

## ON BIRATIONAL MONOMIAL TRANSFORMATIONS OF PLANE

ANATOLY B. KORCHAGIN

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We study birational monomial transformations of the form  $\varphi(x : y : z) = (\varepsilon_1 x^{\alpha_1} y^{\beta_1} z^{\gamma_1} : \varepsilon_2 x^{\alpha_2} y^{\beta_2} z^{\gamma_2} : x^{\alpha_3} y^{\beta_3} z^{\gamma_3})$ , where  $\varepsilon_1, \varepsilon_2 \in \{-1, 1\}$ . These transformations form a group. We describe this group in terms of generators and relations and, for every such transformation  $\varphi$ , we prove a formula, which represents the transformation  $\varphi$  as a product of generators of the group. To prove this formula, we use birationally equivalent polynomials  $Ax + By + C$  and  $Ax^p + By^q + Cx^r y^s$ . If  $\varphi$  is the transformation which carries one polynomial onto another, then the integral powers of generators in the product, which represents the transformation  $\varphi$ , can be calculated by the expansion of  $p/q$  in the continued fraction.

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**1. Introduction.** Birational monomial transformations of the projective plane have already found a lot of applications. For example, such transformations are actively used for construction of real algebraic curves and surfaces (see, e.g., [1, 4, 5, 6, 8, 9]). We think that formula (3.6) will be helpful for description of construction of algebraic objects.

In Section 2 we give a little exposition on projective polynomials in three variables. In Section 3 we describe the birational monomial group in terms of its generators and relations and give the statement of a theorem of decomposition of birational monomial transformations. In Section 4 we give the proof of the theorem.

**2. Preliminaries.** A nonzero homogeneous polynomial of degree  $n$  in three variables,  $x, y, z$ , is the expression

$$f(x, y, z) = \sum_{\omega_1 + \omega_2 + \omega_3 = n} f_\omega x^{\omega_1} y^{\omega_2} z^{\omega_3}, \quad \omega = (\omega_1, \omega_2, \omega_3). \quad (2.1)$$

The convex hull of the set  $\{(\omega_1, \omega_2) \in \mathbb{R}^2 \mid f_\omega \neq 0\}$  is called the *Newton polygon* of the polynomial  $f(x, y, z)$  and is denoted as  $N(f)$ . The plane with coordinates  $(\omega_1, \omega_2)$  is called the *plane of Newton's polygons*.

Every polynomial  $f(x, y, 1)$  can be represented in the form  $f(x, y, 1) = x^i y^j \hat{f}(x, y, 1)$ , where  $i$  and  $j$  are nonnegative integers, and the polynomial  $\hat{f}(x, y, 1)$  has no factors  $x$  and  $y$ . If  $\varphi$  is a transformation, then clearly  $(f^\circ \varphi)^\wedge = (f \circ \varphi)^\wedge$ . It is also clear that the Newton polygon  $N(f^\circ)$  can be obtained from the Newton polygon  $N(f)$  by translation in the plane of Newton's polygons by the vector  $(-i, -j)$ .

**3. The birational monomial group.** Let  $(x : y : z)$  be homogeneous point coordinates in the projective plane  $\mathbb{K}P^2$  over a field  $\mathbb{K}$  and let  $(x, y)$  be affine coordinates in the affine chart  $\mathbb{K}^2 = \mathbb{K}P^2 \setminus \{z = 0\}$ . A projective transformation  $\varphi$  is defined by the

formula  $\varphi(x : y : z) = (\varphi_1(x, y, z) : \varphi_2(x, y, z) : \varphi_3(x, y, z))$ , where  $\varphi_1, \varphi_2, \varphi_3$  are homogeneous polynomials of the same degree, assumed to have no common factors. For the transformation  $\varphi(x : y : z)$ , we define its natural restriction  $\varphi(x, y)$  to the affine chart  $\mathbb{K}^2 = \mathbb{K}P^2 \setminus \{z = 0\}$  by the formula  $\varphi(x, y) = (\varphi_1(x, y, 1) / \varphi_3(x, y, 1), \varphi_2(x, y, 1) / \varphi_3(x, y, 1))$ .

Let  $\text{id} : \mathbb{K}P^2 \rightarrow \mathbb{K}P^2$  be the identity map. If  $\varphi$  is a birational transformation, then we denote as usual

$$\varphi^0 = \text{id}, \quad \underbrace{\varphi \circ \cdots \circ \varphi}_{n \text{ times}} = \varphi^n, \quad \underbrace{\varphi^{-1} \circ \cdots \circ \varphi^{-1}}_{n \text{ times}} = \varphi^{-n}. \quad (3.1)$$

Let  $r_1, r_2, r_3 : \mathbb{K}P^2 \rightarrow \mathbb{K}P^2$  be maps defined by formulas  $r_1(x : y : z) = ((-x) : y : z)$ ,  $r_2(x : y : z) = (x : (-y) : z)$ , and  $r_3(x : y : z) = (x : y : (-z))$ . The set of maps  $R = \{\text{id}, r_1, r_2, r_1 \circ r_2\}$  with the operation of composition of the maps, with generators  $r_1$  and  $r_2$ , and with relations

$$r_1^2 = r_2^2 = \text{id}, \quad r_1 \circ r_2 = r_2 \circ r_1, \quad (3.2)$$

is a group isomorphic to  $\mathbb{Z}_2 \times \mathbb{Z}_2$ . Note that  $r_3 = r_1 \circ r_2$ .

Let  $s_1, s_2, s_3 : \mathbb{K}P^2 \rightarrow \mathbb{K}P^2$  be maps defined by formulas  $s_1(x : y : z) = (x : z : y)$ ,  $s_2(x : y : z) = (z : y : x)$ , and  $s_3(x : y : z) = (y : x : z)$ . The set of maps  $S = \{\text{id}, s_1 \circ s_2, s_2 \circ s_1, s_1, s_2, s_1 \circ s_2 \circ s_1\}$  with the operation of composition of the maps, with generators  $s_1$  and  $s_2$ , and with relations

$$s_1^2 = s_2^2 = \text{id}, \quad s_1 \circ s_2 \circ s_1 = s_2 \circ s_1 \circ s_2, \quad (3.3)$$

is a group isomorphic to the symmetric group  $S_3$ . Note that  $s_3 = s_1 \circ s_2 \circ s_1$ .

Let  $hy$  be the birational transformation defined by the formula  $hy(x : y : z) = (x^2 : yz : xz)$ , whose inverse transformation is  $hy^{-1}(x : y : z) = (xz : xy : z^2)$ . Due to Newton, the transformation  $hy$  is called a *hyperbolism*. The set  $H = \{\dots, hy^{-2}, hy^{-1}, \text{id}, hy, hy^2, \dots\}$  of integral powers of  $hy$  is a free group isomorphic to  $\mathbb{Z}$ .

Let  $G = R * S * H$  be the free product of groups  $R$ ,  $S$ , and  $H$ . This means that the set of generators of  $G$  is the union of the generators of  $R$ ,  $S$ , and  $H$ , and the set of relations of  $G$  is the union of the relations of  $R$ ,  $S$ , and  $H$ .

**DEFINITION 3.1.** The factor group  $G/\mathcal{R}$  with generators  $r_1, r_2, s_1, s_2, hy$ , where  $\mathcal{R}$  is the system of relations

$$\mathcal{R} : \left\{ \begin{array}{l} r_1 \circ s_1 = s_1 \circ r_1, \\ r_2 \circ s_1 = s_1 \circ r_2 \circ r_1, \\ r_1 \circ s_2 = s_2 \circ r_1 \circ r_2, \\ r_2 \circ s_2 = s_2 \circ r_2, \\ r_1 \circ hy = hy \circ r_1 \circ r_2, \\ r_2 \circ hy = hy \circ r_2, \\ s_1 \circ hy = hy \circ s_1 \circ s_2 \circ hy \circ s_1, \\ hy \circ s_2 \circ hy = s_2, \end{array} \right. \quad (3.4)$$

is called the *group of birational monomial transformations* of  $\mathbb{K}P^2$  and denoted by  $T(\mathbb{K}P^2)$ .

The group of birational monomial transformations  $T(\mathbb{K}P^2)$  is a subgroup of the Cremona group  $\text{Cr}(\mathbb{K}P^2)$  (see [2, 3]).

Below in this paper, the word “transformation” without an adjective always means a “birational monomial transformation.”

Every transformation  $\varphi \in T(\mathbb{K}P^2)$  can be represented as a composition  $\varphi_1 \circ \dots \circ \varphi_s$ , where each of  $\varphi_1, \dots, \varphi_s$  is a positive integral power of one of the generators of the group  $T(\mathbb{K}P^2)$ , because  $r_1^{-1} = r_1$ ,  $r_2^{-1} = r_2$ ,  $s_1^{-1} = s_1$ ,  $s_2^{-1} = s_2$ , and  $hy^{-n} = s_2 \circ hy^n \circ s_2$ . Every transformation  $\varphi$  can be represented in the form

$$\varphi(x:y:z) = (\varepsilon_1 x^{\alpha_1} y^{\beta_1} z^{\gamma_1} : \varepsilon_2 x^{\alpha_2} y^{\beta_2} z^{\gamma_2} : x^{\alpha_3} y^{\beta_3} z^{\gamma_3}), \quad (3.5)$$

where  $\varepsilon_1, \varepsilon_2 \in \{-1, 1\}$ ;  $\alpha_i$ ,  $\beta_i$ , and  $\gamma_i$  are nonnegative integers; and the monomials  $x^{\alpha_1} y^{\beta_1} z^{\gamma_1}$ ,  $x^{\alpha_2} y^{\beta_2} z^{\gamma_2}$ ,  $x^{\alpha_3} y^{\beta_3} z^{\gamma_3}$  have no common factors. We stress this convention, for example,  $(hy \circ hy)(x:y:z) = (x^4 : xyz^2 : x^3z) = (x^3 : yz^2 : x^2z)$ , and accept only the last form. It means that one or two of  $\alpha_1, \alpha_2, \alpha_3$ , one or two of  $\beta_1, \beta_2, \beta_3$ , one or two of  $\gamma_1, \gamma_2, \gamma_3$  are equal to 0, and  $\alpha_1 + \beta_1 + \gamma_1 = \alpha_2 + \beta_2 + \gamma_2 = \alpha_3 + \beta_3 + \gamma_3$ . The integer  $\alpha_1 + \beta_1 + \gamma_1$  is a degree of the transformation  $\varphi$ . For example, the degree of the transformations  $r_1, r_2, s_1, s_2$  equals 1, and the degree of  $hy^n$  equals  $|n| + 1$ , where  $n \in \mathbb{Z}$ .

Denote the element  $s_3 \circ hy \circ s_3 \in T(\mathbb{K}P^2)$  as  $hx$ . Its inverse is  $hx^{-1} = s_2 \circ s_1 \circ hy \circ s_1 \circ s_2$ . In homogeneous coordinates it is defined by formulae  $hx(x:y:z) = (xz:yz^2:yz)$  and  $hx^{-1}(x:y:z) = (xy:yz:z^2)$ .

In the following theorem and below, the phrase “a polynomial  $f(x, y, 1)$  subjected to the transformation  $\varphi$  is carried onto the polynomial  $l(x, y, 1)$ ” means that  $l(x, y, 1) = [(f \circ \varphi^{-1})(x, y, z)|_{z=1}]^{\wedge}$ .

**THEOREM 3.2.** *Let  $p$  and  $q$  be mutually prime natural integers,  $0 < q < p$ . Let  $r$  and  $s$  be integers which satisfy the following conditions: (1)  $0 < r < p$ ,  $0 \leq s < q$ , (2)  $r/p + s/q < 1$ , and (3)  $r \equiv -q^{\phi(p)-1} \pmod{p}$  and  $s \equiv -p^{\phi(q)-1} \pmod{q}$ , where  $\phi(m)$  is the Euler function. Then every polynomial  $f(x, y, 1) = Ax^p + By^q + Cx^r y^s$ , where at least two of  $A, B, C$  are not zero, subjected to the transformation*

$$\varphi = \left( hx^{(1-(-1)^{k+1})/2} \circ hy^{(1-(-1)^k)/2} \right)^{a_k} \circ \dots \circ hx^{a_4} \circ hy^{a_3} \circ hx^{a_2} \circ hy^{a_1} \quad (3.6)$$

*is carried onto the polynomial  $l(x, y, 1) = Ax + By + C$ , where the integers  $a_1, a_2, \dots, a_k$  are provided by expansion of  $p/q$  in the continued fraction with adjusted last denominator*

$$\begin{aligned} \frac{p}{q} = a_1 + \frac{1}{a_2 + \frac{1}{\dots + \frac{1}{a_{k-1} + \frac{1}{(a_k + 1)}}}}; \\ (3.7) \end{aligned}$$

*in other words,  $l = (f \circ \varphi^{-1})^{\wedge}$ .*

**COROLLARY 3.3.** (1) If the polynomial  $f(x, y, 1)$  subjected to a transformation  $\psi$  is carried onto the polynomial  $\varepsilon_1 Ax + \varepsilon_2 By + \varepsilon_3 C$ , where  $\varepsilon_1, \varepsilon_2, \varepsilon_3 \in \{1, -1\}$ , then  $\psi = r_1^{(1/2)(1-\varepsilon_1)} \circ r_2^{(1/2)(1-\varepsilon_2)} \circ (r_1 \circ r_2)^{(1/2)(1-\varepsilon_3)} \circ \varphi$ .

(2) If the polynomial  $f(x, y, 1)$  subjected to a transformation  $\psi$  is carried onto the polynomial  $A + Bx + Cy$ ,  $Ay + B + Cx$ ,  $Ax + B + Cy$ ,  $A + By + Cx$ , or  $Ay + Bx + C$ , then  $\psi = s_1 \circ s_2 \circ \varphi$ ,  $\psi = s_2 \circ s_1 \circ \varphi$ ,  $\psi = s_1 \circ \varphi$ ,  $\psi = s_2 \circ \varphi$ , or  $\psi = s_1 \circ s_2 \circ s_1 \circ \varphi$ , respectively.

(3) If condition (2) of [Theorem 3.2](#) is changed to condition (2'),  $r/p + s/q > 1$ , and other conditions and notations are kept, and if the polynomial  $f(x, y, 1)$  subjected to a transformation  $\psi$  is carried onto the polynomial  $Ax + By + C$ , then  $\psi = \varphi \circ \text{tr} = \text{tr} \circ \varphi$ , where  $\text{tr} = s_1 \circ hy^{-1} \circ s_1 \circ s_2 \circ s_1 \circ hy$  is well-known standard (triangular) quadratic transformation  $\text{tr}(x : y : z) = (yz : xz : xy)$ .

**REMARK 3.4.** There is only one more possible case: when  $p = q = 1$ , which does not satisfy the theorem. In this case either  $r = s = 0$ , and the polynomial  $Ax + By + C$  is carried onto itself by the identity transformation:  $\varphi = \text{id}$ , or  $r = s = 1$ , and the polynomial  $Ax + By + Cxy$  is carried onto the polynomial  $Ax + By + C$  by the transformation  $\varphi = s_3 \circ \text{tr} = s_3 \circ s_1 \circ hy^{-1} \circ s_1 \circ s_2 \circ s_1 \circ hy$ .

**4. Proof of the theorem.** A birational monomial transformation  $\psi$  maps a polynomial  $f$  onto a polynomial  $(f \circ \psi^{-1})^\wedge$ . We find the connection between  $N(f \circ \psi^{-1})$  and  $N(f)$ .

Every transformation  $\psi^{-1}$  written in the form (3.5) induces a generic linear mapping  $A(\psi) : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  of the plane of Newton's polygons, which can be defined on any monomial. Namely, if  $g$  is a monomial, say  $g(x, y, z) = x^{\omega_1} y^{\omega_2} z^{\omega_3}$ , then

$$(g \circ \psi^{-1})(x, y, z) = x^{\alpha_1 \omega_1 + \alpha_2 \omega_2 + \alpha_3 \omega_3} y^{\beta_1 \omega_1 + \beta_2 \omega_2 + \beta_3 \omega_3} z^{\gamma_1 \omega_1 + \gamma_2 \omega_2 + \gamma_3 \omega_3}, \quad (4.1)$$

thus, the linear mapping  $A(\psi)$  is defined by the matrix

$$A_\psi = \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{pmatrix}. \quad (4.2)$$

And thus,  $N(f \circ \psi^{-1}) = A_\psi(N(f))$  for every polynomial  $f$ .

Remark that the generators of the birational monomial group have matrices

$$A_{\text{id}} = A_{r_1} = A_{r_2} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (4.3)$$

$$A_{s_1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad A_{s_2} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad A_{hy} = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}.$$

The set of matrices  $A(T(\mathbb{K}P^2)) = \{A_\varphi \mid \varphi \in T(\mathbb{K}P^2)\}$  is a subset in the linear group  $GL(3, \mathbb{R})$  of  $3 \times 3$ -matrices. The operation  $\diamond$  defined by the formula  $A_\varphi \diamond A_\psi = A_{\varphi \circ \psi}$  converts the set  $A(T(\mathbb{K}P^2))$  into a group, which is natural ( $\varphi \mapsto A_\varphi$ ) homomorphic image of  $T(\mathbb{K}P^2)$ .

It is clear that every birational monomial transformation  $\varphi$  induces one-to-one correspondence between monomials of the polynomials  $f$  and  $(f \circ \varphi^{-1})$ . Thus, to represent a transformation  $\varphi$  as a composition of generators of birational monomial group, it is enough to study the action of the transformation  $\varphi$  on a polynomial  $f$  whose Newton's polygon  $N(f)$  is a triangle with the area  $1/2$ .

We consider a polynomial  $f(x, y, 1) = Ax^p + By^q + Cx^r y^s$ , with  $ABC \neq 0$ , where  $p$  and  $q$  are mutually prime integers,  $0 < q < p$ , and  $r$  and  $s$  are integers, which satisfy the following conditions: (1)  $0 < r < p$ ,  $0 \leq s < q$ , (2)  $r/p + s/q < 1$ , and (3)  $r \equiv -q^{\phi(p)-1} \pmod{p}$  and  $s \equiv -p^{\phi(q)-1} \pmod{q}$ , where  $\phi(m)$  is the Euler function. The Newton polygon  $N(f)$  is the triangle with integer vertices  $(p, 0)$ ,  $(0, q)$ ,  $(r, s)$  which has no other integer points belonging to its interior and boundary but its three vertices. According to the Pick theorem [7], the area of such a triangle equals  $1/2$ . The genus of any curve  $f(x, y, 1) = Ax^p + By^q + Cx^r y^s = 0$  with such properties is zero and thus, all such curves are birationally equivalent.

Note that  $hy^{-a}(x : y : 1) = (x : x^a y : 1)$  and  $hx^{-a}(x : y : 1) = (xy^a : y : 1)$ . We evaluate  $(f \circ \varphi^{-1})(x, y, 1)$  as follows.

*The first step.*

$$\begin{aligned} (f \circ hy^{-a_1})(x, y, 1) &= f(x, x^{a_1} y, 1) = x^{a_1 q} (Ax^{b_1} + By^q + Cx^{c+a_1 d - a_1 q} y^d) \\ &= x^{u_1} y^{v_1} (Ax^{b_1} + By^q + Cx^{c_1} y^{d_1}), \end{aligned} \quad (4.4)$$

where  $u_1 = a_1 q$ ,  $v_1 = 0$ ,  $c_1 = c + a_1 d - a_1 q$ , and  $d_1 = d$ .

*The second step.*

$$\begin{aligned} (f \circ hy^{-a_1} \circ hx^{-a_2})(x, y, 1) &= (f \circ hy^{-a_1})(xy^{a_2}, y, 1) \\ &= x^{u_1} y^{a_2 u_1 + v_1 + a_2 b_1} (Ax^{b_1} + By^{b_2} + Cx^{c_1} y^{a_2 c_1 + d_1 - a_2 b_1}) \\ &= x^{u_2} y^{v_2} (Ax^{b_1} + By^{b_2} + Cx^{c_2} y^{d_2}), \end{aligned} \quad (4.5)$$

where  $u_2 = u_1$ ,  $v_2 = a_2 u_1 + v_1 + a_2 b_1$ ,  $c_2 = c_1$ , and  $d_2 = a_2 c_1 + d_1 - a_2 b_1$ .

*The third step.*

$$\begin{aligned} (f \circ hy^{-a_1} \circ hx^{-a_2} \circ hy^{-a_3})(x, y, 1) &= (f \circ hy^{-a_1} \circ hx^{-a_2})(x, x^{a_3} y, 1) \\ &= x^{u_3} y^{v_3} (Ax^{b_3} + By^{b_2} + Cx^{c_3} y^{d_3}), \end{aligned} \quad (4.6)$$

where  $u_3 = u_2 + a_2 v_2 + a_3 b_2$ ,  $v_3 = v_2$ ,  $c_3 = c_2 + a_3 d_2 - a_3 b_2$ , and  $d_3 = d_2$ .

*The fourth step.*

$$\begin{aligned} (f \circ hy^{-a_1} \circ hx^{-a_2} \circ hy^{-a_3} \circ hx^{-a_4})(x, y, 1) &= (f \circ hy^{-a_1} \circ hx^{-a_2} \circ hy^{-a_3})(xy^{a_4}, y, 1) \\ &= x^{u_1} y^{a_2 u_1 + v_1 + a_2 b_1} (Ax^{b_1} + By^{b_2} + Cx^{c_1} y^{a_2 c_1 + d_1 - a_2 b_1}) \\ &= x^{u_2} y^{v_2} (Ax^{b_1} + By^{b_2} + Cx^{c_2} y^{d_2}), \end{aligned} \quad (4.7)$$

where  $u_2 = u_1$ ,  $v_2 = a_2 u_1 + v_1 + a_2 b_1$ ,  $c_2 = c_1$ , and  $d_2 = a_2 c_1 + d_1 - a_2 b_1$ . We then proceed until the  $(k-1)$ th step.

*The  $(k-1)$ th step.* We have two cases.

The first case:  $k$  is even.

$$\begin{aligned} & (f \circ hy^{-a_1} \circ hx^{-a_2} \circ \cdots \circ hx^{-a_{k-2}} \circ hy^{-a_{k-1}})(x, y, 1) \\ &= (f \circ hy^{-a_1} \circ hx^{-a_2} \circ \cdots \circ hx^{-a_{k-2}})(x, x^{a_{k-1}}y, 1) \\ &= x^{u_{k-1}}y^{v_{k-1}}(Ax + By^{b_{k-2}} + Cx^{c_{k-1}}y^{d_{k-1}}), \end{aligned} \quad (4.8)$$

where  $u_{k-1} = u_{k-2} + a_{k-1}v_{k-2} + a_{k-1}b_{k-2}$ ,  $v_{k-1} = v_{k-2}$ ,  $c_{k-1} = c_{k-2} + a_{k-1}d_{k-2} - a_{k-1}b_{k-2}$ , and  $d_{k-1} = d_{k-2}$ .

The second case:  $k$  is odd.

$$\begin{aligned} & (f \circ hy^{-a_1} \circ hx^{-a_2} \circ \cdots \circ hy^{-a_{k-2}} \circ hx^{-a_{k-1}})(x, y, 1) \\ &= (f \circ hy^{-a_1} \circ hx^{-a_2} \circ \cdots \circ hy^{-a_{k-2}})(xy^{a_{k-1}}, y, 1) \\ &= x^{u_{k-1}}y^{v_{k-1}}(Ax^{b_{k-2}} + By + Cx^{c_{k-1}}y^{d_{k-1}}), \end{aligned} \quad (4.9)$$

where  $u_{k-1} = u_{k-2}$ ,  $v_{k-1} = a_{k-1}u_{k-2} + v_{k-2} + a_{k-1}b_{k-2}$ ,  $c_{k-1} = c_{k-2}$ , and  $d_{k-1} = a_{k-1}c_{k-2} + d_{k-2} - a_{k-1}b_{k-2}$ .

*The  $k$ th step.* We have two cases.

The first case:  $k$  is even.

$$\begin{aligned} & (f \circ hy^{-a_1} \circ hx^{-a_2} \circ \cdots \circ hy^{-a_{k-1}} \circ hx^{-a_k})(x, y, 1) \\ &= (f \circ hy^{-a_1} \circ hx^{-a_2} \circ \cdots \circ hy^{-a_{k-1}})(xy^{a_k}, y, 1) \\ &= x^{u_k}y^{v_k}(Ax + By + Cx^{c_k}y^{d_k}), \end{aligned} \quad (4.10)$$

where  $u_k = u_{k-1}$ ,  $v_k = a_ku_{k-1} + v_{k-1} + a_k$ ,  $c_k = c_{k-1}$ , and  $d_k = a_kc_{k-1} + d_{k-1} - a_k$ .

The second case:  $k$  is odd.

$$\begin{aligned} & (f \circ hy^{-a_1} \circ hx^{-a_2} \circ \cdots \circ hx^{-a_{k-1}} \circ hy^{-a_k})(x, y, 1) \\ &= (f \circ hy^{-a_1} \circ hx^{-a_2} \circ \cdots \circ hx^{-a_{k-1}})(x, x^{a_k}y, 1) \\ &= x^{u_k}y^{v_k}(Ax + By + Cx^{c_k}y^{d_k}), \end{aligned} \quad (4.11)$$

where  $u_k = u_{k-1} + a_kv_{k-1} + a_k$ ,  $v_k = v_{k-1}$ ,  $c_k = c_{k-1} + a_kd_{k-1} - a_k$ , and  $d_k = d_{k-1}$ .

This calculation shows that the integers  $a_1, a_2, a_3, \dots, a_k$  satisfy the Euclidean algorithm with adjusted last row

$$\begin{aligned} p &= a_1q + b_1, \\ q &= a_2b_1 + b_2, \\ b_1 &= a_3b_2 + b_3, \\ &\vdots \\ b_{k-4} &= a_{k-2}b_{k-3} + b_{k-2}, \\ b_{k-3} &= a_{k-1}b_{k-2} + 1, \\ b_{k-2} &= a_k + 1, \end{aligned} \quad (4.12)$$

which provides the desired continued fraction.

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Anatoly B. Korchagin: Department of Mathematics and Statistics, Texas Tech University, Lubbock, TX 79409-1042, USA

*E-mail address:* [korchag@math.ttu.edu](mailto:korchag@math.ttu.edu)

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