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ON SEQUENTIALLY RETRACTIVE INDUCTIVE LIMITS

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Every locally complete inductive limit of sequentially complete locally convex spaces, which satisfies Retakh's condition (M) is regular, sequentially complete and sequentially retractive. A quasiconverse for this theorem and a criterion for sequential retractivity of inductive limits of webbed spaces are given.

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1. Introduction. Throughout the paper, $\{(E_n,\tau_n)\}_n$ is an inductive sequence of locally convex spaces and $(E,\tau)=\operatorname{ind}(E_n,\tau_n)$ is its inductive limit. Recall that E is regular if every bounded subset in E is contained and bounded in one of the steps, and E is sequentially retractive if every null sequence in E converges to zero in some step. We say that E satisfies the Retakh's condition (M) if in every space E_n, there is an absolutely convex neighborhood of zero U_n such that

(1) $U_n\subset U_{n+1}$ for every $n\in\mathbb{N}$;

(2) for every $n\in\mathbb{N}$, there is $m>n$ such that all the topologies of the locally convex spaces E_k, for $k\geq m$, coincide on U_n. Equivalently, τ and τ_m coincide on U_n.

We assume that every such U_n is τ_n-closed and that τ_{n+1} and τ induce the same topology on U_n, which we do without loss of generality.

Finally, we say that E satisfies condition (Q) (see [8]) if part (1) in (M) is dropped.

Vogt in [7] studied condition (M) for LF-spaces, that is, for inductive limits of metrizable and complete (equivalently and sequentially complete) locally convex spaces. He obtained several important results about them; for example, that on LF-spaces, condition (M) implies completeness, regularity, and sequential retractivity. Recently, Wengenroth in [8] proved the following very important result on LF-spaces: condition (M), condition (Q), acyclicity and sequential retractivity are equivalent.

On the other hand, Gómez-Wulschner and Kučera in [2, 3] studied sequential completeness and weak regularity conditions for inductive limits of sequentially complete spaces. They have shown that a regular inductive limit of sequentially complete spaces is sequentially complete [3].

In Theorem 2.5 we show a result similar to Vogt's, but in the context of a locally complete inductive limit with condition (M) of a sequence of sequentially complete locally convex spaces.

The last part is devoted to webbed spaces (definitions are recalled in that section). We present a quasiconverse to Theorem 2.5 and a criterion for sequential retractivity.

2. Regularity and sequential retractivity. Recall that a disk D in a locally convex space F is an absolutely convex, bounded and closed subset. We write (F_D,ρ_D) to denote a normed space, where $F_D=\operatorname{span}D$ and ρ_D is the norm topology generated on F_D by the Minkowski's functional of D; equivalently, ρ_D is generated by the basis of neighborhoods $\{\lambda D:\lambda>0\}$. Note that closedness is not necessary for the Minkowski's functional to be a norm.

In order to obtain the first theorem, we need a technical lemma and a pair of useful propositions.

Lemma 2.1. *If $E=\operatorname{ind}E_n$ satisfies condition (M) for the sequence $(U_n)_n$, then $\overline{U_j}^{E}=\bigcup_{k=j}^{\infty}\overline{U_j}^{E_k}$ for every $j\in\mathbb{N}$.*

Proof. Since τ restricted to E_k is coarser than τ_k, we have $\overline{U_j}^{E_k}\subset\overline{U_j}^{E}$ for $k\geq j$. So, $\bigcup_{k=j}^{\infty}\overline{U_j}^{E_k}\subset\overline{U_j}^{E}$. Conversely, let $x\in\overline{U_j}^{E}$. There exists a net $(x_\alpha)_\alpha\subset U_j$ such that $x_\alpha\xrightarrow{\tau}x$. This implies that there exists $n\geq j$ and $\lambda>0$ such that $\lambda\{(x_\alpha)_\alpha,x\}\subset U_n$. Since τ and τ_{n+1} coincide on U_n, $\lambda x_\alpha\xrightarrow{\tau_{n+1}}\lambda x$; so, $x_\alpha\xrightarrow{\tau_{n+1}}x$. Hence, $x\in\overline{U_j}^{E_{n+1}}$. □

The next proposition is the key to Theorem 2.5.

Proposition 2.2. *Let every (E_n,τ_n) be locally complete. If $E=\operatorname{ind}E_n$ satisfies condition (M), then every Banach disk $B\subset E$ is contained and bounded in some E_n.*

Proof. Let $B\subset E$ be a Banach disk. By [5, Proposition 8.5.20], there exists $p\in\mathbb{N}$ such that $B\subset p\overline{U_p}^{E}=p\bigcup_{k=p}^{\infty}\overline{U_p}^{E_k}$, the last identity follows from Lemma 2.1.

Since B is τ-closed and τ-bounded, $B\cap E_k$ is τ_k-closed and $B\cap p\overline{U_p}^{E_k}\subset B$ is τ-bounded, for every $k\geq p$. Let $B_k=B\cap p\overline{U_p}^{E_k}$. We assume that every U_k is τ_k-closed, then $(1/p)B_k\subset\overline{U_p}^{E_k}\subset\overline{U_k}^{E_k}=U_k$ for every $k\geq p$. By condition (M), τ and τ_{k+1} coincide on U_k, then $(1/p)B_k$ is τ_{k+1}-bounded. Now, the local completeness of E_{k+1} implies that $\overline{B_k}^{E_{k+1}}$ is a Banach disk in E_{k+1}, so $(E_{\overline{B_k}^{E_{k+1}}},\rho_{\overline{B_k}^{E_{k+1}}})$ is a Banach space continuously embedded in (E_{k+1},τ_{k+1}).

Note that for every $k\geq p$,

$$\overline{B_k}^{E_{k+1}}=\overline{B\cap p\overline{U_p}^{E_k}}^{E_{k+1}}\subset\overline{B\cap p\overline{U_p}^{E_{k+1}}}^{E_{k+2}}=\overline{B_{k+1}}^{E_{k+2}}. \tag{2.1}$$

This implies that $\overline{B_k}^{E_{k+1}}$ is contained in $\overline{B_{k+1}}^{E_{k+2}}\cap E_{\overline{B_k}^{E_{k+1}}}$; therefore $(E_{\overline{B_k}^{E_{k+1}}},\rho_{\overline{B_k}^{E_{k+1}}})$ is continuously embedded in $(E_{\overline{B_{k+1}}^{E_{k+2}}},\rho_{\overline{B_{k+1}}^{E_{k+2}}})$.

It follows that $\operatorname{ind}(E_{\overline{B_k}^{E_{k+1}}}, \rho_{\overline{B_k}^{E_{k+1}}})$ is an LB-space. In order to finish the proof, we prove that this is a nonproper LB-space. In other words, we show that there exists $k_0 \in \mathbb{N}$ such that $(E_{\overline{B_{k_0}}^{E_{k_0+1}}}, \rho_{\overline{B_{k_0}}^{E_{k_0+1}}}) = (E_B, \rho_B)$.

Since B is τ-closed and $B_k \subset B$, we have $\overline{B_k}^{E_{k+1}} \subset B$. And $\overline{B_k}^{E_{k+1}} \subset B \cap E_{\overline{B_k}^{E_{k+1}}}$ which implies that the identity map $i_k : (E_{\overline{B_k}^{E_{k+1}}}, \rho_{\overline{B_k}^{E_{k+1}}}) \to (E_B, \rho_B)$ is continuous for every $k \geq p$.

On the other hand,

$$B = B \cap p \bigcup_{k=p}^{\infty} \overline{U_p}^{E_k} = \bigcup_{k=p}^{\infty} B \cap p\overline{U_p}^{E_k} = \bigcup_{k=p}^{\infty} B_k \subset \bigcup_{k=p}^{\infty} \overline{B_k}^{E_{k+1}} \subset B. \tag{2.2}$$

This means that $\operatorname{span}(B) = \bigcup_{k=p}^{\infty} \operatorname{span}(\overline{B_k}^{E_{k+1}})$. Therefore, the identity map

$$i : \operatorname{ind}\left(E_{\overline{B_k}^{E_{k+1}}}, \rho_{\overline{B_k}^{E_{k+1}}}\right) \longrightarrow (E_B, \rho_B) \tag{2.3}$$

is continuous and onto. By the open mapping theorem (see [5, Theorem 8.4.11]), the inverse identity map

$$j : (E_B, \rho_B) \longrightarrow \operatorname{ind}\left(E_{\overline{B_k}^{E_{k+1}}}, \rho_{\overline{B_k}^{E_{k+1}}}\right) \tag{2.4}$$

is continuous. By Jarchow [4, Corollary 5.6.4], the space (E_B, ρ_B) is continuously embedded in some $(E_{\overline{B_{k_0}}^{E_{k_0+1}}}, \rho_{\overline{B_{k_0}}^{E_{k_0+1}}})$.

We conclude that B is contained and bounded in $(E_{k_0+1}, \tau_{k_0+1})$. □

Corollary 2.3. *Let every (E_n, τ_n) be locally complete. If $E = \operatorname{ind} E_n$ is locally complete and satisfies condition (M), then E is regular.*

Proposition 2.4. *Let every (E_n, τ_n) be sequentially complete. If $E = \operatorname{ind} E_n$ is regular and satisfies condition (M) for a sequence $(U_n)_n$, then E is sequentially complete and sequentially retractive.*

Proof. Let $(x_l)_l$ be a Cauchy sequence in (E, τ). Then, $A = \{x_l : l \in \mathbb{N}\}$ is a τ-bounded set. So, there exists $n \in \mathbb{N}$, such that $A \subset E_n$ and A is τ_n-bounded. There exists $s > 0$ such that $sA \subset U_n$. Since τ and τ_{n+1} coincide on U_n, it follows that $(sx_l)_l$ is τ_{n+1}-Cauchy, then τ_{n+1}-convergent to sx_0, for some $x_0 \in E_{n+1}$, hence $(x_l)_l$ is convergent to x_0 in (E, τ).

In an analogous way, it is straightforward to show that (E, τ) is sequentially retractive. □

From the preceding results we conclude the following theorem.

Theorem 2.5. *Let every (E_n, τ_n) be sequentially complete. If $E = \operatorname{ind} E_n$ is locally complete and satisfies condition (M), then E is regular, sequentially complete, and sequentially retractive.*

3. Sequential retractivity on webbed spaces. We give now two results on sequential retractivity for certain webbed spaces. For convenience, we recall some basic facts about webs which we need. For more information about the basic properties of webs, we refer the reader to the works of De Wilde [9], Jarchow [4], and Robertson [6].

A strand of a web $\mathbb{W}$ on a locally convex space (F,τ) is a collection of members of $\mathbb{W}$, one from each layer, with the $(k+1)$th member of the strand contained in the kth member. Strands will be denoted by $(W_k)_k$. A web on F is compatible with τ if for each neighborhood of zero U in (F,τ) and for each strand $(W_k)_k$ of $\mathbb{W}$, there is k_0 such that $W_{k_0} \subset U$.

Following the idea of Wengenroth in the proof of [8, Proposition 2.3], we get a quasiconverse for Theorem 2.5. To simplify the notation in this proposition, we use $W_k \in \mathbb{W}$ to denote $W_{k_1,k_2,\dots,k_r} \in \mathbb{W}$, that is, write only one index as for the elements of a specific strand.

Proposition 3.1. Let every (E_n,τ_n) be a webbed space and $E = \operatorname{ind} E_n$ sequentially retractive. Then, for every $N \in \mathbb{N}$ there is $n \geq N$ and an element W_k^N of the web $\mathbb{W}^{(N)}$ on E_N, for some $k = k(N)$, such that τ and τ_n coincide on W_k^N.

Proof. Suppose that this proposition is not true. So, there exists E_{n_0} such that for every element of its web, say $W_k^{n_0} \in \mathbb{W}^{n_0}$, and for each $N \in \mathbb{N}$ there exists $n > N$ such that τ_n restricted to $W_k^{n_0}$ is strictly coarser than τ_N restricted to $W_k^{n_0}$.

For such n_0, fix an element of the web $W_{k_0}^{n_0} \in \mathbb{W}^{n_0}$. Let $N = n_0$, then there exists $n_1 > n_0$ such that τ_{n_1} restricted to $W_{k_0}^{n_0}$ is strictly coarser than τ_{n_0}. So, there is a sequence $(x_l^{k_0})_l \subset W_{k_0}^{n_0}$, which is τ_{n_1}-null but not τ_{n_0}-null. Find an element of the (k_0+1)th layer of the web $W_{k_0+1}^{n_0} \in \mathbb{W}^{n_0}$, such that $W_{k_0+1}^{n_0} + W_{k_0+1}^{n_0} \subset W_{k_0}^{n_0}$, and $n_2 > n_1$ such that τ_{n_2} restricted to $W_{k_0+1}^{n_0}$ is strictly coarser than τ_{n_1} restricted to $W_{k_0+1}^{n_0}$. Then, there is a sequence $(x_l^{k_0+1})_l \subset W_{k_0+1}^{n_0}$, which is τ_{n_2}-null but not τ_{n_1}-null. In this way, determine a strand $(W_{k_0+k}^{n_0})_k$ of the web $\mathbb{W}^{n_0}$, an increasing sequence of natural numbers $(n_k)_k$, and a collection of sequences $[(x_l^{k_0+k})_l]_k$ such that every $(x_l^{k_0+k})_l \subset W_{k_0+k}^{n_0}$ is $\tau_{n_{k+1}}$-null but not τ_{n_k}-null.

Let U be a neighborhood of zero in (E,τ). Then, $U \cap E_{n_0}$ is a neighborhood of zero in (E_{n_0},τ_{n_0}), so there exists $K \in \mathbb{N}$ such that $W_{k_0+k}^{n_0} \subset U \cap E_{n_0} \subset U$ if $k > K$. It implies that $(x_l^{k_0+k})_l \subset U$ for every $l \in \mathbb{N}$ if $k > K$. Now, if $k \leq K$, then $(x_l^{k_0+k})_l$ is a τ-null sequence, since it is a $\tau_{n_{k+1}}$-null sequence. Hence, arranging the double-indexed sequence in any way into a single indexed sequence, it results a τ-null sequence. So, this sequence should be convergent in some E_m since E is sequentially retractive. But this is not possible, since the sequence is not convergent in any E_m. Hence the proposition is true. □

Recall that a space F is strictly barreled if given any ordered web in F, there is a strand $(W_k)_k$ such that for every $k \in \mathbb{N}$, $\overline{W_k}$ is a neighborhood of zero. So,

if in Proposition 3.1 everyone of the corresponding elements from the webs, where the topologies coincide is τ_n-neighborhoods of zero, then E satisfies condition (Q), and hence by [8, Proposition 2.5], E satisfies condition (M).

Following Gilsdorf [1], a locally convex space F is sequentially webbed if it has a compatible web $\mathbb{W}$ such that for every null sequence $(x_m)_m$ in (F,τ), there exist a strand $(W_k)_k$ and a natural number M_k for every $k \in \mathbb{N}$, such that $x_m \in W_k$ for all $m \geq M_k$. To simplify, we denote this condition by $(\#)$.

From [4, Corollary 5.3.3(b)], the inductive limit $E = \operatorname{ind} E_n$ of a numerable sequence of webbed spaces is again webbed and admits a completing web $\mathbb{W}$ such that $\mathbb{W}(n) = E_n$ for every $n \in \mathbb{N}$. Moreover, the kth layer of the web $\mathbb{W}$ on E is the collection of members of the kth layer in the spaces E_n. In the next theorem, we use such a web on $E = \operatorname{ind} E_n$ in order to characterize sequential retractivity for inductive limits of sequentially webbed spaces.

Theorem 3.2. *Let every (E_n,τ_n) be a sequentially webbed space. $E = \operatorname{ind} E_n$ is sequentially retractive if and only if E is sequentially webbed.*

Proof. Suppose that E is sequentially retractive. For any null sequence $(x_m)_m$ in (E,τ), there exists $n \in \mathbb{N}$ such that $(x_m)_m$ is a null sequence in (E_n,τ_n). So, there is a strand $(W_k^{(n)})_k$ of the web $\mathbb{W}^{(n)}$ on E_n satisfying $(\#)$ on E_n. Note that by the form of the web on E, $(W_k^{(n)})_k$ is also a strand for the web $\mathbb{W}$ on E. So, E is sequentially webbed. Conversely, let $(x_m)_m$ be a null sequence in (E,τ); since E is sequentially webbed, there is a strand $(W_k)_k$ of $\mathbb{W}$ on E satisfying $(\#)$. By the form of the web on E, $W_1 = \mathbb{W}(n) = E_n$ for some $n \in \mathbb{N}$. So, this strand is contained in E_n and it is a strand of $\mathbb{W}^{(n)}$ on E_n. Now, since $\mathbb{W}^{(n)}$ is compatible with τ_n, for every U-neighborhood of zero in (E_n,τ_n), there exists $k \in \mathbb{N}$ such that $W_k \subset U$. Hence, $x_m \in U$ for all $m \geq M_k$. □

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