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UNIVERSAL CONNECTIONS ON LIE GROUPOIDS

EFSTATHIOS VASSILIOU and APOSTOLOS NIKOLOPOULOS

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Given a Lie groupoid Ω, we construct a groupoid $J_1\Omega$ equipped with a universal connection from which all the connections of Ω are obtained by certain pullbacks. We show that this general construction leads to universal connections on principal bundles (considered by García (1972)) and universal linear connections on vector bundles (ultimately related with those of Cordero et al. (1989)).

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1. Introduction. Many problems in differential geometry of fiber bundles and applications lead to the consideration of a whole family of connections instead of an individual one. This is, for example, the case of the Chern-Weil homomorphism, the universal holonomy theory, parametric models in statistical theory, variational problems in classical field theory, and so forth. In such cases, things are simplified by constructing universal connections from which we deduce all the connections considered (for relevant comments and examples, we refer to [3, 4]).

Universal connections have been studied in, for example, [4, 8, 12, 13], to name but a few of the earlier attempts to the subject. Narasimhan and Ramanan [12], based on the Stiefel bundles and unitary groups, proved the existence of universal bundles for principal bundles with a compact Lie group as a structure group. This result was extended to the case of arbitrary connected Lie groups in [13]. However, [8] has a rather algebraic nature showing that the geometric construction of [12] corresponds to the identity map of the Weil algebra of the Lie algebra of a connected compact Lie group. On the other hand, a universal connection in the sense of [4] is defined by an appropriate 1-form on the 1-jet bundle of sections of a given principal bundle.

Motivated by this approach, we study the existence of universal connections on Lie groupoids. The geometry of Lie groupoids and algebroids is a topic of current research which, beyond its significance per se, has found numerous applications in many areas (see, e.g., [2, 9, 10, 18]). In this framework, universal connections might be an interesting complement of the theory of groupoid connections.

More explicitly, in Theorem 4.5, we prove that, roughly speaking, a given Lie groupoid Ω determines a Lie groupoid $J_1\Omega$ equipped with a universal connection from which all the connections of Ω are derived (by appropriate pullbacks).

Here, $J_1\Omega$ is the pullback of Ω via $\pi_1 : Q_1\Omega \to B$, where B is the base of the groupoid and $Q_1\Omega$ is the corresponding connection bundle, whose sections are in bijective correspondence with the connections of Ω.

Before giving the proof of the main result, we study in detail the bundle $Q_1\Omega$, the differentiable structure of $J_1\Omega$, as well as the Lie algebroid $\mathbf{L}(J_1\Omega)$. We also give an explicit description of certain mappings involved in the construction of the universal connection.

The paper is completed with two applications. The first one explains how García's universal connections on principal bundles fit in this scheme. Namely, in Theorem 5.2, we show that our universal connection induces the one defined in the sense of [4], by passing to the vertex bundle of the Lie groupoid. The second application deals with universal connections for the frame groupoids of vector bundles. In fact, we prove that such connections lead to the existence of universal linear connections for the corresponding vector bundles (Theorem 5.9). As we remark in Scholium 5.10, the previous result can be related with the universal linear connection considered also in [3].

2. Preliminaries. Throughout this paper, all the manifolds and bundles are supposed to be smooth and finite dimensional. For the standard theory of connections on fiber bundles, we refer to [6, 7]. For groupoids and algebroids, we mainly follow the notation and terminology of [9]; however, for the reader's convenience, in this section, we briefly review some facts needed in the sequel.

A *groupoid* Ω is a small category all the morphisms of which are invertible. Hence, Ω is equipped with two *projections* $\alpha : \Omega \to B$ and $\beta : \Omega \to B$ (called *source* and *target*, respectively), an *inversion* $\tau : \Omega \to \Omega : \xi \mapsto \xi^{-1}$, and a *multiplication* $\gamma : \Omega * \Omega \to \Omega : (\eta, \xi) \mapsto \eta \cdot \xi \equiv \eta\xi$, where $\Omega * \Omega$ is the set of the composable morphisms. Here, B is the set of the objects (*base of the groupoid*). To every element $x \in B$, we assign a corresponding *unity* (or *identity*) $\tilde{x}$, the set of unities being naturally identified with B via the *object inclusion map* $\varepsilon : B \to \Omega : x \mapsto \tilde{x}$. Consequently, B may be viewed as a subset of the *total space* Ω.

Some subsets of the groupoid are of special importance for the study of the latter. Namely, we set

$$\begin{gathered} \Omega_x = \alpha^{-1}(x), \qquad \Omega^y := \beta^{-1}(y), \qquad \Omega_x^y := \alpha^{-1}(x) \cap \beta^{-1}(y), \\ \Omega_U := \alpha^{-1}(U), \qquad \Omega^V := \beta^{-1}(V), \qquad \Omega_U^V := \Omega_U \cap \Omega^V, \end{gathered} \tag{2.1}$$

for every $U, V \subseteq B$ and $x, y \in B$.

As a matter of terminology, Ω_x is the *α-fiber* of Ω at x, Ω^y is the *β-fiber* of Ω at y, and Ω_x^x is the *isotropy group* of Ω at x.

By definition, Ω is a *transitive groupoid* if the *anchor* map

$$(\beta, \alpha) : \Omega \longrightarrow B \times B : \xi \longmapsto (\beta(\xi), \alpha(\xi)) \tag{2.2}$$

is surjective. Moreover, we speak of a *differentiable groupoid* if Ω and B are differentiable manifolds, the mappings γ, ε are differentiable and the projections α, β are differentiable *submersions*. Under these assumptions, the differentiability of the inversion map τ can also be deduced. The α- and β-fibers of a differentiable groupoid are regular submanifolds of the total space.

We need the following typical examples of differentiable groupoids:

(a) the *trivial groupoid* $B\times G\times B$ (B differentiable manifold and G Lie group) with multiplication given by

$$(x,g,y)\cdot(y,h,z)=(x,g\cdot h,z); \tag{2.3}$$

(b) the *frame groupoid* $\Pi(E)$ of a vector bundle E defined as follows: if E_z denotes the fiber of E over $z\in B$, then

$$\Pi(E):=\{f\in \mathrm{Lis}\,(E_x,E_y)\mid x,y\in B\} \tag{2.4}$$

and the multiplication is the usual composition of maps. Here, $\mathrm{Lis}(E_x, E_y)$ denotes the set of linear isomorphisms of E_x onto E_y;

(c) the *inner subgroupoid* $G\Omega$ of a given differentiable groupoid Ω, defined by

$$G\Omega:=\bigcup_{x\in B}\Omega_x^x; \tag{2.5}$$

(d) the sets Ω_U^U, if U is an *open* subset of B and Ω is a differential groupoid.

A *Lie groupoid* is a transitive differentiable groupoid. In this case, (β,α) is also a submersion (see [9, pages 86–89]) and the isotropy groups are mutually isomorphic Lie groups. We note that every trivial groupoid is Lie, while $\Pi(E)$ (see example (b) above) is a Lie groupoid if the base space B is connected.

Defining morphisms of groupoids in the obvious way (see, e.g., [9, Definitions I.2.1 and III.1.1]), a Lie groupoid satisfies the following (equivalent) conditions:

(T.1) there is an open cover $\mathcal{C}$ of B and local trivializing maps (isomorphisms of differentiable groupoids) of the form

$$\Sigma_U:\Omega_U^U\longrightarrow U\times G\times U,\quad U\in\mathcal{C}, \tag{2.6}$$

where, for the sake of simplicity, we have set $G:=\Omega_b^b$ for an arbitrary fixed $b\in B$;

(T.2) there are smooth *local sections* of Ω with respect to $\mathcal{C}$, that is, $\sigma_U:U\to\Omega$ such that $\beta\circ\sigma_U=\mathrm{id}\,|U$ and $\alpha\circ\sigma_U=$ constant.

The maps Σ_U are related with σ_U by the formula

$$\Sigma_U(\xi)=\big(\beta(\xi),\sigma_U(\beta(\xi))^{-1}\cdot\xi\cdot\sigma_U(\alpha(\xi)),\alpha(\xi)\big),\quad \xi\in\Omega. \tag{2.7}$$

For later reference, we note that there is a bijective correspondence between Lie groupoids and principal fiber bundles: given a Lie groupoid Ω, the quadruples $(\Omega_x,\Omega_x^x,B,\beta_x:=\beta|\Omega_x)$, $x\in B$, are mutually isomorphic principal bundles,

called *vertex bundles*. Conversely, a principal bundle P is the vertex bundle of the *gauge groupoid* $P\times P/G$ (see, e.g., [9, Proposition II. 1.19]).

3. Connections on groupoids. Before defining connections, we need the fundamental notion of a Lie algebroid, generalizing that of a Lie algebra, and originally due to Pradines ([15]).

Given a vector bundle (v.b., for short) $E\equiv(E,p,B)$, we denote by $\mathcal{S}(E)$ the module of its sections.

Definition 3.1. A *Lie algebroid* is a vector bundle E, such that the module $\mathcal{S}(E)$ is equipped with a Lie bracket $[\cdot,\cdot]$ and there exists a v.b.-morphism $q^E\equiv q:E\to TB$, satisfying the following properties:

$$q\circ[s,t]=[q\circ s,q\circ t],\qquad [s,ft]=f\cdot[s,t]+(q\circ s)(f)\cdot t, \tag{3.1}$$

for every $s,t\in\mathcal{S}(E)$ and $f\in C^\infty(B)$.

The manifold B is the base of the algebroid and q is its *anchor* (flèche).

The *Lie algebroid* of a differentiable groupoid Ω is defined to be

$$\mathbf{L}\Omega:=\bigcup_{x\in B}T_{\tilde{x}}\Omega_x, \tag{3.2}$$

its projection π_Ω sending every fiber $T_{\tilde{x}}\Omega_x$ to the corresponding $x\in B$, whereas the anchor is $q:=T\beta|\mathbf{L}\Omega$. The bracket of $\mathbf{L}\Omega$ is induced by an appropriate bracket on the module of α-vertical right invariant vector fields of Ω (see, e.g., [9, Section III.3]).

In fact, $\mathbf{L}$ is a functor with $\mathbf{L}\phi:\mathbf{L}\Omega\to\mathbf{L}\Omega'$ given by

$$(\mathbf{L}\phi)|_{T_{\tilde{x}}\Omega_x}:=T_{\tilde{x}}(\phi|_{\Omega_x}),\quad x\in B, \tag{3.3}$$

for any morphism of differentiable groupoids $\phi:\Omega\to\Omega'$.

For a Lie groupoid, the sequence of vector bundles

$$0\longrightarrow\mathbf{L}(G\Omega)\overset{i}{\hookrightarrow}\mathbf{L}\Omega\overset{q}{\longrightarrow}TB\longrightarrow 0 \tag{3.4}$$

is exact, a fact leading to the following basic definition.

Definition 3.2. A *connection* on a Lie groupoid Ω is a right section of q, that is, a vector bundle morphism $\gamma:TB\to\mathbf{L}\Omega$ such that $q\circ\gamma=\mathrm{id}_{TB}$.

In a well-established terminology, γ is a splitting of (3.4). If the base B is paracompact, as usual, connections do exist.

For our purposes, we need another approach to the notion of a connection, involving the sections of an appropriate fiber bundle. Namely, following [11, Section III.1], we set

$$Q_1\Omega|_x := \{j^1_x s \mid s \text{ is a local section of } \Omega : s(x) = \tilde{x}\} \tag{3.5}$$

for every $x \in B$, and

$$Q_1\Omega := \bigcup_{x\in B} Q_1\Omega|_x, \tag{3.6}$$

where $j^1_x s$ is the 1-jet of s at x. We also define the mapping

$$\pi_1 : Q_1\Omega \longrightarrow B : j^1_x s \longmapsto x. \tag{3.7}$$

Proposition 3.3. *The triplet $(Q_1\Omega, \pi_1, B)$ is a fiber bundle.*

Proof. We are based on an explicit use of the local charts of Ω (compare with the proof of [11, Proposition III.1a]).

As in condition (T.2) of Section 2, we may construct smooth local sections $\sigma_U : U \to \Omega$, so that $\mathscr{C}$ is also an atlas of the manifold B. For every $U \in \mathscr{C}$, let

$$\begin{gathered}\Phi_U : U \times \mathbb{B} \longrightarrow TU \quad (\text{where } \mathbb{B} \text{ is the model space of } B),\\ \Sigma_U : \Omega^U_U \longrightarrow U \times G \times U \quad (G \cong \Omega^b_b,\ b \in B)\end{gathered} \tag{3.8}$$

be the induced trivializations of TB and Ω, respectively. We also consider the map

$$k : TU \times TG \times TU \longrightarrow T_eG : (u, v, w) \longmapsto T_g\rho_{g^{-1}}(v) \tag{3.9}$$

if $v \in T_gG$ and ρ_g denotes the right translation of G by g. Taking a fixed linear isomorphism $\ell : T_eG \to \mathbb{G}$, where $\mathbb{G}$ is the model space of G and $e \equiv \tilde{b}$, we can define the smooth map

$$F_U : \pi_1^{-1}(U) \longrightarrow U \times L(\mathbb{B}, \mathbb{G}) : j^1_x s \longmapsto (x, \ell \circ k \circ T_x(\Sigma_U \circ s) \circ (\Phi_U | \{x\} \times \mathbb{B})). \tag{3.10}$$

We obtain a chart of $Q_1\Omega$ since F_U has a smooth inverse. In fact, the latter is given by $(x, f) \mapsto j^1_x s$, the jet $j^1_x s$ being determined by the conditions

$$s(x) = \tilde{x}, \qquad T_x s = T(\Sigma_U)^{-1} \circ \left[\mathrm{id}_{TU} \times \left(\ell^{-1} \circ f \circ (\Phi_U | \{x\} \times \mathbb{B})^{-1}\right) \times 0\right]. \tag{3.11}$$

The compatibility of the previous charts is established by a direct verification; hence, $Q_1\Omega$ becomes a differentiable manifold.

Finally, if $\mathrm{pr}_1 : U \times L(\mathbb{B}, \mathbb{G}) \to U$ is the projection to the first factor, we see that $\mathrm{pr}_1 \circ F_U$ coincides with $\pi_1 | \pi_1^{-1}(U)$, thus concluding the proof. □

Definition 3.4. The fiber bundle $(Q_1\Omega, \pi_1, B)$ is called the *connection bundle* of the groupoid Ω.

The previous terminology is justified by the following proposition.

Proposition 3.5. *There is a bijective correspondence between the smooth sections of $Q_1\Omega$ and the connections of the Lie groupoid Ω.*

Proof. A connection γ induces the section $S^\gamma : B \to Q_1\Omega$ given by $S^\gamma(x) := j^1_x s$, where s is a local section of Ω defined on an open neighborhood of x, with $s(x) = \tilde{x}$ and $T_x s = \gamma | T_x B$.

Conversely, let $S : B \to Q_1\Omega : x \mapsto j^1_x s$ be a smooth section. Then, the map $\gamma^S : TB \to \mathbf{L}\Omega$, with $\gamma^S | T_x B := T_x s$, is a connection on Ω.

It is easily checked that the assignments $\gamma \mapsto S^\gamma$ and $S \mapsto \gamma^S$ are mutually inverse. □

4. Universal connections. We come now to the main part of this paper, treating the construction of universal connections for Lie groupoids. This means, roughly speaking, that we look for a connection (on an appropriate groupoid) from which we can derive all the connections of a given groupoid.

Let Ω be a Lie groupoid. Using the connection bundle $(Q_1\Omega, \pi_1, B)$ (see Proposition 3.3 and Definition 3.4), we give the following definition.

Definition 4.1. The 1-*jet groupoid* of Ω, denoted by $J_1\Omega$, is the pullback groupoid of Ω by π_1, that is, $J_1\Omega := \pi_1^*\Omega$.

We recall that the total space $J_1\Omega \subseteq Q_1\Omega \times \Omega \times Q_1\Omega$ is the set

$$\{(j^1_y s, \xi, j^1_x t) \mid s \in \pi_1^{-1}(y),\ t \in \pi_1^{-1}(x),\ \xi \in \Omega_x^y;\ x, y \in B\} \tag{4.1}$$

and the operations of $J_1\Omega$ are defined similarly to those of a trivial groupoid (see example (a) of Section 2 and [9, Definition I.2.11]). The projections, the inversion, the identity map, and the multiplication of $J_1\Omega$ are denoted by a_1, β_1, τ_1, ε_1, and γ_1, respectively.

We note that $J_1\Omega$ here is not the first prolongation groupoid, often denoted also by the same symbol.

Proposition 4.2. *The 1-jet groupoid $J_1\Omega$ is a Lie groupoid.*

Proof. The groupoid $J_1\Omega$ is differentiable, as a regular submanifold of the trivial groupoid $Q_1\Omega \times \Omega \times Q_1\Omega$. Indeed, using the notations included in the proof of Proposition 3.3, the differential structure is determined by the charts of the form

$$\Psi_U : J_1\Omega \cap (\pi_1^{-1}(U) \times \Omega \times \pi_1^{-1}(U)) \xrightarrow{\simeq} U \times L(\mathbb{B}, \mathbb{G}) \times G \times U \times L(\mathbb{B}, \mathbb{G}), \tag{4.2}$$

given by

$$\Psi_U(j^1_x s,\xi,j^1_y t) = (F_U(j^1_x s), p_2(\Sigma_U(\xi)), F_U(j^1_y t)), \tag{4.3}$$

with U running the domains of the local sections of Ω. The inverse of Ψ_U is

$$\Psi_U^{-1}(x,f,g,y,h) = (F_U^{-1}(x,f), \Sigma_U^{-1}(x,g,y), F_U^{-1}(y,h)). \tag{4.4}$$

On the other hand, the local expressions of α_1, β_1, and (β_1,α_1), relative to the previous chart are given, respectively, by the projections pr_3, pr_1, and $(\mathrm{pr}_1,\mathrm{pr}_3)$ of the following 3-factor product:

$$(U\times L(\mathbb{B},\mathbb{G}))\times G\times(U\times L(\mathbb{B},\mathbb{G})); \tag{4.5}$$

thus, the former maps are (surjective) submersions. Finally, γ_1 and ε_1 are smooth since their local expressions are, respectively,

$$\begin{gathered}((x,f,g_1,y,h),(y,h,g_2,z,k)) \longmapsto (x,f,g_1\cdot g_2,z,k),\\ (y,h) \longmapsto (y,h,1_G,y,h),\end{gathered} \tag{4.6}$$

for every $(x,f),(y,h),(z,k)\in U\times L(\mathbb{B},\mathbb{G})$ and $g_1,g_2\in G$. □

NOTE 4.3. In the previous proof, we have used an explicit form of local charts which will be also needed later.

We will show that the desired universal connection is one defined on $J_1\Omega$. Before its explicit description, we need a few preliminary facts. First, as an immediate consequence of the relative definitions, we obtain that (see also [5, Section 1])

$$\mathbf{L}(J_1\Omega) = \{(u,v) : T\pi_1(u) = T\beta(v)\} \subseteq T(Q_1\Omega)\times\mathbf{L}\Omega. \tag{4.7}$$

Here, $(u,v)\equiv(u,v,0)$, the vector $0\in T(Q_1\Omega)$ being omitted for the sake of convenience. Hence,

$$\mathbf{L}(J_1\Omega) \cong T(Q_1\Omega)\times_{TB}\mathbf{L}\Omega. \tag{4.8}$$

Moreover, $\mathbf{L}(J_1\Omega)$ is a Lie algebroid over $Q_1\Omega$ with anchor (the restriction of) the projection to the first factor $P_1 : \mathbf{L}(J_1\Omega)\to T(Q_1\Omega)$. On the other hand, the restriction of the second projection $P_2 : \mathbf{L}(J_1\Omega)\to\mathbf{L}\Omega$ is a v.b.-morphism covering $\pi_1 : Q_1\Omega\to B$.

We consider now the evaluation map

$$\mathrm{Ev} : \pi_1^*(TB) = Q_1\Omega\times_B TB \longrightarrow \mathbf{L}\Omega : (j^1_x s, v) \longmapsto T_x s(v), \quad v\in T_xB. \tag{4.9}$$

LEMMA 4.4. *The evaluation map* Ev *is differentiable.*

PROOF. The charts F_U and Φ_U of the bundles $Q_1\Omega$ and TB, respectively, induce the corresponding chart $(\pi_1^{-1}(U)\times_U TU, \Psi_U^1)$ of $Q_1\Omega\times_B TB$, with

$$\Psi_U^1(j_x^1 s, v) := (x, D\bar{s}(\phi(x)), (p_2\circ\Phi_U^{-1})(v)) \in U\times L(\mathbb{B},\mathbb{G})\times\mathbb{B}, \tag{4.10}$$

where $\bar{s}$ is the local expression of s and (U,ϕ) is the chart of B inducing the corresponding v.b.-chart Φ_U (see the proof of Proposition 3.3).

Applying now the Lie functor $\mathbf{L}$ on Σ_U, after the identification $\Phi_U : U\times B \xrightarrow{\simeq} TU$, we obtain the diffeomorphism $\psi_U : \mathbf{L}\Omega|_U \to U\times\mathbb{B}\times\mathbb{G}$. The local expression of Ev, relative to Ψ_U^1 and ψ_U, is

$$\mathrm{Ev}_{\mathrm{loc}} : U\times L(\mathbb{B},\mathbb{G})\times\mathbb{B} \longrightarrow U\times\mathbb{B}\times\mathbb{G} : (x,f,m) \longmapsto (x,m,\mathrm{ev}(f,m)), \tag{4.11}$$

where $\mathrm{ev} : L(\mathbb{B},\mathbb{G})\times\mathbb{B}\to\mathbb{G}$ is now the ordinary evaluation map. This proves the desired differentiability. □

Based on Proposition 3.5 and denoting by $\pi_Q : T(Q_1\Omega)\to Q_1\Omega$ the standard projection of the tangent bundle of $Q_1\Omega$, we are in a position to prove the main result of the paper; namely, the following theorem.

THEOREM 4.5 (existence of universal groupoid connections)**.** *Let* Ω *be a Lie groupoid. Then, the mapping*

$$\overline{\Gamma} \equiv \overline{\Gamma}_\Omega : T(Q_1\Omega) \longrightarrow \mathbf{L}(J_1\Omega) : u \longmapsto (u, \mathrm{Ev}(\pi_Q(u), T\pi_1(u))) \tag{4.12}$$

is a connection on $J_1\Omega$ *with the following universal property: if* $\Gamma := P_2\circ\overline{\Gamma}$*, then the equality*

$$\gamma^S = \Gamma\circ TS \tag{4.13}$$

holds for every connection S *on* Ω*. Moreover,* $\overline{\Gamma}$ *is the unique connection satisfying this property.*

PROOF. First, we see that $\overline{\Gamma}$ is well defined, since $(\pi_Q(u), T\pi_1(u)) \in Q_1\Omega\times_B TB$; thus, we can apply Ev on such pairs. Also, $\overline{\Gamma}$ takes values in $\mathbf{L}(J_1\Omega)$, since $\overline{\Gamma}(u)\in T(Q_1\Omega)\times_{TB}\mathbf{L}\Omega \cong \mathbf{L}(J_1\Omega)$, by virtue of (4.8).

The mapping $\overline{\Gamma}$ is differentiable (by Lemma 4.4), as well as a v.b-morphism. Moreover (see also (4.8)), the projection $P_1 : \mathbf{L}(J_1\Omega)\to T(Q_1\Omega)$ satisfies the equality $P_1\circ\overline{\Gamma} = \mathrm{id}$. Since P_1 coincides now with the anchor of $\mathbf{L}(J_1\Omega)$, $\overline{\Gamma}$ is a connection of $J_1\Omega$ (see Definition 3.2).

On the other hand, since $T_\gamma S(v) \in T_{S(\gamma)}Q_1\Omega$, for every $v \in T_\gamma B$ and $\gamma \in B$, Proposition 3.5 implies that

$$\begin{aligned}(\Gamma \circ TS)(v) &= (P_2 \circ \overline{\Gamma} \circ TS)(v)\\ &= \mathrm{Ev}\left((\pi_Q \circ TS)(v), T(\pi_1 \circ S)(v)\right)\\ &= \mathrm{Ev}\left(S(\gamma), v\right)\\ &= \gamma^S(v),\end{aligned} \tag{4.14}$$

for every $v \in TB$; thus, proving the universal property.

Finally, let $\overline{\Gamma}'$ be another connection of $J_1\Omega$ satisfying the analog of (4.13) and $\Gamma' = P_2 \circ \overline{\Gamma}'$. Then, for any $w \in T(Q_1\Omega)$, there is a local section S of $Q_1\Omega$ and a vector $v \in TB$ such that $TS(v) = w$. Therefore, $\Gamma'(w) = \gamma^S(v) = \Gamma(w)$, which proves the uniqueness property. □

Remarks 4.6. (1) For any morphism $f : \mathbf{L}\Omega_1 \to A$ of vector bundles and any morphism of differentiable groupoids $h : \Omega_2 \to \Omega_1$, we set $h^* f := f \circ \mathbf{L}h$. Since, within an isomorphism, $TS = \mathbf{L}(S \times S)$ (see, e.g., [1, Section 4] and [5, Section 1]), (4.13) can be reformulated as

$$\gamma^S = (S \times S)^* \Gamma. \tag{4.15}$$

(2) By virtue of the properties stated in Theorem 4.5, we can legitimately call $\overline{\Gamma} \equiv \overline{\Gamma}_\Omega$ *the universal connection of* Ω.

5. Applications

5.1. Universal connections on principal bundles. Our present aim is to relate universal connections on groupoids with their counterparts defined by García (see [4]) on principal bundles. In fact, we show that $\overline{\Gamma}$ produces the structural infinitesimal connection (form) of the vertex bundle.

First, we note that every connection γ on the Lie groupoid Ω is equivalent to a v.b.-morphism $c : \mathbf{L}\Omega \to \mathbf{L}(G\Omega)$, such that

$$\gamma \circ q + i \circ c = \mathrm{id}. \tag{5.1}$$

In order to define connection forms, we consider the (principal) vertex bundle $\Omega_b \equiv (\Omega_b, \Omega_b^b, B, \beta|\Omega_b)$ for any fixed $b \in B$ (see Section 2). Also, we need the morphism of Lie groupoids $\mathrm{Ad} : \Omega \to \Pi(\mathbf{L}(G\Omega))$ (often called the *adjoint representation* of Ω), defined by $\mathrm{Ad}(\xi) = TL_\xi \circ TR_\xi^{-1}$, as well as the v.b.-morphism

$$\mathfrak{R} : T^\alpha\Omega \longrightarrow \mathbf{L}\Omega : v \longmapsto T_\xi R_\xi^{-1}(v), \quad v \in T_\xi\Omega_{\alpha\xi} \tag{5.2}$$

(see also [9, Proposition III.3.3]).

Lemma 5.1. *Let γ be a connection on Ω. Then, the form $\omega^\gamma \in \Lambda^1(\Omega_b, T_{\tilde{b}}\Omega_b^b)$, given by*

$$\omega^\gamma_\xi(v) = [\mathrm{Ad}(\xi^{-1}) \circ c \circ \mathfrak{R}](v), \quad \xi \in \Omega_b,\ v \in T_\xi\Omega_b, \tag{5.3}$$

is a connection form on the vertex bundle Ω_b.

Proof. By definition,

$$\omega^\gamma_\xi = \left(T_\xi L_\xi^{-1} | T^\alpha_\xi \Omega\right) \circ \left(T_{\widetilde{\beta\xi}} R_\xi | \mathbf{L}(G\Omega)\right) \circ \left(c | (\mathbf{L}\Omega)_{\beta\xi}\right) \circ T_\xi R_\xi^{-1}; \tag{5.4}$$

thus, $\omega^\gamma_\xi \in L(T_\xi\Omega_b, T_{\tilde{b}}\Omega_b^b)$ for every $\xi \in \Omega_b$. Since $\mathfrak{R}$ and Ad are differentiable, ω^γ is a differentiable 1-form.

Now, for any $X \in \mathcal{S}(T(\Omega_b^b))$, we see that $c([\mathrm{Ad}(\xi)](X_{\tilde{b}})) = [\mathrm{Ad}(\xi)](X_{\tilde{b}})$. Consequently, if X^* is the *Killing vector field* corresponding to X,

$$\begin{aligned} \omega^\gamma_\xi(X_b^*) &= \omega^\gamma_\xi(T_{\tilde{b}}L_\xi(X_{\tilde{b}})) = [\mathrm{Ad}(\xi^{-1}) \circ c \circ T_\xi R_\xi^{-1} \circ T_{\tilde{b}}L_\xi](X_b) \\ &= [\mathrm{Ad}(\xi^{-1}) \circ c \circ \mathrm{Ad}(\xi)](X_{\tilde{b}}) = [\mathrm{Ad}(\xi^{-1}) \circ \mathrm{Ad}(\xi)](X_{\tilde{b}}) \\ &= X_{\tilde{b}} \equiv X. \end{aligned} \tag{5.5}$$

On the other hand, for every $(v, g) \in T_\xi\Omega_b \times \Omega_b^b$,

$$\begin{aligned} R_g^* \omega^\gamma(v) &= \omega^\gamma_{\xi g}(T_\xi R_g(v)) = \left[\mathrm{Ad}(\xi g)^{-1} \circ c \circ T_{\xi g} R^{-1}_{(\xi g)} \circ T_\xi R_g\right](v) \\ &= \left[\mathrm{Ad}(g^{-1}) \circ \mathrm{Ad}(\xi^{-1}) \circ c \circ T_\xi R_\xi^{-1}\right](v) = [\mathrm{Ad}(g^{-1})](\omega^\gamma(v)). \end{aligned} \tag{5.6}$$

Equalities (5.5) and (5.6) prove that ω^γ is indeed a connection form. □

In order to apply the preceding lemma to the case of the universal connection $\overline{\Gamma} \equiv \Gamma$, we need the analytic expression of its splitting $C : \mathbf{L}(J_1\Omega) \to \mathbf{L}(G(J_1\Omega))$. As a matter of fact, for every $v \in \mathbf{L}(J_1\Omega)|_Y$, with $Y := j^1_y s \in (Q_1\Omega)|_y = \pi_1^{-1}(y)$ and $y \in B$, we see that

$$v = (v_1, v_2, 0_Y) \in T_Y(Q_1\Omega) \times \mathbf{L}\Omega|_y \times T_Y(Q_1\Omega), \tag{5.7}$$

with $T\pi_1(v_1) = T\beta(v_2)$. Hence, using the analog of (5.1) and the definition of $\overline{\Gamma}$,

$$C(v) = (0_Y, v_2 - T_Y(s \circ \pi_1)(v_1), 0_Y), \tag{5.8}$$

where 0_Y denotes the zero element of $T_Y(Q_1\Omega)$. As a consequence, we obtain the following theorem.

Theorem 5.2. *The connection form ω^{Γ} on the vertex bundle of $J_1\Omega$, induced by the groupoid universal connection $\Gamma \equiv \overline{\Gamma}$ (see Lemma 5.1), coincides—within an isomorphism—with the universal connection (in the sense of [4]) of the vertex bundle Ω_b.*

Proof. We fix a $Z \in Q_1\Omega|_b$ and the vertex bundle $((J_1\Omega)_Z, (J_1\Omega)^Z_Z, Q_1\Omega, p_1)$ of $J_1\Omega$. Then, by virtue of Lemma 5.1, ω^{Γ} may be regarded as a connection on the principal bundle $(J_1\Omega)_Z$.

Every $v \in T_F(J_1\Omega)_Z$, where $F = (X, \xi, Z) \in (J_1\Omega)_Z$, is identified with a triple of the form $v \equiv (v_1, v_2, 0_Z)$, with $0_Z \in T_Z(Q_1\Omega)$, $v_1 \in T_X(Q_1\Omega)$, and $v_2 \in T_\xi\Omega_b$. If we express the maps $T_F R_F^{-1}$ and $\mathrm{Ad}(F^{-1})$ in these terms, then, for $X := j^1_{\beta\xi}s$, Lemma 5.1 and (5.8) imply that

$$\omega^{\Gamma}_F(v) = (0_X, \mathrm{Ad}\,(\xi^{-1})(\mathfrak{R}(v_2) - T_X(s\circ\pi_1)(v_1)), 0_X). \tag{5.9}$$

Also, we consider the 1-jet principal bundle $(J^1(\Omega_b), \Omega_b^b, J^1(\Omega_b)/\Omega_b^b, \pi_1)$ corresponding to the principal bundle Ω_b (see [4, Section 2]), as well as the *isomorphism* (I_1, I_2, I_3) between the bundles $(J_1\Omega)_Z$ and $J^1(\Omega_b)$, determined by

$$I_1 : (J_1\Omega)_Z \longrightarrow J^1(\Omega_b) : (j^1_x s, \xi, Z) \longmapsto j^1_x(s\xi) \tag{5.10}$$

with $(s\xi)(y) := s(y)\cdot\xi$, for every $y \in \mathrm{dom}(s)$,

$$\begin{aligned} I_2 &: (J_1\Omega)^Z_Z \longrightarrow \Omega_b^b : (Z, g, Z) \longmapsto g, \\ I_3 &: Q_1\Omega \longrightarrow J^1(\Omega_b)/\Omega_b^b : j^1_x s \longmapsto [j^1_x(s\eta)] \end{aligned} \tag{5.11}$$

for an arbitrary $\eta \in \Omega_b^x$.

Now, ω^{Γ} induces a differentiable 1-form $\overline{\theta}$ given by

$$\overline{\theta}_K(w) := \left(TI_2 \circ \omega^{\Gamma}_{(X,\xi,Y)} \circ TI_1^{-1}\right)(w), \tag{5.12}$$

where $K := I_1(X, \xi, Z)$ and $w \in T_K(J^1(\Omega_b))$. Hence, comparing (5.9) and (5.12) with the definition of the universal connection θ for ordinary principal bundles (see [4] along with [17]), we conclude that

$$\overline{\theta}_K(w) = (T_\xi L_\xi^{-1} \circ \theta_K)(w), \tag{5.13}$$

which completes the proof. □

Scholium 5.3. Since every principal fiber bundle is the vertex bundle of its gauge groupoid, the previous theorem shows that universal connections on principal bundles (in the sense of [4]) can be derived from the general construction given in Theorem 4.5.

5.2. Universal linear connections. In this section, we prove that the universal connection of the frame groupoid $\Pi(E)$ of a vector bundle $E \equiv (E,\pi,B)$ with connected base B (see example (b) in Section 2) induces a universal *linear* connection on E.

First, we need the following general lemma.

Lemma 5.4. *Let $f : V \to B$ be a differentiable map and f^*E be the pullback vector bundle of E by f. Then,*

$$f^*\Pi(E) \cong \Pi(f^*E). \tag{5.14}$$

Proof. The groupoid $f^*\Pi(E)$ consists of triples of the form (y,g,x) for all $x,y \in V$ and $g \in \mathrm{Lis}(E_{f(x)},E_{f(y)})$. On the other hand, $\Pi(f^*E)$ consists of pairs (c_x^y,g), for all $x,y \in V$ and $g \in \mathrm{Lis}(E_{f(x)},E_{f(y)})$, where c_x^y denotes the mapping $\{x\} \to \{y\}$. We get an isomorphism, say h, by setting $h(y,g,x) = (c_x^y,g)$. □

Applying the previous result to the case of π_1, Definition 4.1 implies also the following corollary.

Corollary 5.5. *The following identification holds true:*

$$J_1(\Pi(E)) = \pi_1^*(\Pi(E)) \cong \Pi(\pi_1^*E). \tag{5.15}$$

It is already known (see, e.g., [11, Section III.2]) that there is a bijective correspondence between the linear connections of the vector bundle E and the connections of $\Pi(E)$. This leads to the following basic proposition.

Proposition 5.6. *The universal connection $\Gamma \equiv \overline{\Gamma}_{\Pi(E)}$ of the Lie groupoid $\Pi(E)$ induces a linear connection C^Γ on the vector bundle π_1^*E.*

Proof. By its construction, Γ is defined on the 1-jet groupoid $J_1(\Pi(E))$. Thus, by virtue of Corollary 5.5, we obtain a connection on $\Pi(\pi_1^*E)$ and, in turn, a linear connection C^Γ on π_1^*E. □

In order to complete the previous statement, by showing that C^Γ has indeed a universal property, we find its analytic expression. This is a consequence of the next result, relating linear connections with connections on frame groupoids by means of a simple formula.

To this end, observe that, since $\Pi(E)$ projects to B via the source map $\alpha_{\Pi(E)}$, we may consider the corresponding fiber product $\Pi(E) \times_B E$ (with respect to $\alpha_{\Pi(E)}$), the evaluation map

$$\overline{\mathrm{ev}} : \Pi(E) \times_B E \longrightarrow E : (f,e) \longmapsto f(e), \tag{5.16}$$

and the partial mappings $\overline{\mathrm{ev}}_e : \Pi(E)|_{\pi(e)} \to E$, for every $e \in E$. Therefore, we have the following lemma.

Lemma 5.7. *Let $\gamma : TB \to \mathbf{L}(\Pi(E))$ be a connection on the frame groupoid $\Pi(E)$. Then, the corresponding linear connection*

$$c^{\gamma} : \pi^*(TB) = E \times_B TB \longrightarrow TE \tag{5.17}$$

on E is given by

$$c^{\gamma}(e,v) := T_{\mathrm{id}}\overline{\mathrm{ev}}_e(\gamma(v)), \quad (e,v) \in \pi^*(TB), \tag{5.18}$$

where id *is the identity map on the fiber E_x, if $x = \pi(e)$.*

Proof. By Proposition 3.5, γ induces the section $S^{\gamma} : B \to Q_1(\Pi(E))$ with $S^{\gamma}(x) = j^1_x s$, where $T_x s = \gamma | T_x B$ and $s(x) = \mathrm{id}$. Now, proceeding as in [11, page 190], S^{γ} gives rise to the section

$$f_S : E \longrightarrow J^1E : e \longmapsto j^1_{\pi(e)}(s \cdot \hat{e}) \tag{5.19}$$

of the canonical morphism $p : J^1E \to E : j^1_z t \mapsto t(z)$. Here, $s \cdot \hat{e}$ is the section of E, given by

$$(s \cdot \hat{e})(y) = \overline{\mathrm{ev}}(s(y), e) \in E_y \tag{5.20}$$

for every $y \in B$. Recall that, by the definition of $\Pi(E)$, $s(y) \in \mathrm{Lis}(E_x, E_y)$ and $s(x)$ is the identity map on E_x.

Finally, based on the relation between linear connections and sections of the canonical morphism p (see, for instance, [4, 16]), we check that

$$\begin{aligned} c^{\gamma}(e,v) &= T_x(s \cdot \hat{e})(v) = T_x\big(\overline{\mathrm{ev}}(s, c_e)\big)(v) \\ &= T_x\big(\overline{\mathrm{ev}}_e \circ s\big)(v) = T_{\mathrm{id}}\overline{\mathrm{ev}}_e\big(T_x s(v)\big) \\ &= T_{\mathrm{id}}\overline{\mathrm{ev}}_e\big(\gamma(v)\big), \end{aligned} \tag{5.21}$$

(where c_e is constant) by which we conclude the proof. □

To obtain the analog of the previous result for the connections Γ and C^{Γ}, we consider the entities π_1^*E, $Q_1(\Pi(E)) \equiv Q_1$, $\overline{\Gamma} \equiv \Gamma$, and $\mathrm{pr}_1 : \pi_1^*E \to Q_1$ in place of E, B, γ, and π, respectively. Also, analogously to $\overline{\mathrm{ev}}$, we define the map

$$\overline{\overline{\mathrm{ev}}} : \Pi(\pi_1^*E) \times_{Q_1} \pi_1^*E \longrightarrow \pi_1^*E : (g, m) \longmapsto g(m). \tag{5.22}$$

Proposition 5.8. *The linear connection $C^{\Gamma} : \mathrm{pr}_1^*(TQ_1) \to T(\pi_1^*E)$ is given by*

$$C^{\Gamma}(a,w) = \big(w, T_{\mathrm{id}}\overline{\mathrm{ev}}_e(T_x s(T_X \pi_1(w)))\big), \tag{5.23}$$

where $a = (X,e) \in \pi_1^*E \subseteq Q_1 \times E$, $X = j^1_x s$, and $w \in T_X Q_1$.

Proof. We outline the main steps of the procedure, omitting the computational details whose verification is a matter of routine.

First, we observe that every $h \in \Pi(\pi_1^* E)$ can be written, by the identification of Corollary 5.5, in the form $h = (Y, \bar{h}, X)$ with $X = j^1_x s$, $Y = j^1_y s'$, and $\bar{h} \in (\Pi(E))^y_x$. In these terms, the map $\overline{\overline{\mathrm{ev}}}$ is equivalently given by

$$\overline{\overline{\mathrm{ev}}}((Y, \bar{h}, X), (X, e)) = (Y, \bar{h}(e)). \tag{5.24}$$

Using the projections $p_1 : J_1(\Pi(E)) \to Q_1(\Pi(E))$ and $p_2 : J_1(\Pi(E)) \to \Pi(E)$, the previous equality becomes

$$\overline{\overline{\mathrm{ev}}}_a = (p_1, \overline{\mathrm{ev}}_e \circ p_2), \quad a = (X, e), \tag{5.25}$$

from which we obtain

$$T_{\mathrm{id}} \overline{\overline{\mathrm{ev}}}_a = (\mathrm{Pr}_1, T_{\mathrm{id}} \overline{\mathrm{ev}}_e \circ \mathrm{Pr}_2), \tag{5.26}$$

where Pr_1 and Pr_2 are the corresponding differentials of p_1 and p_2, respectively, whereas id denotes the identity between appropriate spaces.

On the other hand, by the definition of $\overline{\Gamma}$ and Lemma 5.7, we conclude that

$$C^{\Gamma}(a, w) = T_{\mathrm{id}} \overline{\overline{\mathrm{ev}}}_a (\overline{\Gamma}(w)) = T_{\mathrm{id}} \overline{\overline{\mathrm{ev}}}_a (w, T_x s (T_X \pi_1(w))). \tag{5.27}$$

The desired expression is now an immediate consequence of (5.26) and (5.27). □

For a given section S of $Q_1(\Pi(E))$, we define the map

$$\overline{S} : \pi^* TB \longrightarrow \mathrm{pr}_1^* (T(Q_1(\Pi(E)))) \tag{5.28}$$

with $\mathrm{pr}_1 : \pi_1^*(E) \to Q_1 \equiv Q_1(\Pi(E))$ and

$$\overline{S}(e, v) = (((S \circ \pi)(e), e), TS(v)). \tag{5.29}$$

Also, we denote by

$$\overline{P}_2 : TQ_1 \times_{TB} TE = T(\pi_1^* E) \longrightarrow TE \tag{5.30}$$

the natural projection to the second factor.

With the previous notations, we are in a position to state the universal property (4.13) for linear connections. Namely, we have the following theorem.

Theorem 5.9. *Let (E,π,B) be a vector bundle with connected base and $Q_1(\Pi(E))$ be the connection bundle of the frame groupoid $\Pi(E)$. Then, the linear connection (5.23) of π_1^*E satisfies the universal property*

$$c^S = \overline{P}_2 \circ C^\Gamma \circ \overline{S} \tag{5.31}$$

for any linear connection c^S of E, corresponding to a section S of $Q_1(\Pi(E))$.

Proof. The section S corresponds to the connection γ^S of $\Pi(E)$, which induces on E the linear connection

$$c^S : E \times_B TB \longrightarrow TE : (e,v) \longmapsto T_{\mathrm{id}}\overline{\mathrm{ev}}_e(\gamma^S(v)) \tag{5.32}$$

(see Proposition 3.5 and Lemma 5.7). As in the proof of Proposition 3.5, $S(x) = j^1_x s$ with $x = \pi(e)$. Thus, we get $\gamma^S(v) = T_x s(v)$ and

$$c^S(e,v) = T_{\mathrm{id}}\overline{\mathrm{ev}}_e(T_x s(v)), \quad (e,v) \in E \times_B TB. \tag{5.33}$$

Therefore, we check that

$$\begin{aligned}
(\overline{P}_2 \circ C^\Gamma \circ \overline{S})(e,v) &= (\overline{P}_2 \circ C^\Gamma)((S(x),e),TS(v)) \\
&= \overline{P}_2(TS(v), T_{\mathrm{id}}\overline{\mathrm{ev}}_e(T_x s(T_{S(x)}\pi_1(T_x S(v))))) \\
&= \overline{P}_2(TS(v), T_{\mathrm{id}}\overline{\mathrm{ev}}_e(T_x s(v))) \\
&= T_{\mathrm{id}}\overline{\mathrm{ev}}_e(T_x s(v)) \\
&= c^S(e,v),
\end{aligned} \tag{5.34}$$

by which we conclude the proof. □

For the sake of completeness, we note that, since C^Γ is fully determined by the mapping $\overline{P}_2 \circ C^\Gamma =: \hat{c}$, then the universal property (5.31) can be also written under the form

$$c^S = \overline{S}^* \hat{c}, \tag{5.35}$$

which is analogous to the pullback expression of the universal connection of ordinary bundles.

Scholium 5.10. Under an appropriate interplay between π_1^*E and the ordinary vector bundle of jets J^1E of a vector bundle (E,π,B) (the details of which are beyond the framework of this paper, dealing principally with groupoids), we can show (see [14]) that the universal connections of Section 5.2 are related with the universal connections on frame bundles considered in [3].

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Efstathios Vassiliou: Department of Mathematics, University of Athens, Panepistimiopolis, Athens 15784, Greece

E-mail address: evassil@cc.uoa.gr

Apostolos Nikolopoulos: Department of Mathematics, University of Athens, Panepistimiopolis, Athens 15784, Greece

E-mail address: nikolopoulos@synigoros.gr

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