]](#)). Let  $c = 1 + \sqrt{-5}$  for  $d = 5$  and  $c = \sqrt{-6}$  for  $d = 6$ . In that case, the floor of an isometric fundamental domain of  $B_d$  in  $H^3$  lies in the hemisphere  $\phi_1$ , which is the unit hemisphere defined in [Example 5.3](#), and hemisphere  $\phi_2$  with center  $c/2$  and radius  $1/\sqrt{2}$  (see [\[19, page 308\]](#)). Let

$$U = \begin{bmatrix} 1 & m \\ 0 & 1 \end{bmatrix}, \quad S_6 = \begin{bmatrix} c & 2 \\ 2 & -c \end{bmatrix}, \quad S_5 = \begin{bmatrix} c & \bar{c} \\ 2 & -c \end{bmatrix}, \quad W = \begin{bmatrix} -1 & c \\ 0 & 1 \end{bmatrix}, \quad (5.7)$$

where nonreal  $m = a + b\sqrt{-d} \in \mathbb{Z}_K$ . The axes of reflections  $U' = UW$  and  $S'_d = WS_d$  are perpendicular to the axis  $A_g$  of  $g = US_d = U'S'_d$ . As in [Example 5.3](#), it can be shown that  $g$  is a generator of the torsion-free subgroup of the stabilizer of  $A_g$  in  $B_d$ , provided  $|m| \geq \sqrt{6}$ .

The characteristic polynomial of  $g$  is  $p(x) = x^2 - 2mx + 2$  with discriminant  $d(p) = 4(m^2 - 2)$ . Let  $p(\alpha) = 0$  and  $F = K(\alpha)$ .  $g \notin \text{GL}(\mathbb{Z}_K)$  since  $\det g = 2$ , but  $(1/2)g^2 \in \text{SL}(\mathbb{Z}_K)$ . Hence  $\alpha^2/2 = m\alpha - 1 \in \mathbb{Z}_F^\times$ . (Similarly, the case of  $g' = US'_d = U'S_d$  can be considered. In this case,  $\alpha^2/2 = m\alpha + 1 \in \mathbb{Z}_F^\times$ .) If either  $d = 5$  and  $(a - b)$  is odd, or  $d = 6$  and  $a$  is even, then  $N_{F/K}((\alpha + c)/2) \in \mathbb{Z}_K$ . If  $d(p)/8 = m^2/2 - 1$  is a square-free ideal in  $\mathbb{Z}_K$ , then  $\{(\alpha + c)/2, 1\}$  is a  $\mathbb{Z}_K$ -basis of  $\mathbb{Z}_{F/K}$ ,  $\{(\alpha + c)/2, \omega(\alpha + c)/2, 1, \omega\}$  is a  $\mathbb{Z}$ -basis of  $\mathbb{Z}_F$ , and  $\mathbb{Z}_F^\times/\mu_F = \langle \epsilon \rangle$ , where  $\epsilon = \alpha^2/2 = m\alpha - 1$ . We have proved the following.

**THEOREM 5.5.** Let  $d = 5$  or  $6$ . Let  $c = 1 + \sqrt{-5}$  for  $d = 5$  and  $c = \sqrt{-6}$  for  $d = 6$ . Let  $\{1, \omega\}$  be the standard  $\mathbb{Z}$ -basis of  $\mathbb{Z}_K$ , where  $K = \mathbb{Q}(\sqrt{-d})$ . Let  $\alpha$  be a root of  $p(x) = x^2 - 2mx + 2\delta$ , where nonreal  $m = a + b\sqrt{-d} \in \mathbb{Z}_K$ ,  $|m| \geq \sqrt{6}$ , and  $\delta = \pm 1$ . Let  $F = K(\alpha)$ .

Assume that either  $d = 5$  and  $(a - b)$  is odd, or  $d = 6$  and  $a$  is even. If  $m^2/2 - \delta$  is a square-free integer in  $\mathbb{Z}_K$ , then  $\{(\alpha + c)/2, \omega(\alpha + c)/2, 1, \omega\}$  is a  $\mathbb{Z}$ -basis of  $\mathbb{Z}_F$ , and  $\mathbb{Z}_F^\times/\mu_F = \langle \epsilon \rangle$ , where  $\epsilon = \alpha^2/2 = m\alpha - 1$ .

Note that  $\hat{a} = ((\alpha + c)/2, \omega(\alpha + c)/2, 1, \omega)^T$  is an eigenvector of  $\hat{g}$  corresponding to its eigenvalue  $\alpha$ .

**EXAMPLE 5.6.** Let  $d = 15$ . Let  $K = \mathbb{Q}(\sqrt{-15})$ . The floor of an isometric fundamental domain of  $B_{15}$  in  $H^3$  lies in  $\phi_1$ , which is the unit hemisphere defined in [Example 5.3](#), and the hemisphere  $\phi_2$  with center  $\omega/2$  and radius  $1/\sqrt{2}$  (see [\[22, page 2313\]](#)). Let

$$U = \begin{bmatrix} 1 & m \\ 0 & 1 \end{bmatrix}, \quad S = \begin{bmatrix} -1 - \bar{\omega} & 1 + \omega \\ 1 + \omega & 1 + \bar{\omega} \end{bmatrix}, \quad W = \begin{bmatrix} -1 & \omega \\ 0 & 1 \end{bmatrix}, \quad (5.8)$$

where  $m = a + \omega b \in \mathbb{Z}_K$ . Let  $A_g$  be the axis of  $g = US = U'S'$  in  $H^3$ , where  $U' = UW$  and  $S' = WS$ . As above, it can be shown that the arc  $R = K(\infty) \cap A_g$  is a fundamental domain of the torsion-free subgroup of the stabilizer of  $A_g$  in  $B_{15}$  on  $A_g$ , and that  $g$  is a generator of this subgroup, provided  $|m| \geq 4$ .

The characteristic polynomial of  $g$  is  $p(x) = x^2 - m(1 + \omega)x + 3$  with discriminant  $d(p) = (1 + \omega)^2(m^2 + \omega)$ . Let  $p(\alpha) = 0$ . Let  $F = \mathbb{Q}(\alpha)$ .  $\alpha \notin \mathbb{Z}_F^\times$  since  $\det(g) = 3$  and  $g \notin \text{GL}(\mathbb{Z}_K)$ . But  $(1/3)g^2 \in \text{SL}(\mathbb{Z}_K)$ . Hence  $\epsilon = \alpha^2/3 = m\alpha(1 + \omega)/3 - 1 \in \mathbb{Z}_K^\times$ . Let  $\beta = (\alpha - 1 - \bar{\omega})/(1 + \omega)$ .  $N_{F/K}(\beta) = -1 - 2b + 2(a + b)/\bar{\omega} \in \mathbb{Z}_F$  if and only if  $(a + b) \in 2\mathbb{Z}$ . Assume that  $(a + b)$  is even. Then  $\{\beta, 1\}$  is a  $\mathbb{Z}_K$ -basis of  $\mathbb{Z}_{F/K}$ , and  $\{\beta, \beta\omega, 1, \omega\}$  is a  $\mathbb{Z}$ -basis of  $\mathbb{Z}_F$ , provided  $m^2 + \omega$  is a square-free ideal in  $\mathbb{Z}_K$ . We have obtained the following.

**THEOREM 5.7.** Let  $K = \mathbb{Q}(\sqrt{-15})$ . Let  $m = a + \omega b \in \mathbb{Z}_K$ , where  $(a + b) \in 2\mathbb{Z}$ . Let  $p(x) = x^2 - m(1 + \omega)x + 3$  and  $p(\alpha) = 0$ . Assume that  $|m| \geq 4$  and  $m^2 + \omega$  is a square-free ideal in  $\mathbb{Z}_K$ . Let  $F = \mathbb{Q}(\alpha)$ . Let  $\beta = (\alpha - 1 - \bar{\omega})/(1 + \omega)$ . Then  $\{\beta, \beta\omega, 1, \omega\}$  is a  $\mathbb{Z}$ -basis of  $\mathbb{Z}_L$  and  $\mathbb{Z}_L^\times/\mu_L = \langle \epsilon \rangle$ , where  $\epsilon = \alpha^2/3 = m\alpha(1 + \omega)/3 - 1$ .

Note that  $(\beta, \beta\omega, 1, \omega)^T$  is an eigenvector of  $\hat{g}$  corresponding to its eigenvalue  $\alpha$ .

**6. Complexification of families of totally real cyclic fields.** In this section, systems of fundamental units are found in some families of totally complex fields of degrees 4, 6, and 8, which are cyclic extensions of imaginary quadratic fields. These families are obtained by replacing the real parameter  $t \in \mathbb{Z}$  in [Examples 1 and 2](#) from [\[24\]](#) and [Example 6.4](#) by a nonreal complex parameter  $m \in \mathbb{Z}_K$ .

**EXAMPLE 6.1.** Let  $f(x) = x^2 - mx - 1$ , where  $m \in \mathbb{Z}_K$ . Let  $f(\epsilon) = 0$ . If  $m \in \mathbb{Z}$ , then we obtain the family of real quadratic fields  $\mathbb{Q}(\epsilon)$  considered in [\[24, Example 1\]](#). Assume that  $m \notin \mathbb{Z}$  and that either  $m^2 + 4$  or  $m/4 + 1$  is a square-free ideal in  $\mathbb{Z}_F$ . Then  $\{1, \epsilon\}$  is a  $\mathbb{Z}_K$ -basis of  $\mathbb{Z}_{F/K}$ , where  $F = K(\epsilon)$ . The family of fields considered here is a particular case of the family of fields from [Example 1](#).

Let nonreal  $m = a + ib = a_1 + \omega b_1 = \epsilon - 1/\epsilon$ ,  $\epsilon = u + iv$ , where  $a, b, u, v \in \mathbb{R}$  and  $a_1, b_1 \in \mathbb{Z}$ . Let  $\eta = \eta_0\sqrt{d} = v/(u^2 + v^2 + 1)$  and  $c = |m|^2 + 4$ . Then  $c/b = (1 + 4\eta^2)/\eta$ .

Hence  $\eta_0$  is a root of the polynomial  $r(x) = 4b_1dx^2 - c_1x + b_1$ , where  $c_1 = 2c$  if  $d \equiv 3 \pmod{4}$  and  $c_1 = c$  otherwise. The discriminant of  $r(x)$  is  $d_r = c_1^2 - 16db_1^2 = |d(f)|^2$ , where  $d(f)$  is the discriminant of  $f$ .

Let  $E^*$  be the companion matrix of  $f(x)$ . Let  $L_P$  be the axis of  $E$ . Let  $\Gamma_L$  be the torsion-free subgroup of the stabilizer of  $L_P$  in  $\Gamma$ . Let  $E^*a_i = \epsilon_i a_i$  and  $A_i = a_i a_i^*$ , where  $\epsilon_0 = \epsilon$ ,  $a_i = (1, \epsilon_i)$ ,  $i = 0, 1$ . Then  $q(\mu_0, \mu_1) = \mu_0 A_0 + \mu_1 A_1$ ,  $\mu_i > 0$ ,  $\mu_0 + \mu_1 = 1$  is an equation of  $L_P$ . Let  $F_1$  be the intersection of  $L_P$  and  $L^+(E)$ . Then  $F_1 = q(|\epsilon|^2, 1)$  in the projective coordinates. Let

$$h = \begin{bmatrix} 1 & -\overline{m} \\ 0 & 1 \end{bmatrix}, \quad F_0 = \begin{bmatrix} 1 & 2i\eta \\ -2i\eta & 1 \end{bmatrix}. \quad (6.1)$$

Assume that  $|\epsilon| < 1$ . Then  $|\eta| < |\epsilon| < (|m|^2 - 4)^{-1/2}$  since  $|m| = |\epsilon + 1/\epsilon| \leq |\epsilon| + 1/|\epsilon|$ .

Thus, if  $|m| \geq \sqrt{20}$ , then  $F_0 = F_1[h]$  is Minkowski-reduced. Hence,  $F_1$  and  $F_2 = F_1[E]$  are extremal. Thus, the interval  $R = [F_1, F_2] = L_P \cap K(w)$  is a fundamental domain of  $\Gamma_L$  on  $L_P$ . It follows that  $\Gamma_L = \langle E \rangle$  and, therefore,  $\mathbb{Z}_F^\times / \mu_F = \langle \epsilon \rangle$ .

A point  $X = (x_{ij}) \in \mathcal{P}_n$  is said to be rational over a field  $M$  if all  $x_{ij} \in M$ . A subset  $S$  of  $\mathcal{P}_n$  is rational over  $M$  if the set of rational points of  $S$  is dense in  $S$ . If the summit of the axis  $L_P$  of  $g \in \text{GL}_n(\mathbb{Z})$  is rational over a field  $M$ , then  $L_P$  is rational over  $M$  (see [24, Section 4]). By (3.7), the summit of  $L_P$  is

$$q_m = \begin{bmatrix} 1 & \frac{\overline{m}}{2} \\ \frac{m}{2} & \frac{|m|^2 + |d(f)|}{4} \end{bmatrix}. \quad (6.2)$$

Hence,  $\widehat{L_P} = \{\widehat{X} : X \in L_P\} \subset \mathcal{P}_4$  is rational over the real quadratic field  $\mathbb{Q}(|d(f)|) = \mathbb{Q}(\eta_0)$ .

**EXAMPLE 6.2.** Let  $m \in \mathbb{Z}_K$ . Let  $\Gamma = \text{GL}(3, \mathbb{Z}_K)$ . Here, we consider complexification of the simplest cubic fields (see [15] and [24, Example 2]). These are the relative cyclic fields of relative discriminant  $d_{F/K} = (m^2 + 3m + 9)^2$ . Assume that  $m \notin \mathbb{Z}$ . The sextic field  $F = K(\epsilon_1)$  is generated by a root  $\epsilon_1$  of  $f(x) = x^3 - mx^2 - (m+3)x - 1$ . Assume that  $m^2 + 3m + 9$  is a square-free ideal in  $\mathbb{Z}_K$ . Then  $\{1, \epsilon_1, \epsilon_1^2\}$  is a  $\mathbb{Z}_K$ -basis of  $\mathbb{Z}_F$  and both units  $\epsilon_1$  and  $\epsilon_2 = \sigma(\epsilon_1) = -1/(1 + \epsilon_1)$  are the roots of this polynomial.

Let nonreal  $m = a + ib = a_1 + \omega b_1$ ,  $\epsilon_1 = u + iv$ , where  $a_1, b_1 \in \mathbb{Z}$ ,  $a, b, u, v \in \mathbb{R}$ . Let  $\eta = \eta_0 \sqrt{d} = v/(u^2 + v^2 + u + 1)$ . Since  $b/\eta - a - 3 = |\epsilon_1|^2 + |\epsilon_2|^2 + |\epsilon_3|^2$ ,  $\eta$  does not depend on a chosen root of  $f(x)$ . Denote  $c = |m|^2 + 3a + 9 \in (1/2)\mathbb{Z}$ , so that  $c - b \in \mathbb{Z}$ . Then  $\eta_0$  is the real root of the polynomial  $r(x) = c_1 dx^3 - 9b_1 dx^2 + c_1 x - b_1$ , where  $c_1 = 2c$  if  $d \equiv 3 \pmod{4}$ , and  $c_1 = c$  otherwise. The discriminant of  $r(x)$  is  $d_r = -4d(c_1^2 - 27db_1^2)^2$ .

Let  $E^*$  be the companion matrix of  $f(x)$  and let  $E_1 = E + I$ . Let  $L_P$  be the axis of  $E$ . Let  $\Gamma_L$  be the torsion-free subgroup of the stabilizer of  $L_P$  in  $\Gamma$ . Let  $E^*a_i = \epsilon_i a_i$  and  $A_i = a_i a_i^*$ , where  $a_i = (1, \epsilon_i, \epsilon_i^2)$ ,  $i = 1, 2, 3$ . Then  $q(\mu_1, \mu_2, \mu_3) = \mu_1 A_1 + \mu_2 A_2 + \mu_3 A_3$ ,  $\mu_i > 0$ ,  $\mu_1 + \mu_2 + \mu_3 = 1$ , is an equation of  $L_P$ .

Denote  $E_2 = EE_1^{-1}$ . Let  $F_1$  be the intersection of  $L_P$ ,  $L^+(E)$ , and  $L^+(E_2)$ , and let  $G_1$  be the intersection of  $L_P$ ,  $L^+(E)$ , and  $L^+(E_1)$ . Let  $m = 3n + k$ , where  $n, k = k_1 + \omega k_2 \in \mathbb{Z}_K$ ,  $|k_1| \leq 1$  and  $|k_2| \leq 1$ . Denote

$$h = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 - \bar{m} \\ 0 & 0 & 1 \end{bmatrix}, \quad F_0 = F_1[h] = \begin{bmatrix} 1 & -\alpha & \bar{\alpha} \\ -\bar{\alpha} & 1 & \alpha \\ \alpha & \bar{\alpha} & 1 \end{bmatrix}, \quad (6.3)$$

where  $\alpha = 2\eta/(\eta - i)$ , and

$$h_1 = \begin{bmatrix} 1 & 0 & -1 - \bar{n} \\ 0 & 1 & -2\bar{n} \\ 0 & 0 & 1 \end{bmatrix}, \quad G_0 = \begin{bmatrix} 1 & \gamma_{13} & -i\bar{k}\eta \\ \bar{\gamma}_{13} & 1 & \frac{\bar{k}(1+i\eta)}{2} \\ ik\eta & \frac{k(1-i\eta)}{2} & \gamma_{33} \end{bmatrix}, \quad (6.4)$$

where

$$\gamma_{13} = -\frac{1}{2} - \frac{3}{2}\eta i, \quad \gamma_{33} = \frac{c}{6} \frac{(1-3\eta^2)^2}{1+9\eta^2} + \frac{|k|^2}{3}. \quad (6.5)$$

Assume that  $|\epsilon| < 1/2$ , where  $\epsilon = \epsilon_1$ . Then  $|\eta| < |\epsilon| < ((|m| - 3)^2 - 4)^{-1/2}$  since  $|m| = |\epsilon - 1/(\epsilon + 1) + (\epsilon + 1)/\epsilon| \leq 1 + |\epsilon| + 1/|\epsilon| + 1/(1 - |\epsilon|) \leq 3 + |\epsilon| + 1/|\epsilon|$ . Note that  $\eta \rightarrow 0$  and, therefore,  $\alpha \rightarrow 0$  and  $\gamma_{13} \rightarrow -1/2$ , as  $|m| \rightarrow \infty$ .

Thus, if  $|m| \geq \sqrt{20} + 3$ , then  $F_0 = F_1[h]$  and  $G_0 = G_1[h_1]$  with

$$\det(F_0) = \left(1 - \frac{4\eta^2}{1 + \eta^2}\right)^3, \quad \det(G_0) = \frac{1}{8}(|t|^2 + 3a + 9) \frac{(1 - 3\eta^2)^3}{1 + 9\eta^2} \quad (6.6)$$

are Minkowski-reduced and, therefore,  $F_i, G_i$ ,  $i = 1, 2, 3$ , are extremal. Hence,  $R = L_P \cap K(w)$  is the hexagon with vertices at  $F_1, F_2 = F_1[E]$ ,  $F_3 = F_1[E_2]$ ,  $G_1, G_2 = G_1[E_1]$ ,  $G_3 = G_1[E]$ . The sides of  $R$  are identified as follows:  $E : F_1G_1 \rightarrow F_2G_3$ ;  $E_1 : F_3G_1 \rightarrow F_2G_2$ ;  $E_2 : F_1G_2 \rightarrow F_3G_3$ . Thus,  $R$  is a fundamental domain of  $\Gamma_L = \langle E, E_1 \rangle$  and, therefore,  $\mathbb{Z}_F^\times/\mu_F = \langle \epsilon, \epsilon + 1 \rangle$ . [Theorem 1.2](#) is proved.

Note that  $\widehat{L}_P = \{\widehat{X} : X \in L_P\} \subset \mathcal{P}_6$  is rational over the real cubic field  $\mathbb{Q}(\eta_0)$ . Also, note that  $F_1 = q(|\epsilon_1 + 1|^2, |\epsilon_1(\epsilon_1 + 1)|^2, |\epsilon_1|^2)$  and  $G_1 = q(1, |\epsilon_1 + 1|^2, |\epsilon_1|^2)$  in the projective coordinates, and if  $F_1 = q(\mu_1, \mu_2, \mu_3)$ , then  $F_2 = q(\mu_2, \mu_3, \mu_1)$  and  $F_3 = q(\mu_3, \mu_1, \mu_2)$ . The same relations hold for  $G_1, G_2$ , and  $G_3$ . For the summit  $q_m$  of  $L_P$ , we have

$$q_m = \frac{1}{3} \sum A_i = \frac{1}{3} \sum F_i = \frac{1}{3} \sum G_i. \quad (6.7)$$

**REMARK 6.3.** The properties of the vertices  $F_i$  and  $G_i$  of the fundamental domain  $R$  of  $\Gamma_L$  mentioned above, in the case of the simplest cubic fields, can be explained as follows.

Let  $m = t \in \mathbb{Z}$ . Then  $v = 0$  and  $F$  is the simplest cubic field. Let  $\text{Gal}(F) = \langle \sigma \rangle$ . Since

$$r = t^2 + 3t + 9 = \frac{(u^2 + u + 1)^3}{u^2(u + 1)^2}, \quad (6.8)$$

where  $u = \epsilon_1$ , the divisor  $\rho = u^2 + u + 1$  is ramified in  $F$ . Thus,  $\sigma(\rho)$  and  $\sigma^2(\rho)$  both are divisible by  $\rho$ , and therefore  $t_r = \text{trace}(\rho) = \rho + \sigma(\rho) + \sigma^2(\rho)$  is divisible by  $\rho$ . But,  $t_r \in \mathbb{Z}$ , hence  $t_r$  is divisible by  $r$ . It is easy to verify that  $t_r = r$ . Let  $\mu_i = \sigma^i(\rho)/r$ ,  $i = 0, 1, 2$ . Then

$$\mu_i > 0, \quad \sum \mu_i = 1. \quad (6.9)$$

The point  $F_2 = \sum \mu_i A_i$  belongs to  $L_P$  and it is integral since any entry of  $F_2$  has a form  $\sum \sigma^i(\rho\alpha)/r \in \mathbb{Z}$ . Since  $u, u+1 \in \mathbb{Z}_L^\times$  and  $\text{trace}(\rho^2/\epsilon^2) = 2r$ , where  $\epsilon = u, u+1$ , or  $u(u+1)$ , if we choose  $\mu_i = \sigma^i(\rho^2/\epsilon^2)/(2r)$ , then (6.9) holds, and we obtain one of the points  $G_k$ . Note that (6.9) for  $F_i$  can be written in the form

$$u^2 + (u+1)^2 + (u^2+u)^2 = (u^2+u+1)^2 \quad (6.10)$$

and, for  $G_i$ , in the form

$$u^2 + (u+1)^2 + 1 = 2(u^2+u+1). \quad (6.11)$$

**EXAMPLE 6.4.** Let  $t$  be an odd integer. Let  $f(x) = x^4 - 2tx^3 - 6x^2 + 2tx + 1$ . (Out of the four possible cases enumerated in [10, page 315], here we consider only Case 2.) Let  $f(\epsilon) = 0$  and  $\epsilon_1 = (\epsilon - 1)/(\epsilon + 1)$ . Then  $f(\epsilon_1) = f(-1/\epsilon) = f(-1/\epsilon_1) = 0$ . The discriminant of  $f(x)$  is  $d(f) = 4^4(t^2 + 4)^3$ . Let  $\theta = (\epsilon - \epsilon^{-1})/2$ . Then  $\theta^2 - t\theta - 1 = 0$ ,  $\theta = (t \pm \sqrt{t^2 + 4})/2$ , and  $N = \mathbb{Q}(\sqrt{d})$ ,  $d = t^2 + 4$ , is a quadratic subfield of the cyclic quartic field  $F = \mathbb{Q}(\epsilon)$ . If  $t^2 + 4$  is square-free, then  $\{1, \epsilon\}$  is a  $\mathbb{Z}_N$ -basis of  $\mathbb{Z}_{F/N}$ . Hence  $\{1, \epsilon, \theta, \theta\epsilon\}$  or  $\{1, \epsilon, (\epsilon^2 - 1)/2, (\epsilon^3 - \epsilon)/2\}$  is a  $\mathbb{Z}$ -basis of  $\mathbb{Z}_F$ , provided  $\theta$  is a fundamental unit of  $N$ . Let

$$\tau = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}. \quad (6.12)$$

Then  $a_0 = (1, \epsilon, (\epsilon^2 - 1)/2, (\epsilon^3 - \epsilon)/2)^T = \tau(1, \epsilon, \epsilon^2, \epsilon^3)^T$ . Let  $(E')^T$  be the companion matrix of  $f(x)$  and let  $E^T = \tau(E')^T \tau^{-1}$ . Then  $E^T a_i = \epsilon_i a_i$ , where  $a_i = (1, \epsilon_i, (\epsilon_i^2 - 1)/2, (\epsilon_i^3 - \epsilon_i)/2)$ ,  $\epsilon_0 = \epsilon$ ,  $\epsilon_2 = -1/\epsilon$ ,  $\epsilon_3 = -1/\epsilon_1$ . Denote  $E_1 = (E - I)(E + I)^{-1}$  and  $E_2 = (E - E^{-1})/2$ , where  $I$  is the identity matrix. Let  $L_P$  be the axis of  $E$  in  $\mathcal{P}_4$ . Let

$$E_- = \frac{1}{\sqrt{2}}(E - I), \quad E_+ = \frac{1}{\sqrt{2}}(E + I). \quad (6.13)$$

Then  $E_1 = E_- E_+^{-1}$  and  $E = E_- E_+ E_2^{-1}$ . Let  $\Delta_L = \langle E, E_1, E_2 \rangle$  and  $\Gamma'_L = \langle E_2, E_-, E_+ \rangle$ . Then  $\Delta_L$  is a subgroup of index two in  $\Gamma'_L$ . Thus, if we show that  $\Gamma'_L$  equals the extension of the torsion-free subgroup  $\Gamma_L$  of the stabilizer of  $L_P$  in  $\Gamma$  by  $E_-$ , then  $\Gamma_L = \Delta_L$ .

Let  $(\mu_0, \mu_1, \mu_2, \mu_3)$  be the coordinates of the point  $\sum_{k=1}^4 \mu_k A_k$ , where  $A_k = a_k a_k^T$ , in  $L_P$ . Let  $\delta = u^2 + 1$ , where  $u = \epsilon$ . Let  $i$  be defined modulo 4. Define

$$B_i = \frac{1}{\delta^3}(\beta_i^2, \beta_{i+1}^2, \beta_{i+2}^2, \beta_{i+3}^2), \quad (6.14)$$

where

$$\begin{aligned}\beta_0 &= (u-1)(u+1), & \beta_1 &= \sqrt{2}u(u+1), \\ \beta_2 &= u(u-1)(u+1), & \beta_3 &= \sqrt{2}u(u-1)(u+1).\end{aligned}\quad (6.15)$$

Define

$$C_i = \frac{1}{\delta^3} (y_i^2, y_{i+1}^2, y_{i+2}^2, y_{i+3}^2), \quad (6.16)$$

where

$$\begin{aligned}y_0 &= 2u, & y_1 &= \frac{1}{\sqrt{2}}(u-1)(u+1)^2, \\ y_2 &= 2u^2, & y_3 &= \frac{1}{\sqrt{2}}u(u-1)^2(u+1).\end{aligned}\quad (6.17)$$

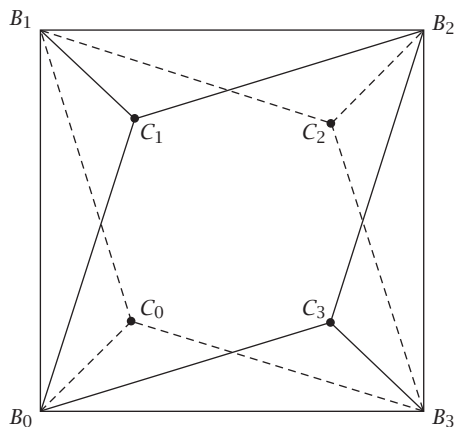
Let

$$h_B = \begin{bmatrix} 1 & 0 & 0 & -\overline{m} \\ 0 & 1 & -\overline{m} & -2 \\ 0 & 0 & 1 & -2\overline{m} \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad h_C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & -2\overline{m} \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad (6.18)$$

where  $m = t$ . Then  $B_0[h_B] = C_0[h_C] = I$ . Thus, all the points  $B_i, C_i$  are integral and, therefore, extremal, with  $\det(B_i) = \det(C_i) = 1$ . The point  $B_0$  is the intersection of  $L_P$ ,  $L^+(E)$ ,  $L^+(E_2^{-1})$ , and  $L^+(EE_2^{-1})$ , and  $B_1 = B_0[EE^{-1}]$ ,  $B_2 = B_0[E]$ ,  $B_3 = B_0[EE_+^{-1}]$ ,  $C_0 = B_0[E_2^{-1}]$ ,  $C_1 = B_0[E_+]$ ,  $C_2 = B_0[EE_2^{-1}]$ , and  $C_3 = B_0[E_-]$ .

The polytope  $R = L_P \cap K(w)$  is bounded by two quadrangles lying in  $L^+(E_2^{\pm 1})$  and eight triangles lying in  $L^+(\mathcal{G}^{\pm 1})$ ,  $\mathcal{G} = E_+, E_-, E_+E_2^{-1}, E_-E_2^{-1}$ . It has 8 vertices, 16 edges, and 10 faces. Note that at any vertex  $B_i$  of  $R$ , four faces of  $R$  meet, but at any  $C_i$ , only three do. The projections of  $R$  into a plane which is “parallel” to its quadrangular faces are shown in Figure 6.1. The faces of  $R$  are identified as follows:  $E_2 : C_0B_1C_2B_3 \rightarrow B_0C_1B_2C_3$ ;  $E_+ : B_0C_0B_3 \rightarrow C_1B_1B_2$ ;  $E_- : B_0C_0B_1 \rightarrow C_3B_3B_2$ ;  $E_2E_+^{-1} : B_1B_2C_2 \rightarrow B_0C_3B_3$ ;  $E_2E_-^{-1} : B_2B_3C_2 \rightarrow C_1B_0B_1$ . Thus,  $R$  is a fundamental domain of  $\Gamma'_L$  in  $L_P$ ,  $\Gamma'_L = \langle E_2, E_-, E_+ \rangle$ , and  $\Gamma_L = \langle E, E_1, E_2 \rangle$ . Hence,  $\mathbb{Z}_F^\times / \{\pm 1\} = \langle \epsilon, (\epsilon - 1)/(\epsilon + 1), (\epsilon - \epsilon^{-1})/2 \rangle$ .

**EXAMPLE 6.5.** Here, we consider complexification of the cyclic quartic fields from Example 6.4. Let  $f(x) = x^4 - 2mx^3 - 6x^2 + 2mx + 1$ , where  $m = a + ib = a_1 + \omega b_1 \in \mathbb{Z}_K$ ,  $\gcd(m, 2) = 1$ ,  $a_1, b_1 \in \mathbb{Z}$ ,  $a, b, u, v \in \mathbb{R}$ , and  $b \neq 0$ . Let  $f(\epsilon) = 0$ . Then  $F = \mathbb{Q}(\epsilon)$  is a totally complex field of degree eight. Let  $\epsilon = u + iv$  and  $\eta = \eta_0\sqrt{d} = v/(u^2 + v^2 + 1)$ . Denote  $c = 2|m|^2 + 8 \in \mathbb{Z}$ . Then  $\eta_0$  is a real root of the polynomial  $r(x) = b_1(16d^2x^4 + 24dx^2 + 1) - c_1(4dx^3 + x)$ , where  $c_1 = 2c$  if  $d \equiv 3 \pmod{4}$ , and  $c_1 = c$  otherwise. The discriminant of  $r(x)$  is  $d_r = 256d^3(64db_1^2 - c_1^2)^3$ . Define  $E, E_1, E_2, E_-, E_+, h_B$ , and  $h_C$  as in Example 6.4.

FIGURE 6.1 Fundamental domain of  $\langle E_2, E_-, E_+ \rangle$ .

The point

$$C_0 = \frac{1}{(1+4\eta^2)(1+|\epsilon|^2)^3} \left( 4|\epsilon|^2, \frac{1}{2}|1+\epsilon|^4|1-\epsilon|^2, 4|\epsilon|^4, \frac{1}{2}|1-\epsilon|^4|1+\epsilon|^2 \right) \quad (6.19)$$

is the intersection of  $L_P$ ,  $L^+(E)$ ,  $L^+(E_+)$ , and  $L^+(EE_2)$ . Let  $C_0[h_C] = (c_{ij}) = (\bar{c}_{ji})$ . Then  $c_{ii} = 1$ ,  $i = 1, 2, 3, 4$ ,

$$\begin{aligned} c_{12} = -c_{34} = -2i\eta, \quad c_{14} = c_{23} = -4i\frac{\eta}{1+4\eta^2}, \\ c_{13} = -c_{24} = c_{12}c_{14} = -8\frac{\eta^2}{1+4\eta^2}. \end{aligned} \quad (6.20)$$

The point

$$B_0 = \frac{1}{(1+4\eta^2)(1+|\epsilon|^2)^3} (|\epsilon^2-1|^2, 2|\epsilon|^2|1+\epsilon|^2, |\epsilon|^2|\epsilon^2-1|^2, 2|\epsilon|^2|1-\epsilon|^2) \quad (6.21)$$

is the intersection of  $L_P$ ,  $L^+(E)$ ,  $L^+(E_+)$ , and  $L^+(E_2^{-1})$ . Let  $B_0[h_B] = (b_{ij}) = (\bar{b}_{ji})$ . Then  $b_{ii} = 1$ ,  $i = 1, 2, 3, 4$ ,

$$\begin{aligned} b_{12} = -b_{34} = -2i\eta, \quad b_{14} = b_{23} = 4i\frac{\eta}{1+4\eta^2}, \\ b_{13} = -b_{24} = b_{12}b_{14} = 8\frac{\eta^2}{1+4\eta^2}. \end{aligned} \quad (6.22)$$

Let  $\alpha = 2\epsilon/(1-\epsilon^2)$ . Then  $|m| = 2|\alpha - 1/\alpha| \leq 2(|\alpha| + 1/|\alpha|)$ . Hence, if  $|\alpha| < 1$ , then  $|\alpha| < 2(|m|^2 - 16)^{-1/2}$  and  $|\epsilon| + 1/|\epsilon| > (|m|^2 - 16)^{1/2}$ . Thus, if  $|\epsilon| < 1$ , then  $|\eta| < |\epsilon| < (|m|^2 - 20)^{-1/2}$ . It follows that  $B_0[h_B] \rightarrow I$  as  $|m| \rightarrow \infty$ . If  $|m| \geq \sqrt{84}$ , then  $|\eta| < |\epsilon| \leq 1/8$ ,  $B_0[h_B]$  and  $C_0[h_C]$  are Minkowski-reduced, and, therefore,  $B_k, C_k$ ,  $k = 1, \dots, 4$ ,



which are defined as in [Example 6.4](#), are extremal. Note that  $\det(B_k) = \det(C_k) = (1 - 4\eta^2)^6 / (1 + 4\eta^2)^4$ . Thus, the polytope  $R = L_P \cap K(w)$  is the same as in [Example 6.4](#),  $R$  is a fundamental domain of  $\Gamma'_L$  in  $L_P$ ,  $\Gamma'_L = \langle E_2, E_-, E_+ \rangle$ , and  $\Gamma_L = \langle E, E_1, E_2 \rangle$ . Hence,  $\mathbb{Z}_F^\times / \mu_F = \langle \epsilon, (\epsilon - 1)/(\epsilon + 1), (\epsilon - \epsilon^{-1})/2 \rangle$ . [Theorem 1.3](#) is proved.

Note that  $\widehat{L}_P = \{\widehat{X} : X \in L_P\} \subset \mathcal{P}_8$  is rational over the real quadric field  $\mathbb{Q}(\eta_0)$ .

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