

## SOME VERSIONS OF ANDERSON'S AND MAHER'S INEQUALITIES II

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We are interested in the investigation of the orthogonality (in the sense of Birkhoff) of the range of an elementary operator and its kernel.

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**1. Introduction.** Let  $H$  be a separable infinite-dimensional complex Hilbert space and let  $B(H)$  denote the algebra of all bounded operators on  $H$  into itself. Given  $A, B \in B(H)$ , we define the generalized derivation  $\delta_{A,B} : B(H) \rightarrow B(H)$  by  $\delta_{A,B}(X) = AX - XB$  and the elementary operator derivation  $\Delta_{A,B} : B(H) \rightarrow B(H)$  by  $\Delta_{A,B}(X) = AXB - X$ . Denote  $\delta_{A,A} = \delta_A$ ,  $\Delta_{A,A} = \Delta_A$ .

In [1, Theorem 1.7], Anderson shows that if  $A$  is normal and commutes with  $T$ , then, for all  $X \in B(H)$ ,

$$\|T + \delta_A(X)\| \geq \|T\|. \quad (1.1)$$

It is shown in [9] that if the pair  $(A, B)$  has the Fuglede-Putnam property (in particular, if  $A$  and  $B$  are normal operators) and  $AT = TB$ , then, for all  $X \in B(H)$ ,

$$\|T + \delta_{A,B}(X)\| \geq \|T\|. \quad (1.2)$$

Duggal [3] showed that the above inequality (1.2) is also true when  $\delta_{A,B}$  is replaced by  $\Delta_{A,B}$ . The related inequality (1.1) was obtained by the author [10], showing that if the pair  $(A, B)$  has the Fuglede-Putnam property  $(FP)_{C_p}$ , then

$$\|T + \delta_{A,B}(X)\|_p \geq \|T\|_p \quad (1.3)$$

for all  $X \in B(H)$ , where  $C_p$  is the von Neumann-Schatten class,  $1 \leq p < \infty$ , and  $\|\cdot\|_p$  is its norm for all  $X \in B(H)$  and for all  $T \in C_p \cap \ker \delta_{A,B}$ . In all of the above results,  $A$  was not arbitrary. In fact, certain normality-like assumptions have been imposed on  $A$ . A characterization of  $T \in C_p$  for  $1 < p < \infty$ , which is orthogonal to  $R(\delta_A|_{C_p})$  (the range of  $\delta_A|_{C_p}$ ) for a general operator  $A$ , has

been carried out by Kittaneh [6], showing that if  $T$  has the polar decomposition  $T = U|T|$ , then

$$\|T + \delta_A(X)\|_p \geq \|T\|_p \quad (1.4)$$

for all  $X \in C_p$  ( $1 < p < \infty$ ) if and only if  $|T|^{p-1}U^* \in \ker \delta_A$ . By a simple modification in the proof of the above inequality, we can prove that this inequality is also true in the general case, that is, if  $T$  has the polar decomposition  $T = U|T|$ , then

$$\|T + \delta_{A,B}(X)\|_p \geq \|T\|_p \quad (1.5)$$

for all  $X \in C_p$  ( $1 < p < \infty$ ) if and only if  $|T|^{p-1}U^* \in \ker \delta_{B,A}$ . In Sections 1, 2, 3, and 4, we prove these results in the case where we consider  $E_{A,B}$  instead of  $\delta_{A,B}$ , which leads us to prove that if  $T \in C_p$  and  $\ker E_{A,B} \subseteq \ker E_{A,B}^*$ , then

$$\|T + E_{A,B}(X)\|_p \geq \|T\|_p \quad (1.6)$$

for all  $X \in C_p$  ( $1 < p < \infty$ ) if and only if  $T \in \ker E_{A,B}$ . In Sections 5, 6, and 7, we minimize the map  $\|S + E_{A,B}(X)\|_p$  and we classify its critical points.

**2. Preliminaries.** Let  $T \in B(H)$  be compact and let  $s_1(X) \geq s_2(X) \geq \dots \geq 0$  denote the singular values of  $T$ , that is, the eigenvalues of  $|T| = (T^*T)^{1/2}$  arranged in their decreasing order. The operator  $T$  is said to belong to the Schatten  $p$ -class  $C_p$  if

$$\|T\|_p = \left[ \sum_{i=1}^{\infty} s_j(T)^p \right]^{1/p} = [\operatorname{tr}(T)^p]^{1/p}, \quad 1 \leq p < \infty, \quad (2.1)$$

where  $\operatorname{tr}$  denotes the trace functional. Hence,  $C_1$  is the trace class,  $C_2$  is the Hilbert-Schmidt class, and  $C_{\infty}$  is the class of compact operators with

$$\|T\|_{\infty} = s_1(T) = \sup_{\|f\|=1} \|Tf\| \quad (2.2)$$

denoting the usual operator norm. For the general theory of the Schatten  $p$ -classes, the reader is referred to [7, 11].

Recall that the norm  $\|\cdot\|$  of the  $B$ -space  $V$  is said to be Gateaux differentiable at nonzero elements  $x \in V$  if

$$\lim_{t \rightarrow 0, t \in \mathbb{R}} \frac{\|x + t\gamma\| - \|x\|}{t} = \Re D_x(\gamma) \quad (2.3)$$

for all  $\gamma \in V$ . Here  $\mathbb{R}$  denotes the set of reals,  $\Re$  denotes the real part, and  $D_x$  is the unique support functional (in the dual space  $V^*$ ) such that  $\|D_x\| = 1$  and  $D_x(x) = \|x\|$ . The Gateaux differentiability of the norm at  $x$  implies that  $x$  is a smooth point of the sphere of radius  $\|x\|$ .

It is well known (see [7] and the references therein) that, for  $1 < p < \infty$ ,  $C_p$  is a uniformly convex Banach space. Therefore, every nonzero  $T \in C_p$  is a smooth point and, in this case, the support functional of  $T$  is given by

$$D_T(X) = \operatorname{tr} \left[ \frac{|T|^{p-1} U X^*}{\|T\|_p^{p-1}} \right] \quad (2.4)$$

for all  $X \in C_p$ , where  $T = U|T|$  is the polar decomposition of  $T$ .

**DEFINITION 2.1.** Let  $E$  be a complex Banach space. We define the orthogonality in  $E$ . We say that  $b \in E$  is orthogonal to  $a \in E$  if, for all complex  $\lambda$ , there holds

$$\|a + \lambda b\| \geq \|a\|. \quad (2.5)$$

This definition has a natural geometric interpretation, namely,  $b \perp a$  if and only if the complex line  $\{a + \lambda b \mid \lambda \in \mathbb{C}\}$  is disjoint with the open ball  $K(0, \|a\|)$ , that is, if and only if this complex line is a tangent one. Note that if  $b$  is orthogonal to  $a$ , then  $a$  needs not be orthogonal to  $b$ . If  $E$  is a Hilbert space, then from (2.5), it follows that  $\langle a, b \rangle = 0$ , that is, orthogonality in the usual sense.

### 3. The elementary operators $AXB - CXD$

**LEMMA 3.1.** Let  $A, B \in B(H)$ . The following statements are equivalent:

- (1) the pair  $(A, B)$  has the property  $(FP)_{C_p}$ ,  $1 \leq p < \infty$ ;
- (2) if  $AT = TB$ , where  $T \in C_p$ , then  $\overline{R(T)}$  reduces  $A$ ,  $\ker(T)^\perp$  reduces  $B$ , and  $A|_{\overline{R(T)}}$  and  $B|_{\ker(T)^\perp}$  are normal operators.

**PROOF.** (1) $\Rightarrow$ (2). Since  $C_p$  is a bilateral ideal and  $T \in C_p$ , then  $AT \in C_p$ . Hence as  $AT = TB$  and  $(A, B)$  satisfies  $(FP)_{C_p}$ ,  $A^*T = TB^*$ , and so,  $\overline{R(T)}$  and  $\ker(T)^\perp$  are reducing subspaces for  $A$  and  $B$ , respectively. Since  $A(AT) = (AT)B$  implies that  $A^*(AT) = (AT)B^*$  by  $(FP)_{C_p}$  and the equality  $A^*T = TB^*$  implies that  $A^*AT = AA^*T$ , thus we see that  $A|_{\overline{R(T)}}$  is normal. Clearly,  $(B^*, A^*)$  satisfies  $(FP)_{C_p}$  and  $B^*T^* = T^*A^*$ . Therefore, it follows from the above argument that  $B^*|_{\overline{R(T^*)}} = B|_{\ker(T)^\perp}$  is normal.

(2) $\Rightarrow$ (1). Let  $T \in C_p$  such that  $AT = TB$ . Taking the two decompositions of  $H$ ,  $H_1 = H = \overline{R(T)} \oplus \overline{R(T)}^\perp$  and  $H_2 = H = \ker(T)^\perp \oplus \ker T$ , then we can write  $A$  and  $B$  on  $H_1$  into  $H_2$ , respectively:

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}, \quad B = \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix}, \quad (3.1)$$

where  $A_1$  and  $B_1$  are normal operators. Also we can write  $T$  and  $X$  on  $H_2$  into  $H_1$ :

$$T = \begin{bmatrix} T_1 & 0 \\ 0 & 0 \end{bmatrix}, \quad X = \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix}. \quad (3.2)$$

It follows from  $AT = TB$  that  $A_1T_1 = T_1B_1$ . Since  $A_1$  and  $B_1$  are normal operators, then, by applying the Fuglede-Putnam theorem, we obtain  $A_1^*T_1 = T_1B_1^*$ , that is,  $A^*T = TB^*$ .  $\square$

**THEOREM 3.2.** *Let  $A, B \in B(H)$ . If  $A$  and  $B$  are normal operators, then*

$$\|S - (AX - XB)\|_p \leq \|S\|_p \quad (3.3)$$

for all  $X \in C_p$  and for all  $S \in \ker \delta_{A,B} \cap C_p$  ( $1 \leq p < \infty$ ).

**PROOF.** Let  $S = U|S|$  be the polar decomposition of  $S$ , where  $U$  is an isometry such that  $\ker U = \ker |S|$ . Since

$$\|U^*S\|_p \leq \|U^*\|_p \|S\|_p = \|S\|_p \quad (3.4)$$

for all  $S \in C_p$ , then

$$\|S - (AX - XB)\|_p^p \geq \|U^*[S - (AX - XB)]\|_p^p = \||S| - U^*(AX - XB)\|_p^p, \quad (3.5)$$

and we have

$$\||S| - U^*(AX - XB)\|_p^p \geq \sum_n |\langle [|S| - U^*(AX - XB)]\varphi_n, \varphi_n \rangle|^p \quad (3.6)$$

for any orthonormal basis  $\{\varphi_n\}_{n \geq 1}$  of  $H$ . Since  $AS = SB$ , and  $A$  and  $B$  are normal operators, it follows from the Fuglede-Putnam theorem that  $S^*A = BS^*$ . Consequently,  $S^*AS = BS^*S$  or  $S^*SB = BS^*S$ , that is,  $B|S| = |S|B$ . Since  $|S|$  is a compact normal operator and commutes with  $B$ , there exists an orthonormal basis  $\{f_k\} \cup \{g_m\}$  of  $H$  such that  $\{f_k\}$  consists of common eigenvectors of  $B$  and  $|S|$ , and  $\{g_m\}$  is an orthonormal basis of  $\ker |S|$ . Since  $\{f_k\}$  is an orthonormal basis of the normal operator  $B$ , then there exists a scalar  $\alpha_k$  such that  $Bf_k = \alpha_k f_k$  and  $B^*f_k = \bar{\alpha}_k f_k$ . Consequently,

$$\begin{aligned} \langle U^*(AX - XB)f_k, |S|f_k \rangle &= \langle S^*(AX - XB)f_k, f_k \rangle \\ &= \langle (B(S^*X) - (S^*X)B)f_k, f_k \rangle = 0, \end{aligned} \quad (3.7)$$

that is,  $\langle U^*(AX - XB)f_k, f_k \rangle = 0$ .

In (3.6) take  $\{\varphi_n\} = \{f_k\} \cup \{g_m\}$  as an orthonormal basis of  $H$ , then

$$\begin{aligned} &\sum_n |\langle [|S| - U^*(AX - XB)]\varphi_n, \varphi_n \rangle|^p \\ &\geq \sum_k |\langle |S|f_k, f_k \rangle|^p + \sum_m |\langle U^*(AX - XB)g_m, g_m \rangle|^p \\ &\geq \sum_k |\langle |S|f_k, f_k \rangle|^p = \|S\|_p^p. \end{aligned} \quad (3.8)$$

$\square$

**LEMMA 3.3.** *Let  $A, B \in B(H)$  satisfying  $(FP)_{C_p}$ . Then*

$$\|S + AX - XB\|_p^p \geq \|S\|_p^p \quad (3.9)$$

for every operator  $S \in \ker \delta_{A,B} \cap C_p$  ( $1 < p < \infty$ ) and for all  $X \in C_p$ .

**PROOF.** If the pair  $(A, B)$  satisfies the  $(FP)_{C_p}$  property, then  $\overline{R(S)}$  reduces  $A$ ,  $\ker^\perp S$  reduces  $B$ , and  $A|_{\overline{R(S)}}$  and  $B|_{\ker^\perp S}$  are normal operators. Letting  $S_0 : \ker^\perp S \rightarrow \overline{R(S)}$  be the quasiaffinity defined by setting  $S_0 x = Sx$  for each  $x \in \ker^\perp S$ , it results that  $\delta_{A_1, B_1}(S_0) = \delta_{A_1^*, B_1^*}(S_0) = 0$ . Let  $A = A_1 \oplus A_2$ , with respect to  $H = \overline{R(S)} \oplus \overline{R(S)}^\perp$ ,  $A = B_1 \oplus B_2$ , with respect to  $H = \ker(S)^\perp \oplus \ker S$ , and  $X : \overline{R(S)} \oplus \overline{R(S)}^\perp \rightarrow \ker(S)^\perp \oplus \ker S$  have the matrix representation

$$X = \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix}. \quad (3.10)$$

Then we have

$$\|S - (AX - XB)\|_p = \left\| \begin{bmatrix} S_1 - (A_1 X_1 - X_1 B_1) & * \\ * & * \end{bmatrix} \right\|_p. \quad (3.11)$$

The result of Gohberg and Kreĭn [4] guarantees that

$$\|S - (AX - XB)\|_p \geq \|S_1 - (A_1 X_1 - X_1 B_1)\|_p. \quad (3.12)$$

Since  $A_1$  and  $B_1$  are two normal operators, then it results from Theorem 3.5 that

$$\|S_1 - (A_1 X_1 - X_1 B_1)\|_p \geq \|S_1\|_p = \|S\|_p. \quad (3.13)$$

□

**LEMMA 3.4** [6]. *Let  $u$  and  $v$  be two elements of a Banach space  $V$  with norm  $\|\cdot\|$ . If  $u$  is a smooth point, then  $D_u(v) = 0$  if and only if*

$$\|u + zv\| \geq \|u\| \quad (3.14)$$

for all  $z \in \mathbb{C}$  (the complex numbers).

**THEOREM 3.5.** *Let  $A, B \in B(H)$  and  $T \in C_p$  ( $1 < p < \infty$ ). Then*

$$\|T + \delta_{A,B}(X)\|_p \geq \|T\|_p \quad (3.15)$$

for all  $X \in B(H)$  with  $\Delta_{A,B}(X) \in C_p$  if and only if

$$\text{tr}(|T|^{p-1} U^* \delta_{A,B}(X)) = 0 \quad (3.16)$$

for all such  $X$ .

**PROOF.** The theorem is an immediate consequence of equality (2.4) and Lemma 3.4.  $\square$

**THEOREM 3.6.** *Let  $A, B \in B(H)$  and  $T \in C_p$  ( $1 < p < \infty$ ). Then*

$$\|T + \delta_{A,B}(X)\|_p \geq \|T\|_p \quad (3.17)$$

*for all  $X \in C_p$  if and only if  $\tilde{T} = |T|^{p-1}U^* \in \ker \delta_{B,A}$ .*

**PROOF.** By virtue of Theorem 3.5, it is sufficient to show that  $\text{tr}(\tilde{T}\delta_{A,B}(X)) = 0$  for all  $X \in C_p$  if and only if  $\tilde{T} \in \ker \delta_{B,A}$ .

Choose  $X$  to be the rank-one operator  $f \otimes g$  for some arbitrary elements  $f$  and  $g$  in  $H$ . Then  $\text{tr}(\tilde{T}(AX - XB)) = \text{tr}(B\tilde{T} - \tilde{T}A)X = 0$  implies that  $\langle \delta_{B,A}(\tilde{T})f, g \rangle = 0 \Leftrightarrow \tilde{T} \in \ker \delta_{B,A}$ .

Conversely, assume that  $\tilde{T} \in \ker \delta_{B,A}$ , that is,  $B\tilde{T} = \tilde{T}A$ . Since  $\tilde{T}X$  and  $\tilde{T}\delta_{B,A}$  are trace classes, then for all  $X \in C_p$ , we get

$$\begin{aligned} \text{tr}(\tilde{T}(AX - XB)) &= \text{tr}(\tilde{T}AX - \tilde{T}XB) = \text{tr}(XB\tilde{T} - X\tilde{T}A) \\ &= \text{tr}(X\delta_{B,A}(\tilde{T})) = 0. \end{aligned} \quad (3.18) \quad \square$$

**LEMMA 3.7.** *Let  $A, B \in B(H)$  and  $S \in C_p$  such that  $\delta_{A,B}(T) = 0 = \delta_{A,B}^*(T)$ .*

*If  $A|S|^{p-1}U^* = |S|^{p-1}U^*B$ , where  $p > 1$  and  $S = U|S|$  is the polar decomposition of  $S$ , then  $A|S|U^* = |S|U^*B$ .*

**PROOF.** If  $T = |S|^{p-1}$ , then

$$ATU^* = TU^*B. \quad (3.19)$$

We prove that

$$AT^nU^* = T^nU^*B \quad (3.20)$$

for all  $n \geq 1$ . If  $S = U|S|$ , then

$$\begin{aligned} \ker U &= \ker |S| = \ker |S|^{p-1} = \ker T, \\ (\ker U)^\perp &= (\ker T)^\perp = \overline{R(T)}. \end{aligned} \quad (3.21)$$

This shows that the projection  $U^*U$  onto  $(\ker T)^\perp$  satisfies  $U^*UT = T$  and  $TU^*UT = T^2$ . By taking the adjoints of (3.19) and since  $A$  and  $B$  are normal operators applying Fuglede-Putnam theorem, we get  $BUT = UTA$  and  $AT^2 = ATU^*UT = TU^*BUT = TU^*UTA = T^2A$ .

Since  $A$  commutes with the positive operator  $T^2$ ,  $A$  commutes with its square roots, that is,

$$AT = TA. \quad (3.22)$$

By (3.19) and (3.22) we obtain (3.20). Let  $f(t)$  be the map defined on  $\sigma(T) \subset \mathbb{R}^+$  by  $f(t) = t^{1/(p-1)}$  ( $1 < p < \infty$ ). Since  $f$  is the uniform limit of a sequence

( $P_i$ ) of polynomials without constant term (since  $f(0) = 0$ ), it follows from (3.20) that  $AP_i(T)U^* = P_i(T)U^*B$ . Therefore,  $AT^{1/(p-1)}U^* = U^*T^{1/(p-1)}B$ .  $\square$

**THEOREM 3.8.** *Let  $A$  and  $B$  be operators in  $B(H)$  such that  $\delta_{A,B}(T) = 0 = \delta_{A,B}^*(T)$ . Then  $T \in \ker \Delta_{A,B} \cap C_p$  if and only if*

$$\|S + \delta_{A,B}(X)\|_p \geq \|S\|_p \quad (3.23)$$

for all  $X \in C_p$ .

**PROOF.** If  $S \in \ker \Delta_{A,B}$ , then it follows from Lemma 3.3 that

$$\|S + \delta_{A,B}(X)\|_p \geq \|S\|_p \quad (3.24)$$

for all  $X \in C_p$ . Conversely, if

$$\|S + \delta_{A,B}(X)\|_p \geq \|S\|_p \quad (3.25)$$

for all  $X \in C_p$ , then, from Theorem 3.6,

$$A|S|^{p-1}U^* = |S|^{p-1}U^*B. \quad (3.26)$$

Since  $\delta_{A,B}(S) = 0 = \delta_{A,B}^*(S)$ ,

$$A^*|S|^{p-1}U^* = |S|^{p-1}U^*B^*. \quad (3.27)$$

By taking adjoints, we get

$$AU|S|^{p-1} = U|S|^{p-1}B. \quad (3.28)$$

From Lemma 3.7, it follows that  $AU|S| = U|S|B$ , that is,  $S \in \ker \Delta_{A,B}$ .  $\square$

**REMARK 3.9.** (1) It is well known that the Hilbert-Schmidt class  $C_2$  is a Hilbert space under the inner product  $\langle Y, Z \rangle = \text{tr } Z^*Y$ .

We remark here that for the Hilbert-Schmidt norm  $\|\cdot\|_2$ , the orthogonality result in Theorem 3.8 is to be understood in the usual Hilbert-space sense. Note in the case where  $I = C_2$  that

$$\|T + \delta_{A,B}(X)\|_2^2 = \|\delta_{A,B}(X)\|_2^2 + \|T\|_2^2 \quad (3.29)$$

for all  $X \in C_2$  if and only if  $AT^* = T^*B$ . This can be seen as an immediate consequence of the fact that

$$R(\delta_{A,B}|C_2)^\perp = \ker(\delta_{A,B}|C_2)^* = \ker(\delta_{B^*,A^*}|C_2). \quad (3.30)$$

(2) It is known [2] that if  $A$  and  $B$  are contractions and  $S \in C_p$ , then  $\delta_{A^*,B^*}(S) = \delta_{A,B}(S) = 0$ . Hence

$$\|S + \delta_{A,B}(X)\|_p \geq \|S\|_p \quad (3.31)$$

holds for all  $X \in C_p$  if and only if  $S \in \ker(\delta_{A,B}|C_p)$ .

(3) If  $A = B$ , then the following counterexample shows that [Theorem 3.8](#) does not hold if  $p < 1$ . Take  $p = 1/2$  and

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad S = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad X = \begin{bmatrix} 0 & -\alpha \\ \alpha & 0 \end{bmatrix}, \quad (3.32)$$

where  $\alpha$  is real such that  $0 < \alpha < 1$ . We have

$$S - (AX - XA) = \begin{bmatrix} 1 & \alpha \\ \alpha & 1 \end{bmatrix} \quad (3.33)$$

and, for eigenvectors  $\beta_1 = 1 - \alpha$ ,  $\beta_2 = 1 + \alpha$ . Then

$$\|S - (AX - XA)\|_{1/2} = [(1 - \alpha)^{1/2} + (1 + \alpha)^{1/2}]^2 < 4 = \|S\|_{1/2}. \quad (3.34)$$

**COROLLARY 3.10.** *Let  $A, B \in L(H)$ . Then*

$$\|S + AX - XB\|_p \geq \|S\|_p \quad (3.35)$$

*if and only if  $S \in \ker \delta_{A,B} \cap C_p$  and for all  $X \in C_p$ , in each of the following cases:*

- (1) *if  $A, B \in L(H)$  such that  $\|Ax\| \geq \|x\| \geq \|Bx\|$  for all  $x \in H$ ,*
- (2) *if  $A$  is invertible and  $B$  is such that  $\|A^{-1}\| \|B\| \leq 1$ .*

**PROOF.** The result of Tong [[13](#), Lemma 1] guarantees that the above condition implies that, for all  $T \in \ker(\delta_{A,B}|_{K(H)})$ ,  $\overline{R(T)}$  reduces  $A$ ,  $\ker(T)^\perp$  reduces  $B$ , and  $A|_{\overline{R(T)}}$  and  $B|_{\ker(T)^\perp}$  are unitary operators. Hence it results from [Lemma 3.1](#) that the pair  $(A, B)$  has the property  $(FP)_{K(H)}$  and the results hold by [Theorem 3.8](#). Here  $K(H)$  is the ideal of compact operators.

The above inequality holds in particular if  $A = B$  is isometric; in other words,  $\|Ax\| = \|x\|$  for all  $x \in H$ .

- (2) In this case, it suffices to take  $A_1 = \|B\|^{-1}A$ ,  $B_1 = \|B\|^{-1}B$ .

Then  $\|A_1x\| \geq \|x\| \geq \|B_1x\|$  and the result holds by (1) for all  $x \in H$ .  $\square$

**4. Orthogonality and the elementary operators  $AXB - CXD$ .** Let  $H$  be a separable infinite-dimensional complex Hilbert space and let  $B(H)$  denote the algebra of all bounded operators on  $H$  into itself. Given  $A, B, C$ , and  $D$  normal operators in  $B(H)$  such that  $AC = CA$ ,  $BD = DB$ , we define the elementary operator  $\Psi : B(H) \rightarrow B(H)$  by  $\Psi(X) = AXB - CXD$ . We prove that if  $T \in C_p$  ( $1 < p < \infty$ ), then  $\|T + \Phi(X)\|_p \geq \|T\|_p$  if and only if  $T \in \ker \Phi$  for all  $X \in C_p$ .

By the same argument used in the proofs of [Theorems 3.5](#) and [3.6](#), we prove the following theorems.

**THEOREM 4.1.** *Let  $A, B, C, D \in B(H)$  and  $T \in C_p$  ( $1 < p < \infty$ ). Then*

$$\|T + \Psi(X)\|_p \geq \|T\|_p \quad (4.1)$$



for all  $X \in B(H)$  with  $\Psi(X) \in C_p$  if and only if

$$\operatorname{tr}(|T|^{p-1}U^*\Psi(X)) = 0 \quad (4.2)$$

for all such  $X$ .

**THEOREM 4.2.** Let  $A, B, C, D \in B(H)$  and  $T \in C_p$  ( $1 < p < \infty$ ). Then

$$\|T + \Psi(X)\|_p \geq \|T\|_p \quad (4.3)$$

for all  $X \in C_p$  if and only if  $\tilde{T} = |T|^{p-1}U^* \in \ker \Psi$ .

**LEMMA 4.3.** Let  $A, B \in B(H)$  be normal operators and  $AB = BA$ . Suppose that  $ASB = BSA$ ,  $S \in C_p$  ( $1 < p < \infty$ ). If

$$AU|S|^{p-1}B = BU|S|^{p-1}A, \quad (4.4)$$

then

$$AU|S|B = BU|S|A. \quad (4.5)$$

**PROOF.** Assume that  $B^{-1} \in B(H)$ . Then, from  $ASB = BSA$  and  $AB = BA$ , we get  $AB^{-1}S = SB^{-1}A$ . Hence, applying the above lemma to the operators  $AB^{-1}$ ,  $B^{-1}A$ , and  $S$ , we get

$$AB^{-1}U|S|^{p-1} = U|S|^{p-1}B^{-1}A, \quad (4.6)$$

which implies that

$$AB^{-1}U|S| = U|S|B^{-1}A. \quad (4.7)$$

Multiply (4.6) and (4.7) at right and left by  $B$  to obtain

$$BAB^{-1}U|S|^{p-1}B = BU|S|^{p-1}B^{-1}AB \quad (4.8)$$

or

$$ABB^{-1}U|S|^{p-1}B = BU|S|^{p-1}B^{-1}BA, \quad (4.9)$$

that is,

$$AU|S|^{p-1}B = BU|S|^{p-1}A, \quad (4.10)$$

which implies that

$$AU|S|B = BU|S|A. \quad (4.11)$$

Consider now the case when  $B$  is injective, that is,  $\ker B = \{0\}$ . Let

$$\Delta_n = \left\{ \lambda \in \mathbb{C} : |\lambda| \leq \frac{1}{n} \right\} \quad (4.12)$$

and let  $E_B(\Delta_n)$  be the corresponding spectral projector.

Putting

$$P_n = I - E_B(\Delta_n), \quad (4.13)$$

the subspace  $P_n H$  reduces both operators  $A$  and  $B$  (since they commute and are normal). Hence, with respect to the decomposition

$$\begin{aligned} H &= (I - P_n)H \oplus P_n H, \\ A &= \begin{bmatrix} A_1^{(n)} & 0 \\ 0 & A_2^{(n)} \end{bmatrix}, \quad B = \begin{bmatrix} B_1^{(n)} & 0 \\ 0 & B_2^{(n)} \end{bmatrix}, \\ S &= \begin{bmatrix} S_{11}(n) & S_{12}(n) \\ S_{21}(n) & S_{22}(n) \end{bmatrix}, \quad X = \begin{bmatrix} X_{11}(n) & X_{12}(n) \\ X_{21}(n) & X_{22}(n) \end{bmatrix}, \end{aligned} \quad (4.14)$$

it is easy to see that  $B_2^{(n)}$  acting on  $P_n H$  is invertible. Then, from  $ASB = BSA$ , it follows that

$$A_2^{(n)} S_{22}(n) B_2^{(n)} = B_2^{(n)} S_{22}(n) A_2^{(n)}, \quad (4.15)$$

and, from  $AB = BA$ , we get  $A_2 B_2 = B_2 A_2$ . Since

$$AU|S|^{p-1}B = BU|S|^{p-1}A, \quad (4.16)$$

according to the first part of the proof, it follows that

$$A_2^{(n)} U |S_{22}(n)|^{p-1} B_2^{(n)} = B_2^{(n)} U |S_{22}(n)|^{p-1} A_2^{(n)}, \quad (4.17)$$

which implies that

$$A_2^{(n)} U |S_{22}(n)| B_2^{(n)} = B_2^{(n)} U |S_{22}(n)| A_2^{(n)}, \quad (4.18)$$

so we have  $AU|S|B = BU|S|A$ . Assume now  $\ker A \cap \ker B = \{0\}$ .

Then  $\ker B$  reduces  $A$  and  $P_{\ker B} A P_{\ker B}$  is injective. Let  $H = \ker B \oplus H_1$  ( $H_1 = H \ominus \ker B$ ). Then we have

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 0 & B_2 \end{bmatrix}, \quad S = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix}, \quad (4.19)$$

where  $A_1, B_2$  are injective and their ranges are dense in subspaces they act on. We have

$$ASB - BSA = \begin{bmatrix} 0 & A_1 S_{12} B_2 \\ -B_2 S_{21} A_1 & A_2 S_{22} B_2 - B_2 S_{22} A_2 \end{bmatrix}. \quad (4.20)$$

Now, if  $ASB = BSA$ , then  $A_2S_{22}B_2 = B_2S_{22}A_2$ ,  $B_2S_{21}A_1 = 0$ , and  $A_1S_{12}B_2 = 0$ , that is,  $S_{21} = S_{12} = 0$ . It follows that

$$S = \begin{bmatrix} S_{11} & 0 \\ 0 & S_{22} \end{bmatrix}. \quad (4.21)$$

Since  $A_2B_2 = B_2A_2$ ,  $A_2S_{22}B_2 = B_2S_{22}A_2$ , and  $B_2$  is injective, and we have already proved that

$$A_2U|S_{22}|^{p-1}B_2 = B_2U|S_{22}|^{p-1}A_2 \quad (4.22)$$

implies

$$A_2U|S_{22}|B_2 = B_2U|S_{22}|A_2, \quad (4.23)$$

so we have  $AU|S|B = BU|S|A$ .  $\square$

Let  $\Phi(X) = AXB - BXA$ . We prove the following theorem.

**THEOREM 4.4.** *Let  $A, B \in B(H)$  be normal operators,  $AB = BA$ , and  $S \in C_p$  ( $1 < p < \infty$ ). Then  $S \in \ker \Phi$  if and only if*

$$\|S - (AXB - BXA)\|_p \geq \|S\|_p \quad (4.24)$$

for all  $X \in C_p$ .

**PROOF.** If  $S \in \ker \Phi$ , then, from [13, Theorem 3.4], it follows that

$$\|S + \Phi(X)\|_p \geq \|S\|_p \quad (4.25)$$

for all  $X \in C_p$ . Conversely, if

$$\|S + \Phi(X)\|_p \geq \|S\|_p \quad (4.26)$$

for all  $X \in C_p$ , then, from Theorem 4.2,

$$A|S|^{p-1}U^*B = B|S|^{p-1}U^*A. \quad (4.27)$$

Since  $A$  and  $B$  are normal operators applying Fuglede-Putnam theorem, we get  $A^*|S|^{p-1}U^*B^* = B^*|S|^{p-1}U^*A^*$ . By taking adjoints, we get  $AU|S|^{p-1}B = BU|S|^{p-1}A$ .

From Lemma 4.3, it follows that  $AU|S|B = BU|S|A$ , that is,  $S \in \ker \Phi$ .  $\square$

Let  $\Psi(X) = AXB - CXD$ .

**THEOREM 4.5.** *Let  $A, B, C, D \in B(H)$  be normal operators,  $AC = CA$ ,  $BD = DB$ , and  $S \in C_p$  ( $1 < p < \infty$ ). Then  $S \in \ker \Psi$  if and only if*

$$\|S - (AXB - CXD)\|_p \geq \|S\|_p \quad (4.28)$$

for all  $X \in C_p$ .

**PROOF.** It suffices to take the Hilbert space  $H \oplus H$  and the operators

$$\begin{aligned} A^\sim &= \begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix}, & B^\sim &= \begin{bmatrix} C & 0 \\ 0 & B \end{bmatrix}, \\ S^\sim &= \begin{bmatrix} 0 & S \\ 0 & 0 \end{bmatrix}, & X^\sim &= \begin{bmatrix} 0 & X \\ 0 & 0 \end{bmatrix} \end{aligned} \quad (4.29)$$

and apply [Theorem 4.4](#).  $\square$

**REMARK 4.6.** The results of the above theorems can be obtained when the normality of  $A$  and  $B$  is replaced by some other condition, in particular, if  $|A| = |B|$ ,  $|A^*| = |B^*|$ . In this case, it suffices to take

$$\begin{aligned} A^\sim &= \begin{bmatrix} 0 & A^* \\ B & 0 \end{bmatrix}, & B^\sim &= \begin{bmatrix} 0 & B^* \\ A & 0 \end{bmatrix}, \\ S^\sim &= \begin{bmatrix} 0 & S \\ 0 & 0 \end{bmatrix}, & X^\sim &= \begin{bmatrix} 0 & X \\ 0 & 0 \end{bmatrix} \end{aligned} \quad (4.30)$$

and apply [Lemma 4.3](#) and [Theorem 4.4](#).

**5. On minimizing  $\|AX - XB - T\|_p^p$ .** Maher [8, Theorem 3.2] shows that if  $A$  is normal and  $S \in \ker \delta_A \cap C_p$  ( $1 \leq p < \infty$ ), then the map  $F_p$  defined by  $F_p(X) = \|S - (AX - XA)\|_p^p$  has a global minimizer at  $V$  if, and for  $1 < p < \infty$  only if,  $AV - VA = 0$ .

In this section, we prove that if the pair  $(A, B)$  has the property  $(FP)_{C_p}$  (i.e.,  $AT = TB$ , where  $T \in C_p$ , implies  $A^*T = TB^*$ ),  $1 \leq p < \infty$ , and  $S \in \ker \delta_{A,B} \cap C_p$ , then the map  $F_p$  defined by  $F_p(X) = \|S - (AX - XB)\|_p^p$  has a global minimizer at  $V$  if, and for  $1 < p < \infty$  only if,  $AV - VB = 0$ . In other words, we have

$$\|S - (AX - XB)\|_p^p \geq \|T\|_p^p \quad (5.1)$$

if, and for  $1 < p < \infty$  only if,  $AV - VB = 0$ . Thus in Halmos' terminology [5], the zero commutator is the commutator approximant in  $C_p$  of  $T$ . Additionally, we show that if the pair  $(A, B)$  has the property  $(FP)_{C_p}$  and  $S \in \ker \delta_{A,B} \cap C_p$  ( $1 < p < \infty$ ), then the map  $F_p$  has a critical point at  $W$  if and only if  $AW - WB = 0$ , that is, if  $\mathcal{D}_W F_p$  is the Frechet derivative at  $W$  of  $F_p$ , the set  $\{W \in B(H) : \mathcal{D}_W F_p = 0\}$  coincides with  $\ker \delta_{A,B}$  (the kernel of  $\delta_{A,B}$ ).

**THEOREM 5.1** [9]. *If  $1 < p < \infty$ , then the map  $F_p : C_p \mapsto \mathbb{R}^+$  defined by  $X \mapsto \|X\|_p^p$  is differentiable at every  $X \in C_p$  with derivative  $\mathcal{D}_X F_p$  given by  $\mathcal{D}_X F_p(T) = p \operatorname{Re} \operatorname{tr}(|X|^{p-1} U^* T)$ , where  $\operatorname{tr}$  denotes trace,  $\operatorname{Re} z$  is the real part of a complex number  $z$ , and  $X = U|X|$  is the polar decomposition of  $X$ . If  $\dim H < \infty$ , then the same result holds for  $0 < p \leq 1$  at every invertible  $X$ .*

**THEOREM 5.2** [9]. *If  $\mathcal{U}$  is a convex set of  $C_p$ ,  $1 < p < \infty$ , then the map  $X \mapsto \|X\|_p^p$ , where  $X \in \mathcal{U}$ , has at most a global minimizer.*

**DEFINITION 5.3.** Let  $\mathcal{U}(A, B) = \{X \in B(H) : AX - XB \in C_p\}$  and let  $F_p : \mathcal{U} \rightarrow \mathbb{R}^+$  be the map defined by  $F_p(X) = \|T - (AX - XB)\|_p^p$ , where  $T \in \ker \delta_{A,B} \cap C_p$ ,  $1 \leq p < \infty$ .

**6. Main results.** By simple modifications in the proof of [Lemma 3.7](#), we can prove the following lemma.

**LEMMA 6.1.** Let  $A, B \in B(H)$  and  $C \in B(H)$  such that the pair  $(A, B)$  has the property  $(FP)_{B(H)}$ . If  $A|S|^{p-1}U^* = |S|^{p-1}U^*B$ , where  $p > 1$  and  $S = U|S|$  is the polar decomposition of  $S$ , then  $A|S|U^* = |S|U^*B$ .

**THEOREM 6.2.** Let  $A, B \in \mathcal{L}(H)$ . If the pair  $(A, B)$  has the property  $(FP)_{C_p}$  and  $S \in C_p$  such that  $AS = SB$ , then

- (1) for  $1 \leq p < \infty$ , the map  $F_p$  has a global minimizer at  $W$  if, and for  $1 < p < \infty$  only if,  $AW - WB = 0$ ;
- (2) for  $1 < p < \infty$ , the map  $F_p$  has a critical point at  $W$  if and only if  $AW - WB = 0$ ;
- (3) for  $0 < p \leq 1$   $\dim \mathcal{H} < \infty$  and  $S - (AW - WB)$  is invertible, then  $F_p$  has a critical point at  $W$  if  $AW - WB = 0$ .

**PROOF.** Since the pair  $(A, B)$  has the property  $(FP)_{C_p}$ , it follows from [Lemma 3.3](#) that

$$\|S - (AX - XB)\|_p^p \geq \|S\|_p^p, \quad (6.1)$$

that is,  $F_p(X) \geq F_p(W)$ .

Conversely, if  $F_p$  has a minimum, then

$$\|S - (AW - WB)\|_p^p = \|S\|_p^p. \quad (6.2)$$

Since  $\mathcal{U}$  is convex, the set  $\mathcal{V} = \{S - (AX - XB); X \in \mathcal{U}\}$  is also convex. Thus [Theorem 5.2](#) implies that

$$S - (AW - WB) = S. \quad (6.3)$$

(2) Let  $W, S \in \mathcal{U}$  and let  $\phi$  and  $\varphi$  be two maps defined, respectively, by  $\phi : X \mapsto S - (AX - XB)$  and  $\varphi : X \mapsto \|X\|_p^p$ .

Since the Frechet derivative of  $F_p$  is given by

$$\mathcal{D}_W F_p(T) = \lim_{h \rightarrow 0} \frac{F_p(W + hT) - F_p(W)}{h}, \quad (6.4)$$

it follows that

$$\mathcal{D}_W F_p(T) = [\mathcal{D}_{S-(AW-WB)}](TB - AT). \quad (6.5)$$

If  $W$  is a critical point of  $F_p$ , then  $\mathcal{D}_W F_p(T) = 0$  for all  $T \in \mathcal{U}$ . By applying [Theorem 5.1](#), we get

$$\begin{aligned}\mathcal{D}_W F_p(T) &= p \operatorname{Retr} [|S - (AW - WB)|^{p-1} W^* (TB - AT)] \\ &= p \operatorname{Retr} [Y(TB - AT)] = 0,\end{aligned}\quad (6.6)$$

where  $S - (AW - WB) = W|S - (AW - WB)|$  is the polar decomposition of the operator  $S - (AW - WB)$  and  $Y = |S - (AW - WB)|^{p-1} W^*$ . An easy calculation shows that  $BY - YA = 0$ , that is,

$$A|S - (AW - WB)|^{p-1} W^* = |S - (AW - WB)|^{p-1} W^* B. \quad (6.7)$$

It follows from [Lemma 6.1](#) that

$$A|S - (AW - WB)|W^* = |S - (AW - WB)|W^* B. \quad (6.8)$$

By taking adjoints and since the pair  $(A, B)$  has the property  $(FP)_{C_p}$ , we get  $A(T - (AW - WB)) = (T - (AW - WB))B$ . Then  $A(AW - WB) = (AW - WB)B$ . Hence

$$AW - WB \in R(\delta_{A,B}) \cap \ker \delta_{A,B}. \quad (6.9)$$

By the same argument used in the proof of [Lemma 6.1](#) we can prove that

$$\| |S - (AX - XB)| \| \geq \|S\| \quad (6.10)$$

for all  $X \in B(H)$  and for all  $T \in B(H)$  and it results that  $AW - WB = 0$ .

Conversely, if  $AW = WB$ , then  $W$  is a minimum, and since  $F_p$  is differentiable, then  $W$  is a critical point.

(3) Suppose that  $\dim H < \infty$ . If  $AW - WB = 0$ , then  $S$  is invertible by hypothesis. Also  $|S|$  is invertible, hence  $|S|^{p-1}$  exists for  $0 < p \leq 1$  taking  $Y = |S|^{p-1} U^*$ , where  $S = U|S|$  is the polar decomposition of  $S$ . Since  $AS = SB$  implies that  $S^*A = BS^*$ , then  $S^*AS = BS^*S$ , and this implies that  $|S|^2 B = B|S|^2$  and  $|S|B = B|S|$ .

Since  $S^*A = BS^*$ , that is,  $|S|U^*A = B|S|U^*$ , then  $|S|(U^*A - BU^*) = 0$ , and since  $B|S|^{p-1} = |S|^{p-1}B$ , then

$$BY - YA = B|S|^{p-1}U^* - |S|^{p-1}U^*A = |S|^{p-1}(BU^* - U^*A) \quad (6.11)$$

so that  $BY - YA = 0$  and  $\operatorname{tr}[(BY - YA)T] = 0$  for every  $T \in B(H)$ . Since  $S = S - (AW - WB)$ , then

$$\begin{aligned}0 &= \operatorname{tr}[YTB - YAT] = \operatorname{tr}[Y(TB - AT)] \\ &= p \operatorname{Retr}[Y(TB - AT)] = p \operatorname{Retr}[|S|^{p-1}U^*(TB - AT)] \\ &= (\mathcal{D}_T \phi)(TB - AT) = (\mathcal{D}_W F_p)(T).\end{aligned}\quad (6.12)$$

□

**REMARK 6.3.** In [Theorem 6.2](#), the implication “ $W$  is a critical point implies  $AW - WB = 0$ ” does not hold in the case  $0 < p \leq 1$  because the functional calculus argument involving the function  $t \mapsto t^{1/(p-1)}$ , where  $0 \leq t < \infty$ , is only valid for  $1 < p < \infty$ .

**7. On minimizing  $\|T - (AXB - CXD)\|_p^p$ .** In this section, we consider the elementary operator  $\Phi(X) = AXB - CXD$  and we prove that if  $AC = CA$ ,  $BD = DB$ , and  $ASB = CSD$ ,  $S \in C_p$ , then, for  $1 < p < \infty$ , the map  $F_p$  defined by  $F_p(X) = \|T - (AXB - CXD)\|_p^p$  has a global minimizer at  $V$  if, and for  $1 < p < \infty$  only if,  $AVB - CVD = 0$ . In other words, we have  $\|T - (AXB - CXD)\|_p^p \geq \|T\|_p^p$  if, and for  $1 < p < \infty$  only if,  $AVB - CVD = 0$ . Additionally, we show that if  $AC = CA$ ,  $BD = DB$ , and  $T \in \ker \Delta_{A,B} \cap C_p$ ,  $1 < p < \infty$ , then the map  $F_p$  has a critical point at  $W$  if and only if  $AWB - CWD = 0$ , that is, if  $\mathcal{D}_W F_p$  is the Frechet derivative at  $W$  of  $F_p$ , the set  $\{W \in B(H) : \mathcal{D}_W F_p = 0\}$  coincides with  $\ker \Phi$  (the kernel of  $\Phi$ ).

**DEFINITION 7.1.** Let  $\mathcal{U}(A, B) = \{X \in B(H) : AXB - CXD \in C_p\}$  and let  $F_p : \mathcal{U} \mapsto \mathbb{R}^+$  be the map defined by  $F_p(X) = \|T - (AXB - CXD)\|_p^p$ , where  $T \in \ker \Phi \cap C_p$ ,  $1 \leq p < \infty$ .

The proof of the following lemma is similar to the proof of [Lemma 4.3](#).

**LEMMA 7.2.** Let  $A, B \in B(H)$  be normal commuting operators. Suppose that  $ASB = BSA$ ,  $S \in C_p$  ( $1 < p < \infty$ ). If

$$A|S|^{p-1}U^*B = B|S|^{p-1}U^*A, \quad (7.1)$$

then

$$A|S|U^*B = B|S|U^*A. \quad (7.2)$$

**THEOREM 7.3.** Let  $A, B, C, D \in B(H)$  be normal operators such that  $AC = CA$  and  $BD = DB$ . Assume that  $ASB = CSD$ ,  $S \in C_p$  ( $1 < p < \infty$ ). If  $A|S|^{p-1}U^*B = C|S|^{p-1}U^*D$ , then  $A|S|U^*B = C|S|U^*D$ .

**PROOF.** It suffices to take the Hilbert space  $H \oplus H$  and the operators

$$A^\sim = \begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix}, \quad B^\sim = \begin{bmatrix} C & 0 \\ 0 & B \end{bmatrix}, \quad S^\sim = \begin{bmatrix} 0 & S \\ 0 & 0 \end{bmatrix} \quad (7.3)$$

and apply [Lemma 7.2](#). □

**THEOREM 7.4.** Let  $A, B, C, D \in B(H)$  be normal operators,  $AC = CA$ , and  $BD = DB$ . Suppose that  $ASB = CSD$ ,  $S \in C_p$ . Then, for  $1 \leq p < \infty$ , the map  $F_p$  has a global minimizer at  $W$  if, and for  $1 < p < \infty$  only if,  $AWB - CWD = 0$ .

**PROOF.** If  $AC = CA$ ,  $BD = DB$ , and  $ASB = CSD$ ,  $S \in C_p$ , then, for  $1 < p < \infty$ , the result of Turnšek [14, Theorem 3.4] guarantees that

$$\|T - (AXB - CXD)\|_p^p \geq \|T\|_p^p, \quad (7.4)$$

that is,  $F_p(X) \geq F_p(W)$ . Conversely, if  $F_p$  has a minimum, then

$$\|T - (AWB - CWD)\|_p^p = \|S\|_p^p. \quad (7.5)$$

Since  $\mathcal{U}$  is convex, then the set  $\mathcal{V} = \{T - (AXB - CXD); X \in \mathcal{U}\}$  is also convex. Thus Theorem 5.2 implies that  $S - (AWB - CWD) = 0$ .  $\square$

**THEOREM 7.5.** *Let  $A, B, C$ , and  $D$  be normal operators in  $B(H)$  such that  $AC = CA$  and  $BD = DB$ . If  $S \in \ker \Phi \cap C_p$ , then, for  $1 < p < \infty$ , the map  $F_p$  has a critical point at  $W$  if and only if  $AWB - CWD = 0$ .*

**PROOF.** Let  $W, S \in \mathcal{U}$  and let  $\phi$  and  $\varphi$  be two maps defined, respectively, by  $\phi : X \mapsto S - (AXB - CXD)$  and  $\varphi : X \mapsto \|X\|_p^p$ . Since the Frechet derivative of  $F_p$  is given by

$$\mathcal{D}_W F_p(T) = \lim_{h \rightarrow 0} \frac{F_p(W + hT) - F_p(W)}{h}, \quad (7.6)$$

it follows that  $\mathcal{D}_W F_p(T) = [\mathcal{D}_{S - (AWB - CWD)}](BTA - DTC)$ . If  $W$  is a critical point of  $F_p$ , then  $\mathcal{D}_W F_p(T) = 0$  for all  $T \in \mathcal{U}$ . By applying Theorem 5.1, we get

$$\begin{aligned} \mathcal{D}_W F_p(T) &= p \operatorname{Retr} [|S - (AWB - CWD)|^{p-1} W^* (BTA - DTC)] \\ &= p \operatorname{Retr} [Y(BTA - DTC)] = 0, \end{aligned} \quad (7.7)$$

where  $S - (AWB - CWD) = W|S - (AWB - CWD)|$  is the polar decomposition of the operator  $S - (AWB - CWD)$  and  $Y = |S - (AWB - CWD)|^{p-1} W^*$ . An easy calculation shows that  $BYA - DYC = 0$ , that is,

$$A|S - (AWB - CWD)|^{p-1} W^* B = C|S - (AWB - CWD)|^{p-1} W^* D. \quad (7.8)$$

It follows from Theorem 7.3 that

$$A|S - (AWB - CWD)| W^* B = C|S - (AWB - CWD)| W^* D. \quad (7.9)$$

By taking adjoints and since  $A$  and  $B$  are normal operators, applying Fuglede-Putnam theorem, we get  $A(T - (AWB - CWD))B = C(T - (AWB - CWD))D$ . Then  $A(AW - WB)B = C(AWB - CWD)D$ . Hence  $AWB - CWD \in R(\Phi) \cap \ker \Phi$ . By the same argument used in the proof of [13, Theorem 3.4], we can prove that

$$\|T - (AXB - CXD)\| \geq \|T\| \quad (7.10)$$

for all  $T \in B(H)$ . Hence  $AWB - CWD = 0$ .



Conversely, if  $AWB = CWD$ , then  $W$  is a minimum, and since  $F_p$  is differentiable, then  $W$  is a critical point.  $\square$

**THEOREM 7.6.** *Let  $A, B, C$ , and  $D$  be normal operators in  $B(H)$  such that  $AC = CA$  and  $BD = DB$ . If  $S \in \ker \Phi \cap C_p$ ,  $0 < p \leq 1$ ,  $\dim H < \infty$ , and  $S - (AWB - CWD)$  is invertible, then  $F_p$  has a critical point at  $W$  if  $AWB - CWD = 0$ .*

**PROOF.** Suppose that  $\dim H < \infty$ . If  $AWB - CWD = 0$ , then  $S$  is invertible by hypothesis. Also  $|S|$  is invertible, hence  $|S|^{p-1}$  exists for  $0 < p \leq 1$ . Taking  $Y = |S|^{p-1}U^*$ , where  $S = U|S|$  is the polar decomposition of  $S$ , choose  $X$  to be the rank-one operator  $f \otimes g$  for some arbitrary elements  $f$  and  $g$  in  $H \oplus H$ . Then  $\text{tr}(Y(AXB - CXD)) = \text{tr}(AYB - CYD)X = 0$  implies that  $\langle \Psi(Y)f, g \rangle = 0 \Leftrightarrow Y \in \ker \Phi$ , that is,  $AYB - CYD = 0$  and  $\text{tr}[(DYC - AYB)T] = 0$  for every  $T \in B(H)$ . Since  $S = S - (AWB - CWD)$ , then

$$\begin{aligned} 0 &= \text{tr}[YDTC - YATB] = \text{tr}[Y(DTC - ATB)] \\ &= p \text{Re tr}[Y(DTC - ATB)] = p \text{Re tr}[|S|^{p-1}U^*(DTC - ATB)] \\ &= (\mathcal{D}_T \phi)(DTC - ATB) = (\mathcal{D}_W F_p)(T). \end{aligned} \quad (7.11)$$

$\square$

**REMARK 7.7.** The set  $\mathcal{S} = \{X : AXB - CXD \in C_p\}$  contains  $C_p$ ; if  $X \in C_p$ , then  $X \in \mathcal{S}$  and, for example,  $I \in \mathcal{S}$  but  $I \notin C_p$ . If  $A \in C_p$ , the conclusions of Theorems 7.3, 7.4, 7.5, and 7.6 hold for all  $X \in B(H)$ .

For  $n > 2$  the generalization of the above results to the elementary operators  $\sum_{i=1}^n A_i X B_i$  is not possible. In [12], Shul'man stated that there exists a normally represented elementary operator of the form  $\sum_{i=1}^n A_i X B_i$  with  $n > 2$  such that  $\text{asc } E > 1$ , that is, the range and kernel have no trivial intersection.

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