

SOME VERSIONS OF ANDERSON'S AND MAHER'S INEQUALITIES II

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We are interested in the investigation of the orthogonality (in the sense of Birkhoff) of the range of an elementary operator and its kernel.

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1. Introduction. Let H be a separable infinite-dimensional complex Hilbert space and let $B(H)$ denote the algebra of all bounded operators on H into itself. Given $A, B \in B(H)$, we define the generalized derivation $\delta_{A,B} : B(H) \rightarrow B(H)$ by $\delta_{A,B}(X) = AX - XB$ and the elementary operator derivation $\Delta_{A,B} : B(H) \rightarrow B(H)$ by $\Delta_{A,B}(X) = AXB - X$. Denote $\delta_{A,A} = \delta_A$, $\Delta_{A,A} = \Delta_A$.

In [1, Theorem 1.7], Anderson shows that if A is normal and commutes with T , then, for all $X \in B(H)$,

$$\|T + \delta_A(X)\| \geq \|T\|. \quad (1.1)$$

It is shown in [9] that if the pair (A, B) has the Fuglede-Putnam property (in particular, if A and B are normal operators) and $AT = TB$, then, for all $X \in B(H)$,

$$\|T + \delta_{A,B}(X)\| \geq \|T\|. \quad (1.2)$$

Duggal [3] showed that the above inequality (1.2) is also true when $\delta_{A,B}$ is replaced by $\Delta_{A,B}$. The related inequality (1.1) was obtained by the author [10], showing that if the pair (A, B) has the Fuglede-Putnam property $(FP)_{C_p}$, then

$$\|T + \delta_{A,B}(X)\|_p \geq \|T\|_p \quad (1.3)$$

for all $X \in B(H)$, where C_p is the von Neumann-Schatten class, $1 \leq p < \infty$, and $\|\cdot\|_p$ is its norm for all $X \in B(H)$ and for all $T \in C_p \cap \ker \delta_{A,B}$. In all of the above results, A was not arbitrary. In fact, certain normality-like assumptions have been imposed on A . A characterization of $T \in C_p$ for $1 < p < \infty$, which is orthogonal to $R(\delta_A|C_p)$ (the range of $\delta_A|C_p$) for a general operator A , has

been carried out by Kittaneh [6], showing that if T has the polar decomposition $T = U|T|$, then

$$\|T + \delta_A(X)\|_p \geq \|T\|_p \quad (1.4)$$

for all $X \in C_p$ ($1 < p < \infty$) if and only if $|T|^{p-1}U^* \in \ker \delta_A$. By a simple modification in the proof of the above inequality, we can prove that this inequality is also true in the general case, that is, if T has the polar decomposition $T = U|T|$, then

$$\|T + \delta_{A,B}(X)\|_p \geq \|T\|_p \quad (1.5)$$

for all $X \in C_p$ ($1 < p < \infty$) if and only if $|T|^{p-1}U^* \in \ker \delta_{B,A}$. In Sections 1, 2, 3, and 4, we prove these results in the case where we consider $E_{A,B}$ instead of $\delta_{A,B}$, which leads us to prove that if $T \in C_p$ and $\ker E_{A,B} \subseteq \ker E_{A,B}^*$, then

$$\|T + E_{A,B}(X)\|_p \geq \|T\|_p \quad (1.6)$$

for all $X \in C_p$ ($1 < p < \infty$) if and only if $T \in \ker E_{A,B}$. In Sections 5, 6, and 7, we minimize the map $\|S + E_{A,B}(X)\|_p$ and we classify its critical points.

2. Preliminaries. Let $T \in B(H)$ be compact and let $s_1(X) \geq s_2(X) \geq \dots \geq 0$ denote the singular values of T , that is, the eigenvalues of $|T| = (T^*T)^{1/2}$ arranged in their decreasing order. The operator T is said to belong to the Schatten p -class C_p if

$$\|T\|_p = \left[\sum_{i=1}^{\infty} s_j(T)^p \right]^{1/p} = [\operatorname{tr}(T^p)]^{1/p}, \quad 1 \leq p < \infty, \quad (2.1)$$

where tr denotes the trace functional. Hence, C_1 is the trace class, C_2 is the Hilbert-Schmidt class, and C_∞ is the class of compact operators with

$$\|T\|_\infty = s_1(T) = \sup_{\|f\|=1} \|Tf\| \quad (2.2)$$

denoting the usual operator norm. For the general theory of the Schatten p -classes, the reader is referred to [7, 11].

Recall that the norm $\|\cdot\|$ of the B -space V is said to be Gateaux differentiable at nonzero elements $x \in V$ if

$$\lim_{t \rightarrow 0, t \in \mathbb{R}} \frac{\|x + t\gamma\| - \|x\|}{t} = \mathcal{R}D_x(\gamma) \quad (2.3)$$

for all $\gamma \in V$. Here \mathbb{R} denotes the set of reals, \mathcal{R} denotes the real part, and D_x is the unique support functional (in the dual space V^*) such that $\|D_x\| = 1$ and $D_x(x) = \|x\|$. The Gateaux differentiability of the norm at x implies that x is a smooth point of the sphere of radius $\|x\|$.

It is well known (see [7] and the references therein) that, for $1 < p < \infty$, C_p is a uniformly convex Banach space. Therefore, every nonzero $T \in C_p$ is a smooth point and, in this case, the support functional of T is given by

$$D_T(X) = \text{tr} \left[\frac{|T|^{p-1} UX^*}{\|T\|_p^{p-1}} \right] \quad (2.4)$$

for all $X \in C_p$, where $T = U|T|$ is the polar decomposition of T .

DEFINITION 2.1. Let E be a complex Banach space. We define the orthogonality in E . We say that $b \in E$ is orthogonal to $a \in E$ if, for all complex λ , there holds

$$\|a + \lambda b\| \geq \|a\|. \quad (2.5)$$

This definition has a natural geometric interpretation, namely, $b \perp a$ if and only if the complex line $\{a + \lambda b \mid \lambda \in \mathbb{C}\}$ is disjoint with the open ball $K(0, \|a\|)$, that is, if and only if this complex line is a tangent one. Note that if b is orthogonal to a , then a needs not be orthogonal to b . If E is a Hilbert space, then from (2.5), it follows that $\langle a, b \rangle = 0$, that is, orthogonality in the usual sense.

3. The elementary operators $AXB - CXD$

LEMMA 3.1. Let $A, B \in B(H)$. The following statements are equivalent:

- (1) the pair (A, B) has the property $(FP)_{C_p}$, $1 \leq p < \infty$;
- (2) if $AT = TB$, where $T \in C_p$, then $\overline{R(T)}$ reduces A , $\ker(T)^\perp$ reduces B , and $A|_{\overline{R(T)}}$ and $B|_{\ker(T)^\perp}$ are normal operators.

PROOF. (1) \Rightarrow (2). Since C_p is a bilateral ideal and $T \in C_p$, then $AT \in C_p$. Hence as $AT = TB$ and (A, B) satisfies $(FP)_{C_p}$, $A^*T = TB^*$, and so, $\overline{R(T)}$ and $\ker(T)^\perp$ are reducing subspaces for A and B , respectively. Since $A(AT) = (AT)B$ implies that $A^*(AT) = (AT)B^*$ by $(FP)_{C_p}$ and the equality $A^*T = TB^*$ implies that $A^*AT = AA^*T$, thus we see that $A|_{\overline{R(T)}}$ is normal. Clearly, (B^*, A^*) satisfies $(FP)_{C_p}$ and $B^*T^* = T^*A^*$. Therefore, it follows from the above argument that $B^*|_{\overline{R(T^*)}} = B|_{\ker(T)^\perp}$ is normal.

(2) \Rightarrow (1). Let $T \in C_p$ such that $AT = TB$. Taking the two decompositions of H , $H_1 = H = \overline{R(T) \oplus R(T)^\perp}$ and $H_2 = H = \ker(T)^\perp \oplus \ker T$, then we can write A and B on H_1 into H_2 , respectively:

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}, \quad B = \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix}, \quad (3.1)$$

where A_1 and B_1 are normal operators. Also we can write T and X on H_2 into H_1 :

$$T = \begin{bmatrix} T_1 & 0 \\ 0 & 0 \end{bmatrix}, \quad X = \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix}. \quad (3.2)$$

It follows from $AT = TB$ that $A_1 T_1 = T_1 B_1$. Since A_1 and B_1 are normal operators, then, by applying the Fuglede-Putnam theorem, we obtain $A_1^* T_1 = T_1 B_1^*$, that is, $A^* T = TB^*$. \square

THEOREM 3.2. *Let $A, B \in B(H)$. If A and B are normal operators, then*

$$\|S - (AX - XB)\|_p \geq \|S\|_p \quad (3.3)$$

for all $X \in C_p$ and for all $S \in \ker \delta_{A,B} \cap C_p$ ($1 \leq p < \infty$).

PROOF. Let $S = U|S|$ be the polar decomposition of S , where U is an isometry such that $\ker U = \ker |S|$. Since

$$\|U^*S\|_p \leq \|U^*\|_p \|S\|_p = \|S\|_p \quad (3.4)$$

for all $S \in C_p$, then

$$\|S - (AX - XB)\|_p^p \geq \|U^*[S - (AX - XB)]\|_p^p = \||S| - U^*(AX - XB)\|_p^p, \quad (3.5)$$

and we have

$$\||S| - U^*(AX - XB)\|_p^p \geq \sum_n |\langle [||S| - U^*(AX - XB)]\varphi_n, \varphi_n \rangle|^p \quad (3.6)$$

for any orthonormal basis $\{\varphi_n\}_{n \geq 1}$ of H . Since $AS = SB$, and A and B are normal operators, it follows from the Fuglede-Putnam theorem that $S^*A = BS^*$. Consequently, $S^*AS = BS^*S$ or $S^*SB = BS^*S$, that is, $B|S| = |S|B$. Since $|S|$ is a compact normal operator and commutes with B , there exists an orthonormal basis $\{f_k\} \cup \{g_m\}$ of H such that $\{f_k\}$ consists of common eigenvectors of B and $|S|$, and $\{g_m\}$ is an orthonormal basis of $\ker |S|$. Since $\{f_k\}$ is an orthonormal basis of the normal operator B , then there exists a scalar α_k such that $Bf_k = \alpha_k f_k$ and $B^*f_k = \bar{\alpha}_k f_k$. Consequently,

$$\begin{aligned} \langle U^*(AX - XB)f_k, |S|f_k \rangle &= \langle S^*(AX - XB)f_k, f_k \rangle \\ &= \langle (B(S^*X) - (S^*X)B)f_k, f_k \rangle = 0, \end{aligned} \quad (3.7)$$

that is, $\langle U^*(AX - XB)f_k, f_k \rangle = 0$.

In (3.6) take $\{\varphi_n\} = \{f_k\} \cup \{g_m\}$ as an orthonormal basis of H , then

$$\begin{aligned} &\sum_n |\langle [||S| - U^*(AX - XB)]\varphi_n, \varphi_n \rangle|^p \\ &\geq \sum_k |\langle |S|f_k, f_k \rangle|^p + \sum_m |\langle U^*(AX - XB)g_m, g_m \rangle|^p \\ &\geq \sum_k |\langle |S|f_k, f_k \rangle|^p = \|S\|_p^p. \end{aligned} \quad (3.8)$$

\square

LEMMA 3.3. *Let $A, B \in B(H)$ satisfying $(FP)_{C_p}$. Then*

$$\|S + AX - XB\|_p^p \geq \|S\|_p^p \quad (3.9)$$

for every operator $S \in \ker \delta_{A,B} \cap C_p$ ($1 < p < \infty$) and for all $X \in C_p$.

PROOF. If the pair (A, B) satisfies the $(FP)_{C_p}$ property, then $\overline{R(S)}$ reduces $A, \ker^\perp S$ reduces B , and $A|_{\overline{R(S)}}$ and $B|_{\ker^\perp S}$ are normal operators. Letting $S_0 : \ker^\perp S \rightarrow \overline{R(S)}$ be the quasiaffinity defined by setting $S_0x = Sx$ for each $x \in \ker^\perp S$, it results that $\delta_{A_1, B_1}(S_0) = \delta_{A_1^*, B_1^*}(S_0) = 0$. Let $A = A_1 \oplus A_2$, with respect to $H = \overline{R(S)} \oplus \overline{R(S)}^\perp$, $A = B_1 \oplus B_2$, with respect to $H = \ker(S)^\perp \oplus \ker S$, and $X : \overline{R(S)} \oplus \overline{R(S)}^\perp \rightarrow \ker(S)^\perp \oplus \ker S$ have the matrix representation

$$X = \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix}. \quad (3.10)$$

Then we have

$$\|S - (AX - XB)\|_p = \left\| \begin{bmatrix} S_1 - (A_1 X_1 - X_1 B_1) & * \\ * & * \end{bmatrix} \right\|_p. \quad (3.11)$$

The result of Gohberg and Krein [4] guarantees that

$$\|S - (AX - XB)\|_p \geq \|S_1 - (A_1 X_1 - X_1 B_1)\|_p. \quad (3.12)$$

Since A_1 and B_1 are two normal operators, then it results from **Theorem 3.5** that

$$\|S_1 - (A_1 X_1 - X_1 B_1)\|_p \geq \|S_1\|_p = \|S\|_p. \quad (3.13)$$

□

LEMMA 3.4 [6]. *Let u and v be two elements of a Banach space V with norm $\|\cdot\|$. If u is a smooth point, then $D_u(v) = 0$ if and only if*

$$\|u + zv\| \geq \|u\| \quad (3.14)$$

for all $z \in \mathbb{C}$ (the complex numbers).

THEOREM 3.5. *Let $A, B \in B(H)$ and $T \in C_p$ ($1 < p < \infty$). Then*

$$\|T + \delta_{A,B}(X)\|_p \geq \|T\|_p \quad (3.15)$$

for all $X \in B(H)$ with $\Delta_{A,B}(X) \in C_p$ if and only if

$$\text{tr}(|T|^{p-1} U^* \delta_{A,B}(X)) = 0 \quad (3.16)$$

for all such X .

PROOF. The theorem is an immediate consequence of equality (2.4) and Lemma 3.4. \square

THEOREM 3.6. *Let $A, B \in B(H)$ and $T \in C_p$ ($1 < p < \infty$). Then*

$$\|T + \delta_{A,B}(X)\|_p \geq \|T\|_p \quad (3.17)$$

for all $X \in C_p$ if and only if $\tilde{T} = |T|^{p-1}U^* \in \ker \delta_{B,A}$.

PROOF. By virtue of Theorem 3.5, it is sufficient to show that $\text{tr}(\tilde{T}\delta_{A,B}(X)) = 0$ for all $X \in C_p$ if and only if $\tilde{T} \in \ker \delta_{B,A}$.

Choose X to be the rank-one operator $f \otimes g$ for some arbitrary elements f and g in H . Then $\text{tr}(\tilde{T}(AX - XB)) = \text{tr}(B\tilde{T} - \tilde{T}A)X = 0$ implies that $\langle \delta_{B,A}(\tilde{T})f, g \rangle = 0 \Leftrightarrow \tilde{T} \in \ker \delta_{B,A}$.

Conversely, assume that $\tilde{T} \in \ker \delta_{B,A}$, that is, $B\tilde{T} = \tilde{T}A$. Since $\tilde{T}X$ and $\tilde{T}\delta_{B,A}$ are trace classes, then for all $X \in C_p$, we get

$$\begin{aligned} \text{tr}(\tilde{T}(AX - XB)) &= \text{tr}(\tilde{T}AX - \tilde{T}XB) = \text{tr}(XB\tilde{T} - X\tilde{T}A) \\ &= \text{tr}(X\delta_{B,A}(\tilde{T})) = 0. \end{aligned} \quad (3.18) \quad \square$$

LEMMA 3.7. *Let $A, B \in B(H)$ and $S \in C_p$ such that $\delta_{A,B}(T) = 0 = \delta_{A,B}^*(T)$.*

If $A|S|^{p-1}U^ = |S|^{p-1}U^*B$, where $p > 1$ and $S = U|S|$ is the polar decomposition of S , then $A|S|U^* = |S|U^*B$.*

PROOF. If $T = |S|^{p-1}$, then

$$ATU^* = TU^*B. \quad (3.19)$$

We prove that

$$AT^nU^* = T^nU^*B \quad (3.20)$$

for all $n \geq 1$. If $S = U|S|$, then

$$\begin{aligned} \ker U &= \ker |S| = \ker |S|^{p-1} = \ker T, \\ (\ker U)^\perp &= (\ker T)^\perp = \overline{R(T)}. \end{aligned} \quad (3.21)$$

This shows that the projection U^*U onto $(\ker T)^\perp$ satisfies $U^*UT = T$ and $TU^*UT = T^2$. By taking the adjoints of (3.19) and since A and B are normal operators applying Fuglede-Putnam theorem, we get $BUT = UTA$ and $AT^2 = ATU^*UT = TU^*BUT = TU^*UTA = T^2A$.

Since A commutes with the positive operator T^2 , A commutes with its square roots, that is,

$$AT = TA. \quad (3.22)$$

By (3.19) and (3.22) we obtain (3.20). Let $f(t)$ be the map defined on $\sigma(T) \subset \mathbb{R}^+$ by $f(t) = t^{1/(p-1)}$ ($1 < p < \infty$). Since f is the uniform limit of a sequence

(P_i) of polynomials without constant term (since $f(0) = 0$), it follows from (3.20) that $AP_i(T)U^* = P_i(T)U^*B$. Therefore, $AT^{1/(p-1)}U^* = U^*T^{1/(p-1)}B$. \square

THEOREM 3.8. *Let A and B be operators in $B(H)$ such that $\delta_{A,B}(T) = 0 = \delta_{A,B}^*(T)$. Then $T \in \ker \Delta_{A,B} \cap C_p$ if and only if*

$$\|S + \delta_{A,B}(X)\|_p \geq \|S\|_p \quad (3.23)$$

for all $X \in C_p$.

PROOF. If $S \in \ker \Delta_{A,B}$, then it follows from Lemma 3.3 that

$$\|S + \delta_{A,B}(X)\|_p \geq \|S\|_p \quad (3.24)$$

for all $X \in C_p$. Conversely, if

$$\|S + \delta_{A,B}(X)\|_p \geq \|S\|_p \quad (3.25)$$

for all $X \in C_p$, then, from Theorem 3.6,

$$A|S|^{p-1}U^* = |S|^{p-1}U^*B. \quad (3.26)$$

Since $\delta_{A,B}(S) = 0 = \delta_{A,B}^*(S)$,

$$A^*|S|^{p-1}U^* = |S|^{p-1}U^*B^*. \quad (3.27)$$

By taking adjoints, we get

$$AU|S|^{p-1} = U|S|^{p-1}B. \quad (3.28)$$

From Lemma 3.7, it follows that $AU|S| = U|S|B$, that is, $S \in \ker \Delta_{A,B}$. \square

REMARK 3.9. (1) It is well known that the Hilbert-Schmidt class C_2 is a Hilbert space under the inner product $\langle Y, Z \rangle = \text{tr } Z^*Y$.

We remark here that for the Hilbert-Schmidt norm $\|\cdot\|_2$, the orthogonality result in Theorem 3.8 is to be understood in the usual Hilbert-space sense. Note in the case where $I = C_2$ that

$$\|T + \delta_{A,B}(X)\|_2^2 = \|\delta_{A,B}(X)\|_2^2 + \|T\|_2^2 \quad (3.29)$$

for all $X \in C_2$ if and only if $AT^* = T^*B$. This can be seen as an immediate consequence of the fact that

$$R(\delta_{A,B}|C_2)^\perp = \ker(\delta_{A,B}|C_2)^* = \ker(\delta_{B^*,A^*}|C_2). \quad (3.30)$$

(2) It is known [2] that if A and B are contractions and $S \in C_p$, then $\delta_{A^*,B^*}(S) = \delta_{A,B}(S) = 0$. Hence

$$\|S + \delta_{A,B}(X)\|_p \geq \|S\|_p \quad (3.31)$$

holds for all $X \in C_p$ if and only if $S \in \ker(\delta_{A,B}|C_p)$.

(3) If $A = B$, then the following counterexample shows that [Theorem 3.8](#) does not hold if $p < 1$. Take $p = 1/2$ and

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad S = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad X = \begin{bmatrix} 0 & -\alpha \\ \alpha & 0 \end{bmatrix}, \quad (3.32)$$

where α is real such that $0 < \alpha < 1$. We have

$$S - (AX - XA) = \begin{bmatrix} 1 & \alpha \\ \alpha & 1 \end{bmatrix} \quad (3.33)$$

and, for eigenvectors $\beta_1 = 1 - \alpha$, $\beta_2 = 1 + \alpha$. Then

$$\|S - (AX - XA)\|_{1/2} = [(1 - \alpha)^{1/2} + (1 + \alpha)^{1/2}]^2 < 4 = \|S\|_{1/2}. \quad (3.34)$$

COROLLARY 3.10. *Let $A, B \in L(H)$. Then*

$$\|S + AX - XB\|_p \geq \|S\|_p \quad (3.35)$$

if and only if $S \in \ker \delta_{A,B} \cap C_p$ and for all $X \in C_p$, in each of the following cases:

- (1) if $A, B \in L(H)$ such that $\|Ax\| \geq \|x\| \geq \|Bx\|$ for all $x \in H$,
- (2) if A is invertible and B is such that $\|A^{-1}\| \|B\| \leq 1$.

PROOF. The result of Tong [\[13, Lemma 1\]](#) guarantees that the above condition implies that, for all $T \in \ker(\delta_{A,B}|K(H))$, $\overline{R(T)}$ reduces A , $\ker(T)^\perp$ reduces B , and $A|_{\overline{R(T)}}$ and $B|_{\ker(T)^\perp}$ are unitary operators. Hence it results from [Lemma 3.1](#) that the pair (A, B) has the property $(FP)_{K(H)}$ and the results hold by [Theorem 3.8](#). Here $K(H)$ is the ideal of compact operators.

The above inequality holds in particular if $A = B$ is isometric; in other words, $\|Ax\| = \|x\|$ for all $x \in H$.

(2) In this case, it suffices to take $A_1 = \|B\|^{-1}A$, $B_1 = \|B\|^{-1}B$.

Then $\|A_1x\| \geq \|x\| \geq \|B_1x\|$ and the result holds by (1) for all $x \in H$. \square

4. Orthogonality and the elementary operators $AXB - CXD$. Let H be a separable infinite-dimensional complex Hilbert space and let $B(H)$ denote the algebra of all bounded operators on H into itself. Given A, B, C , and D normal operators in $B(H)$ such that $AC = CA$, $BD = DB$, we define the elementary operator $\Psi : B(H) \rightarrow B(H)$ by $\Psi(X) = AXB - CXD$. We prove that if $T \in C_p$ ($1 < p < \infty$), then $\|T + \Phi(X)\|_p \geq \|T\|_p$ if and only if $T \in \ker \Phi$ for all $X \in C_p$.

By the same argument used in the proofs of [Theorems 3.5](#) and [3.6](#), we prove the following theorems.

THEOREM 4.1. *Let $A, B, C, D \in B(H)$ and $T \in C_p$ ($1 < p < \infty$). Then*

$$\|T + \Psi(X)\|_p \geq \|T\|_p \quad (4.1)$$

for all $X \in B(H)$ with $\Psi(X) \in C_p$ if and only if

$$\text{tr}(|T|^{p-1}U^*\Psi(X)) = 0 \quad (4.2)$$

for all such X .

THEOREM 4.2. *Let $A, B, C, D \in B(H)$ and $T \in C_p$ ($1 < p < \infty$). Then*

$$\|T + \Psi(X)\|_p \geq \|T\|_p \quad (4.3)$$

for all $X \in C_p$ if and only if $\tilde{T} = |T|^{p-1}U^* \in \ker \Psi$.

LEMMA 4.3. *Let $A, B \in B(H)$ be normal operators and $AB = BA$. Suppose that $ASB = BSA$, $S \in C_p$ ($1 < p < \infty$). If*

$$AU|S|^{p-1}B = BU|S|^{p-1}A, \quad (4.4)$$

then

$$AU|S|B = BU|S|A. \quad (4.5)$$

PROOF. Assume that $B^{-1} \in B(H)$. Then, from $ASB = BSA$ and $AB = BA$, we get $AB^{-1}S = SB^{-1}A$. Hence, applying the above lemma to the operators AB^{-1} , $B^{-1}A$, and S , we get

$$AB^{-1}U|S|^{p-1} = U|S|^{p-1}B^{-1}A, \quad (4.6)$$

which implies that

$$AB^{-1}U|S| = U|S|B^{-1}A. \quad (4.7)$$

Multiply (4.6) and (4.7) at right and left by B to obtain

$$BAB^{-1}U|S|^{p-1}B = BU|S|^{p-1}B^{-1}AB \quad (4.8)$$

or

$$ABB^{-1}U|S|^{p-1}B = BU|S|^{p-1}B^{-1}BA, \quad (4.9)$$

that is,

$$AU|S|^{p-1}B = BU|S|^{p-1}A, \quad (4.10)$$

which implies that

$$AU|S|B = BU|S|A. \quad (4.11)$$

Consider now the case when B is injective, that is, $\ker B = \{0\}$. Let

$$\Delta_n = \left\{ \lambda \in \mathbb{C} : |\lambda| \leq \frac{1}{n} \right\} \quad (4.12)$$

and let $E_B(\Delta_n)$ be the corresponding spectral projector.

Putting

$$P_n = I - E_B(\Delta_n), \quad (4.13)$$

the subspace $P_n H$ reduces both operators A and B (since they commute and are normal). Hence, with respect to the decomposition

$$\begin{aligned} H &= (I - P_n)H \oplus P_n H, \\ A &= \begin{bmatrix} A_1^{(n)} & 0 \\ 0 & A_2^{(n)} \end{bmatrix}, \quad B = \begin{bmatrix} B_1^{(n)} & 0 \\ 0 & B_2^{(n)} \end{bmatrix}, \\ S &= \begin{bmatrix} S_{11}(n) & S_{12}(n) \\ S_{21}(n) & S_{22}(n) \end{bmatrix}, \quad X = \begin{bmatrix} X_{11}(n) & X_{12}(n) \\ X_{21}(n) & X_{22}(n) \end{bmatrix}, \end{aligned} \quad (4.14)$$

it is easy to see that $B_2^{(n)}$ acting on $P_n H$ is invertible. Then, from $ASB = BSA$, it follows that

$$A_2^{(n)} S_{22}(n) B_2^{(n)} = B_2^{(n)} S_{22}(n) A_2^{(n)}, \quad (4.15)$$

and, from $AB = BA$, we get $A_2 B_2 = B_2 A_2$. Since

$$AU|S|^{p-1}B = BU|S|^{p-1}A, \quad (4.16)$$

according to the first part of the proof, it follows that

$$A_2^{(n)} U |S_{22}(n)|^{p-1} B_2^{(n)} = B_2^{(n)} U |S_{22}(n)|^{p-1} A_2^{(n)}, \quad (4.17)$$

which implies that

$$A_2^{(n)} U |S_{22}(n)| B_2^{(n)} = B_2^{(n)} U |S_{22}(n)| A_2^{(n)}, \quad (4.18)$$

so we have $AU|S|B = BU|S|A$. Assume now $\ker A \cap \ker B = \{0\}$.

Then $\ker B$ reduces A and $P_{\ker B} A P_{\ker B}$ is injective. Let $H = \ker B \oplus H_1$ ($H_1 = H \ominus \ker B$). Then we have

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 0 & B_2 \end{bmatrix}, \quad S = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix}, \quad (4.19)$$

where A_1, B_2 are injective and their ranges are dense in subspaces they act on. We have

$$ASB - BSA = \begin{bmatrix} 0 & A_1 S_{12} B_2 \\ -B_2 S_{21} A_1 & A_2 S_{22} B_2 - B_2 S_{22} A_2 \end{bmatrix}. \quad (4.20)$$

Now, if $ASB = BSA$, then $A_2S_{22}B_2 = B_2S_{22}A_2$, $B_2S_{21}A_1 = 0$, and $A_1S_{12}B_2 = 0$, that is, $S_{21} = S_{12} = 0$. It follows that

$$S = \begin{bmatrix} S_{11} & 0 \\ 0 & S_{22} \end{bmatrix}. \quad (4.21)$$

Since $A_2B_2 = B_2A_2$, $A_2S_{22}B_2 = B_2S_{22}A_2$, and B_2 is injective, and we have already proved that

$$A_2U|S_{22}|^{p-1}B_2 = B_2U|S_{22}|^{p-1}A_2 \quad (4.22)$$

implies

$$A_2U|S_{22}|B_2 = B_2U|S_{22}|A_2, \quad (4.23)$$

so we have $AU|S|B = BU|S|A$. \square

Let $\Phi(X) = AXB - BXA$. We prove the following theorem.

THEOREM 4.4. *Let $A, B \in B(H)$ be normal operators, $AB = BA$, and $S \in C_p$ ($1 < p < \infty$). Then $S \in \ker\Phi$ if and only if*

$$\|S - (AXB - BXA)\|_p \geq \|S\|_p \quad (4.24)$$

for all $X \in C_p$.

PROOF. If $S \in \ker\Phi$, then, from [13, Theorem 3.4], it follows that

$$\|S + \Phi(X)\|_p \geq \|S\|_p \quad (4.25)$$

for all $X \in C_p$. Conversely, if

$$\|S + \Phi(X)\|_p \geq \|S\|_p \quad (4.26)$$

for all $X \in C_p$, then, from [Theorem 4.2](#),

$$A|S|^{p-1}U^*B = B|S|^{p-1}U^*A. \quad (4.27)$$

Since A and B are normal operators applying Fuglede-Putnam theorem, we get $A^*|S|^{p-1}U^*B^* = B^*|S|^{p-1}U^*A^*$. By taking adjoints, we get $AU|S|^{p-1}B = BU|S|^{p-1}A$.

From [Lemma 4.3](#), it follows that $AU|S|B = BU|S|A$, that is, $S \in \ker\Phi$. \square

Let $\Psi(X) = AXB - CXD$.

THEOREM 4.5. *Let $A, B, C, D \in B(H)$ be normal operators, $AC = CA$, $BD = DB$, and $S \in C_p$ ($1 < p < \infty$). Then $S \in \ker\Psi$ if and only if*

$$\|S - (AXB - CXD)\|_p \geq \|S\|_p \quad (4.28)$$

for all $X \in C_p$.

PROOF. It suffices to take the Hilbert space $H \oplus H$ and the operators

$$\begin{aligned} A^{\sim} &= \begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix}, & B^{\sim} &= \begin{bmatrix} C & 0 \\ 0 & B \end{bmatrix}, \\ S^{\sim} &= \begin{bmatrix} 0 & S \\ 0 & 0 \end{bmatrix}, & X^{\sim} &= \begin{bmatrix} 0 & X \\ 0 & 0 \end{bmatrix} \end{aligned} \quad (4.29)$$

and apply [Theorem 4.4](#). \square

REMARK 4.6. The results of the above theorems can be obtained when the normality of A and B is replaced by some other condition, in particular, if $|A| = |B|$, $|A^*| = |B^*|$. In this case, it suffices to take

$$\begin{aligned} A^{\sim} &= \begin{bmatrix} 0 & A^* \\ B & 0 \end{bmatrix}, & B^{\sim} &= \begin{bmatrix} 0 & B^* \\ A & 0 \end{bmatrix}, \\ S^{\sim} &= \begin{bmatrix} 0 & S \\ 0 & 0 \end{bmatrix}, & X^{\sim} &= \begin{bmatrix} 0 & X \\ 0 & 0 \end{bmatrix} \end{aligned} \quad (4.30)$$

and apply [Lemma 4.3](#) and [Theorem 4.4](#).

5. On minimizing $\|AX - XB - T\|_p^p$. Maher [8, Theorem 3.2] shows that if A is normal and $S \in \ker \delta_A \cap C_p$ ($1 \leq p < \infty$), then the map F_p defined by $F_p(X) = \|S - (AX - XA)\|_p^p$ has a global minimizer at V if, and for $1 < p < \infty$ only if, $AV - VA = 0$.

In this section, we prove that if the pair (A, B) has the property $(FP)_{C_p}$ (i.e., $AT = TB$, where $T \in C_p$, implies $A^*T = TB^*$), $1 \leq p < \infty$, and $S \in \ker \delta_{A,B} \cap C_p$, then the map F_p defined by $F_p(X) = \|S - (AX - XB)\|_p^p$ has a global minimizer at V if, and for $1 < p < \infty$ only if, $AV - VB = 0$. In other words, we have

$$\|S - (AX - XB)\|_p^p \geq \|T\|_p^p \quad (5.1)$$

if, and for $1 < p < \infty$ only if, $AV - VB = 0$. Thus in Halmos' terminology [5], the zero commutator is the commutator approximant in C_p of T . Additionally, we show that if the pair (A, B) has the property $(FP)_{C_p}$ and $S \in \ker \delta_{A,B} \cap C_p$ ($1 < p < \infty$), then the map F_p has a critical point at W if and only if $AW - WB = 0$, that is, if $\mathcal{D}_W F_p$ is the Frechet derivative at W of F_p , the set $\{W \in B(H) : \mathcal{D}_W F_p = 0\}$ coincides with $\ker \delta_{A,B}$ (the kernel of $\delta_{A,B}$).

THEOREM 5.1 [9]. *If $1 < p < \infty$, then the map $F_p : C_p \mapsto \mathbb{R}^+$ defined by $X \mapsto \|X\|_p^p$ is differentiable at every $X \in C_p$ with derivative $\mathcal{D}_X F_p$ given by $\mathcal{D}_X F_p(T) = p \operatorname{Re} \operatorname{tr}(|X|^{p-1} U^* T)$, where tr denotes trace, $\operatorname{Re} z$ is the real part of a complex number z , and $X = U|X|$ is the polar decomposition of X . If $\dim H < \infty$, then the same result holds for $0 < p \leq 1$ at every invertible X .*

THEOREM 5.2 [9]. *If \mathcal{U} is a convex set of C_p , $1 < p < \infty$, then the map $X \mapsto \|X\|_p^p$, where $X \in \mathcal{U}$, has at most a global minimizer.*

DEFINITION 5.3. Let $\mathcal{U}(A, B) = \{X \in B(H) : AX - XB \in C_p\}$ and let $F_p : \mathcal{U} \rightarrow \mathbb{R}^+$ be the map defined by $F_p(X) = \|T - (AX - XB)\|_p^p$, where $T \in \ker \delta_{A,B} \cap C_p$, $1 \leq p < \infty$.

6. Main results. By simple modifications in the proof of [Lemma 3.7](#), we can prove the following lemma.

LEMMA 6.1. Let $A, B \in B(H)$ and $C \in B(H)$ such that the pair (A, B) has the property $(FP)_{B(H)}$. If $A|S|^{p-1}U^* = |S|^{p-1}U^*B$, where $p > 1$ and $S = U|S|$ is the polar decomposition of S , then $A|S|U^* = |S|U^*B$.

THEOREM 6.2. Let $A, B \in \mathcal{L}(H)$. If the pair (A, B) has the property $(FP)_{C_p}$ and $S \in C_p$ such that $AS = SB$, then

- (1) for $1 \leq p < \infty$, the map F_p has a global minimizer at W if, and for $1 < p < \infty$ only if, $AW - WB = 0$;
- (2) for $1 < p < \infty$, the map F_p has a critical point at W if and only if $AW - WB = 0$;
- (3) for $0 < p \leq 1$ $\dim \mathcal{H} < \infty$ and $S - (AW - WB)$ is invertible, then F_p has a critical point at W if $AW - WB = 0$.

PROOF. Since the pair (A, B) has the property $(FP)_{C_p}$, it follows from [Lemma 3.3](#) that

$$\|S - (AX - XB)\|_p^p \geq \|S\|_p^p, \quad (6.1)$$

that is, $F_p(X) \geq F_p(W)$.

Conversely, if F_p has a minimum, then

$$\|S - (AW - WB)\|_p^p = \|S\|_p^p. \quad (6.2)$$

Since \mathcal{U} is convex, the set $\mathcal{V} = \{S - (AX - XB); X \in \mathcal{U}\}$ is also convex. Thus [Theorem 5.2](#) implies that

$$S - (AW - WB) = S. \quad (6.3)$$

(2) Let $W, S \in \mathcal{U}$ and let ϕ and φ be two maps defined, respectively, by $\phi : X \mapsto S - (AX - XB)$ and $\varphi : X \mapsto \|X\|_p^p$.

Since the Frechet derivative of F_p is given by

$$\mathcal{D}_W F_p(T) = \lim_{h \rightarrow 0} \frac{F_p(W + hT) - F_p(W)}{h}, \quad (6.4)$$

it follows that

$$\mathcal{D}_W F_p(T) = [\mathcal{D}_{S - (AW - WB)}](TB - AT). \quad (6.5)$$

If W is a critical point of F_p , then $\mathcal{D}_W F_p(T) = 0$ for all $T \in \mathcal{U}$. By applying [Theorem 5.1](#), we get

$$\begin{aligned}\mathcal{D}_W F_p(T) &= p \operatorname{Re} \operatorname{tr} [|S - (AW - WB)|^{p-1} W^* (TB - AT)] \\ &= p \operatorname{Re} \operatorname{tr} [Y(TB - AT)] = 0,\end{aligned}\tag{6.6}$$

where $S - (AW - WB) = W|S - (AW - WB)|$ is the polar decomposition of the operator $S - (AW - WB)$ and $Y = |S - (AW - WB)|^{p-1} W^*$. An easy calculation shows that $BY - YA = 0$, that is,

$$A |S - (AW - WB)|^{p-1} W^* = |S - (AW - WB)|^{p-1} W^* B.\tag{6.7}$$

It follows from [Lemma 6.1](#) that

$$A |S - (AW - WB)| W^* = |S - (AW - WB)| W^* B.\tag{6.8}$$

By taking adjoints and since the pair (A, B) has the property $(FP)_{C_p}$, we get $A(T - (AW - WB)) = (T - (AW - WB))B$. Then $A(AW - WB) = (AW - WB)B$. Hence

$$AW - WB \in R(\delta_{A,B}) \cap \ker \delta_{A,B}.\tag{6.9}$$

By the same argument used in the proof of [Lemma 6.1](#) we can prove that

$$\|S - (AX - XB)\| \geq \|S\|\tag{6.10}$$

for all $X \in B(H)$ and for all $T \in B(H)$ and it results that $AW - WB = 0$.

Conversely, if $AW = WB$, then W is a minimum, and since F_p is differentiable, then W is a critical point.

(3) Suppose that $\dim H < \infty$. If $AW - WB = 0$, then S is invertible by hypothesis. Also $|S|$ is invertible, hence $|S|^{p-1}$ exists for $0 < p \leq 1$ taking $Y = |S|^{p-1} U^*$, where $S = U|S|$ is the polar decomposition of S . Since $AS = SB$ implies that $S^* A = BS^*$, then $S^* AS = BS^* S$, and this implies that $|S|^2 B = B|S|^2$ and $|S|B = B|S|$.

Since $S^* A = BS^*$, that is, $|S|U^* A = B|S|U^*$, then $|S|(U^* A - BU^*) = 0$, and since $B|S|^{p-1} = |S|^{p-1}B$, then

$$BY - YA = B|S|^{p-1}U^* - |S|^{p-1}U^*A = |S|^{p-1}(BU^* - U^*A)\tag{6.11}$$

so that $BY - YA = 0$ and $\operatorname{tr}[(BY - YA)T] = 0$ for every $T \in B(H)$. Since $S = S - (AW - WB)$, then

$$\begin{aligned}0 &= \operatorname{tr}[YTB - YAT] = \operatorname{tr}[Y(TB - AT)] \\ &= p \operatorname{Re} \operatorname{tr} [Y(TB - AT)] = p \operatorname{Re} \operatorname{tr} [|S|^{p-1} U^* (TB - AT)] \\ &= (\mathcal{D}_T \phi)(TB - AT) = (\mathcal{D}_W F_p)(T).\end{aligned}\tag{6.12}$$

□

REMARK 6.3. In [Theorem 6.2](#), the implication “ W is a critical point implies $AW - WB = 0$ ” does not hold in the case $0 < p \leq 1$ because the functional calculus argument involving the function $t \mapsto t^{1/(p-1)}$, where $0 \leq t < \infty$, is only valid for $1 < p < \infty$.

7. On minimizing $\|T - (AXB - CXD)\|_p^p$. In this section, we consider the elementary operator $\Phi(X) = AXB - CXD$ and we prove that if $AC = CA$, $BD = DB$, and $ASB = CSD$, $S \in C_p$, then, for $1 < p < \infty$, the map F_p defined by $F_p(X) = \|T - (AXB - CXD)\|_p^p$ has a global minimizer at V if, and for $1 < p < \infty$ only if, $AVB - CVD = 0$. In other words, we have $\|T - (AXB - CXD)\|_p^p \geq \|T\|_p^p$ if, and for $1 < p < \infty$ only if, $AVB - CVD = 0$. Additionally, we show that if $AC = CA$, $BD = DB$, and $T \in \ker \Delta_{A,B} \cap C_p$, $1 < p < \infty$, then the map F_p has a critical point at W if and only if $AWB - CWD = 0$, that is, if $\mathcal{D}_W F_p$ is the Frechet derivative at W of F_p , the set $\{W \in B(H) : \mathcal{D}_W F_p = 0\}$ coincides with $\ker \Phi$ (the kernel of Φ).

DEFINITION 7.1. Let $\mathcal{U}(A, B) = \{X \in B(H) : AXB - CXD \in C_p\}$ and let $F_p : \mathcal{U} \mapsto \mathbb{R}^+$ be the map defined by $F_p(X) = \|T - (AXB - CXD)\|_p^p$, where $T \in \ker \Phi \cap C_p$, $1 \leq p < \infty$.

The proof of the following lemma is similar to the proof of [Lemma 4.3](#).

LEMMA 7.2. Let $A, B \in B(H)$ be normal commuting operators. Suppose that $ASB = BSA$, $S \in C_p$ ($1 < p < \infty$). If

$$A|S|^{p-1}U^*B = B|S|^{p-1}U^*A, \quad (7.1)$$

then

$$A|S|U^*B = B|S|U^*A. \quad (7.2)$$

THEOREM 7.3. Let $A, B, C, D \in B(H)$ be normal operators such that $AC = CA$ and $BD = DB$. Assume that $ASB = CSD$, $S \in C_p$ ($1 < p < \infty$). If $A|S|^{p-1}U^*B = C|S|^{p-1}U^*D$, then $A|S|U^*B = C|S|U^*D$.

PROOF. It suffices to take the Hilbert space $H \oplus H$ and the operators

$$A^\sim = \begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix}, \quad B^\sim = \begin{bmatrix} C & 0 \\ 0 & B \end{bmatrix}, \quad S^\sim = \begin{bmatrix} 0 & S \\ 0 & 0 \end{bmatrix} \quad (7.3)$$

and apply [Lemma 7.2](#). □

THEOREM 7.4. Let $A, B, C, D \in B(H)$ be normal operators, $AC = CA$, and $BD = DB$. Suppose that $ASB = CSD$, $S \in C_p$. Then, for $1 \leq p < \infty$, the map F_p has a global minimizer at W if, and for $1 < p < \infty$ only if, $AWB - CWD = 0$.

PROOF. If $AC = CA$, $BD = DB$, and $ASB = CSD$, $S \in C_p$, then, for $1 < p < \infty$, the result of Turnšek [14, Theorem 3.4] guarantees that

$$\|T - (AXB - CXD)\|_p^p \geq \|T\|_p^p, \quad (7.4)$$

that is, $F_p(X) \geq F_p(W)$. Conversely, if F_p has a minimum, then

$$\|T - (AWB - CWD)\|_p^p = \|S\|_p^p. \quad (7.5)$$

Since \mathcal{U} is convex, then the set $\mathcal{V} = \{T - (AXB - CXD); X \in \mathcal{U}\}$ is also convex. Thus Theorem 5.2 implies that $S - (AWB - CWD) = S$. \square

THEOREM 7.5. *Let A , B , C , and D be normal operators in $B(H)$ such that $AC = CA$ and $BD = DB$. If $S \in \ker\Phi \cap C_p$, then, for $1 < p < \infty$, the map F_p has a critical point at W if and only if $AWB - CWD = 0$.*

PROOF. Let $W, S \in \mathcal{U}$ and let ϕ and φ be two maps defined, respectively, by $\phi : X \mapsto S - (AXB - CXD)$ and $\varphi : X \mapsto \|X\|_p^p$. Since the Frechet derivative of F_p is given by

$$\mathcal{D}_W F_p(T) = \lim_{h \rightarrow 0} \frac{F_p(W + hT) - F_p(W)}{h}, \quad (7.6)$$

it follows that $\mathcal{D}_W F_p(T) = [\mathcal{D}_{S - (AWB - CWD)}](BTA - DTC)$. If W is a critical point of F_p , then $\mathcal{D}_W F_p(T) = 0$ for all $T \in \mathcal{U}$. By applying Theorem 5.1, we get

$$\begin{aligned} \mathcal{D}_W F_p(T) &= p \operatorname{Re} \operatorname{tr} [|S - (AWB - CWD)|^{p-1} W^* (BTA - DTC)] \\ &= p \operatorname{Re} \operatorname{tr} [Y(BTA - DTC)] = 0, \end{aligned} \quad (7.7)$$

where $S - (AWB - CWD) = W|S - (AWB - CWD)|$ is the polar decomposition of the operator $S - (AWB - CWD)$ and $Y = |S - (AWB - CWD)|^{p-1} W^*$. An easy calculation shows that $BYA - DYC = 0$, that is,

$$A|S - (AWB - CWD)|^{p-1} W^* B = C|S - (AWB - CWD)|^{p-1} W^* D. \quad (7.8)$$

It follows from Theorem 7.3 that

$$A|S - (AWB - CWD)|W^* B = C|S - (AWB - CWD)|W^* D. \quad (7.9)$$

By taking adjoints and since A and B are normal operators, applying Fuglede-Putnam theorem, we get $A(T - (AWB - CWD))B = C(T - (AWB - CWD))D$. Then $A(AW - WB)B = C(AWB - CWD)D$. Hence $AWB - CWD \in R(\Phi) \cap \ker\Phi$. By the same argument used in the proof of [13, Theorem 3.4], we can prove that

$$\|T - (AXB - CXD)\| \geq \|T\| \quad (7.10)$$

for all $T \in B(H)$. Hence $AWB - CWD = 0$.

Conversely, if $AWB = CWD$, then W is a minimum, and since F_p is differentiable, then W is a critical point. \square

THEOREM 7.6. *Let A, B, C , and D be normal operators in $B(H)$ such that $AC = CA$ and $BD = DB$. If $S \in \ker \Phi \cap C_p$, $0 < p \leq 1$, $\dim H < \infty$, and $S - (AWB - CWD)$ is invertible, then F_p has a critical point at W if $AWB - CWD = 0$.*

PROOF. Suppose that $\dim H < \infty$. If $AWB - CWD = 0$, then S is invertible by hypothesis. Also $|S|$ is invertible, hence $|S|^{p-1}$ exists for $0 < p \leq 1$. Taking $Y = |S|^{p-1}U^*$, where $S = U|S|$ is the polar decomposition of S , choose X to be the rank-one operator $f \otimes g$ for some arbitrary elements f and g in $H \oplus H$. Then $\text{tr}(Y(AXB - CXD)) = \text{tr}(AYB - CYD)X = 0$ implies that $\langle \Psi(Y)f, g \rangle = 0 \Leftrightarrow Y \in \ker \Phi$, that is, $AYB - CYD = 0$ and $\text{tr}[(DYC - AYB)T] = 0$ for every $T \in B(H)$. Since $S = S - (AWB - CWD)$, then

$$\begin{aligned} 0 &= \text{tr}[YDTC - YATB] = \text{tr}[Y(DTC - ATB)] \\ &= p \text{Re} \text{tr}[Y(DTC - ATB)] = p \text{Re} \text{tr}[|S|^{p-1}U^*(DTC - ATB)] \quad (7.11) \\ &= (\mathcal{D}_T \phi)(DTC - ATB) = (\mathcal{D}_W F_p)(T). \end{aligned}$$

\square

REMARK 7.7. The set $\mathcal{S} = \{X : AXB - CXD \in C_p\}$ contains C_p ; if $X \in C_p$, then $X \in \mathcal{S}$ and, for example, $I \in \mathcal{S}$ but $I \notin C_p$. If $A \in C_p$, the conclusions of Theorems 7.3, 7.4, 7.5, and 7.6 hold for all $X \in B(H)$.

For $n > 2$ the generalization of the above results to the elementary operators $\sum_{i=1}^n A_i X B_i$ is not possible. In [12], Shul'man stated that there exists a normally represented elementary operator of the form $\sum_{i=1}^n A_i X B_i$ with $n > 2$ such that $\text{asc} E > 1$, that is, the range and kernel have no trivial intersection.

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