A DISTRIBUTIONAL HARDY TRANSFORMATION

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(Received January 23, 1979)

ABSTRACT: The Hardy's F-transform

$$F(t) = \int_0^\infty F_\nu(ty) \, yf(y) \, dy$$

is extended to distributions. The corresponding inversion formula

$$f(x) = \int_0^\infty C_\nu(tx) \, t \, F(t) dt$$

is shown to be valid in the weak distributional sense. This is accomplished by

transferring the inversion formula onto the testing function space for the

generalized functions under consideration and then showing that the limiting

process in the resulting formula converges with respect to the topology of

the testing function space.

KEY WORDS AND PHRASES. *Integral Transform, Hardy Transform, Hankel Transform,*
Distributions, Generalized Functions.

AMS (MOS) SUBJECT CLASSIFICATION (1970) CODES. *Primary 44A20, Secondary 46F05.*

1. INTRODUCTION.

The Hardy transforms with their inversion formulae are represented by the

following two integral equations:

$$f(x) = \int_0^\infty F_\nu(tx)t dt \int_0^\infty C_\nu(ty)y f(y) dy \tag{1}$$

and

$$f(x) = \int_0^\infty C_\nu(tx)t dt \int_0^\infty F_\nu(ty)y\, f(y)\, dy \tag{2}$$

where

$$C_\nu(z) = \cos p\pi\, J_\nu(z) + \sin p\pi\, Y_\nu(z) \tag{3}$$

and

$$F_\nu(z) = \sum_{m=0}^\infty \frac{(-1)^m (z/2)^{\nu+2p+2m}}{\Gamma(p+m+1)\,\Gamma(p+m+\nu+1)} \tag{4}$$

$$= 2^{2-\nu-2p}\, s_{\nu+2p-1,\ \nu}(z) / \{\Gamma(p)\,\Gamma(\nu+p)\}\ [2,\ p.\ 40].$$

The theory of the inversion formulae (1) and (2) has been given by Cooke [1].

The Hankel transform with its inversion formula can be deduced as a special case

of both (1) and (2) by taking p = 0. The Y-transform [3, p. 93] is a special

case of (1) whereas H-transform [3, p. 155] is a special case of (2) for $p = \frac{1}{2}$.

Recently the inversion formula (1) was proved to be valid for the generalized

function space $H'_\alpha(I)$ by Pathak and Pandey [7] in the weak distributional sense.

It turns out that the kernal $y\, F_\nu(ty)$ of F_ν-transform does not belong to the space

H_α(I) and therefore the inversion formula (2) cannot be proved to be valid for

the space of distributions directly as a corollary to theorems proved in [7]. We

will therefore extend briefly the inversion formula (2) to a generalized function

space essentially by following the techniques and results proved in [7].

2. <u>TESTING FUNCTION SPACE</u> $H_{\alpha,\beta}^{\nu,p}$ (I). For $-\frac{1}{2} \leq \nu \leq \frac{1}{2}$ and real p let $F_\nu(z)$ be

the function defined in (4) and let α be a fixed number satisfying $\alpha + \nu + 2p \geq 0$.

Assume that β is also a fixed number satisfying $\beta \geq \sigma = \max (\nu + 2p - 2, -\frac{1}{2})$.

For each k = 0, 1, 2, ... define a positive and continuous function $\xi_k(x)$ on

I = {x; $0 < x < \infty$} satisfying

$$
\xi_k(x) = \begin{cases} x^{2k + \alpha} & 0 < x \leq 1 \\ x^{-\beta - 2} & x > 1. \end{cases}
$$

An infinitely differentiable complex-valued function ϕ (x) defined over I is

said to belong to the space $H_{\alpha,\beta}^{\nu,p}$ (I) if

$$
\nu_k(\phi) \equiv \sup_{0 < x < \infty} \left| \xi_k(x) \, \Delta_x^k \left(\frac{\phi(x)}{x} \right) \right| < \infty
$$

for each k = 0, 1, 2, 3, ... where Δ_x stands for the differentiation operator

$\left(D_x^2 + \frac{1}{x} D_x - \frac{\nu^2}{x^2} \right)$, $D_x = \frac{d}{dx}$. It can be readily seen that $H_{\alpha,\beta}^{\nu,p}$ (I) is a vector

space. The topology over $H_{\alpha,\beta}^{\nu,p}$ (I) is generated by the sequence of seminorms

$\{\nu_k\}_{k=0}^{\infty}$ [9; p. 8].

A sequence $\{\phi_\nu\}$ in this space is said to converge to the element ϕ if

$\gamma_k(\emptyset_\nu - \emptyset) \to 0$ as $\nu \to \infty$ for each $k = 0, 1, 2, 3, \ldots$ A sequence $\emptyset_\nu$ in $H^{\nu,p}_{\alpha,\beta}$ (I)

is said to be a Cauchy sequence if $\gamma_k(\emptyset_m - \emptyset_n) \to 0$ as $m, n \to \infty$ independently

of each other. It is a simple exercise to verify that the space $H^{\nu,p}_{\alpha,\beta}$ (I) is

sequentially complete and so it is a Fréchet space. Since $D(I) \subset H^{\nu,p}_{\alpha,\beta}$ (I) and

the topology of $D(I)$ is stronger than that induced on $D(I)$ by $H^{\nu,p}_{\alpha,\beta}$ (I), it

follows that the restriction of any $f \in H^{\nu,p}_{\alpha,\beta}$ (I) to $D(I)$ is in D' (I).

In view of the fact that

$$\Delta^k_x \left[F_\nu (xt) \right] = (-1)^k t^{2k} F_\nu (xt) - P(x,t)$$

where

$$P(x,t) = t^{\nu+2p} \sum_{i=1}^{k} a_i x^{\nu+2p-2i} t^{2k-2i} \tag{5}$$

a_k being certain constants depending on ν and p, and the asymptotic orders

[9, p. 345]

$$F_\nu(z) = \begin{cases} 0 & |z|^{\nu + 2p} & |z| \to 0 \\ 0 & |z|^{\sigma} & |z| \to \infty \end{cases} \tag{6}$$

where

$$\sigma = \max (\nu + 2p - 2, -\tfrac{1}{2}) \qquad [8, \text{pp. } 347, 351]$$

it follows that for fixed $t > 0$, $x F_\nu (tx)$ belongs to the space $H^{\nu,p}_{\alpha,\beta}$ (I) when

treated as a function of x. Therefore, Hardy's F_ν-transform $F(y)$ of a generalized

function $f \in H^{\nu,p}_{\alpha,\beta}$ (I) can be defined by

$$F(y) = \langle f(x), \ x F_\nu (xy) \rangle, \qquad y > 0. \tag{7}$$

By following the technique as used in [7] it can be shown that $F(y)$ is differ-

entiable for each $y > 0$ and that

$$F'(y) = \langle f(x), \ \frac{\partial}{\partial y} \left\{ x F_\nu (xy) \right\} \rangle \tag{8}$$

Note that

$$\frac{\partial}{\partial y} [x \, F_\nu \, (xy)] \text{ also belongs to } H^{\nu,p}_{\alpha,\beta} (I).$$

We now state some results which will be used in the sequel.

Define

$$H_N \, (t,x) = \int_0^N C_\nu \, (tx) \, C_\nu \, (xy) \, y \, dy$$

$$= \frac{N}{x^2 - t^2} \left[x \, C_{\nu+1}(xN) \, C_\nu(tN) - t \, C_{\nu+1}(tN) \, C_\nu(xN) \right] - Q \, (x,t)$$

where

$$Q \, (x, t) = \frac{2 \sin p \, \pi}{\pi \sin \nu \, \pi} \, \sin (p + \nu) \, \pi \, \frac{x^{2\nu} - t^{2\nu}}{x^\nu \, t^\nu (x^2 - t^2)} \, , \quad x \neq t \qquad (10)$$

$$= \frac{2 \sin p \, \pi \sin (p+\nu) \, \pi}{\pi \sin \nu \, \pi} \, \frac{\nu}{x^2} \qquad \text{when} \quad t = x$$

$$[8, \text{p. } 466].$$

Using the technique employed in proving Lemma 2 in [7] it can be proved

that for fixed t , $x.Q \, (x,t) \in H^{\nu,p}_{\alpha,\beta} \, (I)$.

It is now a simple exercise to prove for $\alpha \geq |\nu|$, $\beta \leq \nu - 4$ and $\emptyset \in D(I)$ that

$$x \int_a^b Q(x,y) \, \emptyset \, (y) \, y \, dy \text{ also belongs to } H^{\nu,p}_{\alpha,\beta} \, (I).$$

LEMMA 2. Let $\xi_k(t)$ be defined as in section 2. Then for $0 < y < 1$

$$\sup_{0 < t < \infty} \left| \frac{\xi_k(t)}{\xi_k(ty)} \right| = y^{\min (\beta + 2, \, -\alpha - 2k)}, \quad k = 0, 1, 2, \ldots$$

PROOF.   The result follows by dividing the t- line into three parts

$0 < t < 1$, $1 < t < 1/y$, $1/y < t < \infty$ and considering the corresponding

$$\sup_{0 < t < \infty} \left| \frac{\xi_k(t)}{\xi_k(ty)} \right|$$

LEMMA 3.   Let $C_\nu(z)$ be the function as defined in (3) and let

$-\frac{1}{2} \leq \nu \leq \frac{1}{2}$, $-\nu - 2p \leq \alpha \leq 3/2$, $\beta \geq$ max ($\nu + 2p - 2$, $-\frac{1}{2}$).

Then for fixed $x > 0$,

$$t \int_0^\eta F_\nu (ty)\, C_\nu (xy)\, ydy \rightarrow 0 \text{ in } H_{\alpha,\beta}^{\nu,p} (I) \text{ as } \eta \rightarrow 0 +$$

PROOF.   The lemma can be proved by using lemma 2 and a variation of the

technique used in proving lemma 4 of [7].

LEMMA 4.   Let α, β, ν and p be restricted as in Lemma 3 and let

$f \in H_{\alpha,\beta}^{\nu,p}$,   then

$$\int_0^N < f(t),\, t\, F_\nu (ty) >\ C_\nu (xy)\, ydy = \ < f(t),\, t \int_0^N F_\nu (ty)\, C_\nu (xy)\, ydy >.$$

PROOF.   The result follows in view of Lemma 3. The details of the

technique to be used can be found in [7, Lemma 5].

LEMMA 5.   Let $b > a > 0$ and $H_N (t,x)$, $Q (t,x)$ be the functions as

defined by (9) and (10). Then

$$\lim_{N \rightarrow \infty} \int_a^b [\ H_N (t,x) + Q (t,x)\ ]\, xdx = \begin{array}{ll} 1 & t \in [a,b] \\ 0 & t \notin [a,b]. \end{array}$$

PROOF.   See Lemma 6 in [7].

LEMMA 6.   Let the support of $\varphi \in D(I)$ be contained in (a,b) where

$b > a > 0$. Let $H_N (t,x)$, $Q (t,x)$ be the functions as defined in (9) and (10).

Assume that $-\frac{1}{2} \leq \nu \leq \frac{1}{2}$, max $(- \nu - 2p, \nu) \leq \alpha \leq 3/2$ and $\beta \geq$ max $(\nu + 2p - 2, -\frac{1}{2})$.

Then

$$t \int_a^b \left[H_N (t,x) + Q (x,t) \right] \varphi (x)\, xdx \rightarrow t\, \varphi (t) \text{ in } H_{\alpha,\beta}^{\nu,p} (I) \underline{\text{as}} N \rightarrow \infty.$$

PROOF.   The proof can be given only by using Lemma 3 and a simple variations of the techniques used in proving [7, Lemma 7] and so the details are omitted.

3. INVERSION OF THE DISTRIBUTIONAL

F_ν - transform: We now state and prove our main result.

THEOREM.   Let $-\frac{1}{2} \leq \nu \leq \frac{1}{2}$, $\max(-\nu - 2p, |\nu|) \leq \alpha \leq 3/2$ and $\beta \geq \max$ $(\nu + 2p - 2, -\frac{1}{2})$.

Assume that F(y) is the distributional F_ν-transform of $f \in H^{\nu,p}_{\alpha,\beta}$ (I) as defined by (7). Then

$$\lim_{N \to \infty} \left\langle \int_0^N F(y) \, C_\nu \, (x,y) \, y \, dy, \, \emptyset \, (x) \right\rangle = \langle f, \emptyset \rangle \quad \text{for each} \quad \emptyset \in D(I).$$

PROOF. Let the support of $\emptyset$ be [a,b] where $b > a > 0$.

Since $F(y) \, C_\nu \, (xy) \, y$ generates a regular distribution we have

$$\int_0^N \left\langle F(y) \, C_\nu \, (xy) \, y \, dy, \, \emptyset \, (x) \right\rangle = \int_a^b \emptyset \, (x) \, dx \int_0^N F(y) \, C_\nu \, (xy) \, y \, dy$$

$$= \int_a^b \left\langle f(t), \, t \int_0^N F_\nu \, (ty) \, C_\nu \, (xy) \, y \, dy \right\rangle \, \emptyset \, (x) \, dx$$

<div align="center">[Lemma 4]</div>

$$= \int_a^b \left\langle f(t), \, t \left\{ H_N(t,x) + Q_N(t,x) \right\} \right\rangle \, \emptyset \, (x) \, dx$$

<div align="center">[7, Lemma 8]</div>

$$= \int_a^b \left\langle f(t), \, t \left\{ H_N(t,x) + Q(t,x) \right\} \right\rangle \, \emptyset \, (x) \, dx$$

for $N > b > 0$ [1, Lemma ϵ p. 394]

$$= \left\langle f(t), \quad t \int_a^b \left\{ H_N(t,x) + Q(t,x) \right\} \right\rangle \quad \frac{\phi(x)}{x} \quad x \, dx$$

by Riemann Sums technique   [9, p. 148]

$$\rightarrow \left\langle f(t), \quad t \cdot \frac{\phi(t)}{t} \right\rangle \quad = \left\langle f, \phi \right\rangle$$

$$\text{[Lemma 6]}$$

This completes the proof of the theorem.

Taking $p = 0$ and $p = \frac{1}{2}$ in the above theorem we derive

COROLLARY 1.   Let $f \in H_{\alpha,\beta}^{\nu,0}$ (I) <u>where</u> $-\frac{1}{2} \leq \nu \leq \frac{1}{2}$, $|\nu| \leq \alpha \leq 3/2$ and

$\beta \geq -\frac{1}{2}$. <u>Define the distributional Hankel transform</u> of f by

$$F(y) = \left\langle f(t), \quad t \, J_\nu \, (ty) \right\rangle$$

then

$$\lim_{N \to \infty} \left\langle \int_0^N F(y) \, y \, J_\nu(xy) \, dy, \quad \phi(x) \right\rangle = \left\langle f, \phi \right\rangle$$

<u>for all</u> $\phi \in D(I)$.

COROLLARY 2.   <u>Let</u> $f \in H_{\alpha,\beta}^{\nu,\frac{1}{2}}$ (I)   <u>where</u> $-\frac{1}{2} \leq \nu \leq \frac{1}{2}$, $|\nu| \leq \alpha \leq 3/2$ and

$\beta \geq -\frac{1}{2}$. <u>Define the distributional Struve transform</u> (H_ν-transform) of f by

$$F(y) = \left\langle f(t), \quad t \quad H_\nu \, (ty) \right\rangle$$

<u>then</u>

$$\lim_{N \to \infty} \left\langle \int_0^N F(y) \, Y_\nu(xy) \, y \, dy, \quad \phi(x) \right\rangle = \left\langle f, \phi \right\rangle$$

<u>for all</u> $\phi \in D(I)$.

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Manuscript Due	December 1, 2008
First Round of Reviews	March 1, 2009
Publication Date	June 1, 2009

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