

EXISTENCE OF PERIODIC SOLUTIONS FOR NONLINEAR LIENARD SYSTEMS

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ABSTRACT. We prove the existence and multiplicity of periodic solutions for nonlinear Lienard System of the type

$$x''(t) + \frac{d}{dt}[\nabla F(x(t))] + g(x(t)) + h(t, x(t)) = e(t)$$

under various conditions upon the functions g , h and e .

KEY WORDS AND PHRASES: Nonlinear Lienard system, multiplicity of periodic solution.

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1. INTRODUCTION

Let R^n be n -dimensional Euclidean space. We define $\|x\| = [\sum_{i=1}^n |x_i|^2]^{1/2}$ for $x = (x_1, x_2, \dots, x_n) \in R^n$. By $L^2([0, 2\pi], R^n)$ we denote the space of all measurable functions $x: [0, 2\pi] \rightarrow R^n$ for which $\|x(t)\|^2$ is integrable. The norm is given by

$$\|x\|_{L^2} = \left[\sum_{i=1}^n \|x_i\|_{L^2}^2 \right]^{1/2}.$$

By $C^k([0, 2\pi], R^n)$ we denote the Banach space of 2π -periodic continuous functions $x: [0, 2\pi] \rightarrow R^n$ whose derivatives up to order k are continuous. The norm is given by

$$\|x\|_{C^k} = \sum_{i=0}^k \|x^{(i)}\|_{\infty}$$

where $\|y\|_{\infty} = \sup_{t \in [0, 2\pi]} \|y(t)\|$ which is a norm in $C([0, 2\pi], R^n)$. We use the symbol (\cdot, \cdot) for the Euclidean inner product in the space R^n . For x, y in $C([0, 2\pi], R^n)$ we define the L^2 -inner product as follows

$$\langle x, y \rangle = \int_0^{2\pi} (x(t), y(t)) dt.$$

The mean value \bar{x} of x and the function of mean value zero are defined by $\bar{x} = \frac{1}{2\pi} \int_0^{2\pi} x(t) dt$ and $\tilde{x}(t) = x(t) - \bar{x}$, respectively.

We define inequalities in R^n componentwise, i.e. $x, y \in R^n$, $x \leq y$ if and only if $x_i \leq y_i$ for $i = 1, 2, \dots, n$, and $x < y$ if and only if $x_i < y_i$ for $i = 1, 2, \dots, n$. In this work, we will study the existence of periodic solutions and multiple periodic solutions for the problem

$$(E) \quad x''(t) + \frac{d}{dt}[\nabla F(x(t))] + g(x) + h(t, x) = e(t)$$

$$(B) \quad x(0) - x(2\pi) = x'(0) - x'(2\pi) = 0$$

where $F : R^n \rightarrow R$ is a C^2 -function, $g : R^n \rightarrow R^n$ is continuous, $h : [0, 2\pi] \times R^n \rightarrow R$ is continuous in both variables and 2π -periodic in t , and $e : [0, 2\pi] \rightarrow R$ is in $L^2([0, 2\pi], R^n)$. We assume that $g(x) = (g_1(x_1), g_2(x_2), \dots, g_n(x_n))$ for all $x = (x_1, x_2, \dots, x_n) \in R^n$ and $h(t, x) = (h_1(t, x), h_2(t, x), \dots, h_n(t, x))$ for all $(t, x) \in [0, 2\pi] \times R^n$.

Moreover, we assume the following:

(H₁) h is bounded; i.e., for each $i = 1, 2, 3, \dots, n$, there exists $K_i > 0$ such that

$$|h_i(t, x)| \leq K_i$$

for all $(t, x) \in [0, 2\pi] \times R^n$.

(H₂) for each $i = 1, 2, \dots, n$,

$$\frac{d}{dt} \frac{\partial F(x)}{\partial x_i} = \frac{\partial^2 F(x)}{\partial x_i^2} x_i'$$

and there exists $C_i > 0$ such that

$$\left| \frac{\partial^2 F(x)}{\partial x_i^2} \right| \geq C_i$$

for all $x = (x_1, x_2, \dots, x_n) \in R^n$.

The purpose of this work is to give existence and multiplicity results for periodic solutions of coupled Lienard system in R^n . This paper was motivated by the results in [1] and so our results in this work extend some results in [1]. To prove our results we adapt Mawhin's continuation theorem in [2], and we give appropriate region for the system's multiplicity by finding an a'priori bound.

2. A'priori Bound

To prove our assertion, we consider the following homotopy:

$$(E_\lambda) \quad x''(t) + \lambda \frac{d}{dt} [\nabla F(x(t))] + \lambda g(x) + \lambda h(t, x) = \lambda e(t).$$

Let $\lambda \in (0, 1)$ and let $x(t)$ be a possible solution of the problem $(E_\lambda)(B)$. Taking L^2 -inner product by $x'(t)$ on both sides of (E_λ) , we have

$$\begin{aligned} & \lambda \sum_{i=1}^n \int_0^{2\pi} \frac{\partial^2 F(x(t))}{\partial x_i^2} [x_i'(t)]^2 dt + \lambda \sum_{i=1}^n \int_0^{2\pi} g_i(x_i(t)) x_i'(t) dt \\ & + \lambda \sum_{i=1}^n \int_0^{2\pi} h_i(t, x(t)) x_i'(t) dt - \lambda \sum_{i=1}^n \int_0^{2\pi} e_i(t) x_i'(t) dt. \end{aligned}$$

By the continuity of $\frac{\partial^2 F(x)}{\partial x_i^2}$, (H₂) and the periodicity of $x_i(t)$ in t , we have

$$\begin{aligned} & \sum_{i=1}^n C_i \int_0^{2\pi} [x_i'(t)]^2 dt \leq \left| \sum_{i=1}^n \int_0^{2\pi} \frac{\partial^2 F(x)}{\partial x_i^2} [x_i'(t)]^2 dt \right| \\ & \leq \sum_{i=1}^n \sqrt{2\pi} \left[\sum_{i=1}^n K_i^2 \right]^{1/2} \left[\int_0^{2\pi} |x_i'(t)|^2 dt \right]^{1/2} + \left[\sum_{i=1}^n \int_0^{2\pi} |\bar{e}_i(t)|^2 dt \right]^{1/2} \left[\sum_{i=1}^n \int_0^{2\pi} [x_i'(t)]^2 dt \right]^{1/2}. \end{aligned}$$

Hence

$$\|x'\|_{L^2} \leq \left(\frac{1}{\min_{1 \leq i \leq n} C_i} \right) \left[\sqrt{2\pi} \left[\sum_{i=1}^n K_i^2 \right]^{1/2} + \|\bar{e}\|_{L^2} \right] = M_0.$$

By the Sobolev inequality, we have

$$\|\bar{x}\|_\infty \leq \sqrt{\frac{\pi}{6}} M_0 = M_1.$$

Suppose there exist $a = (a_1, a_2, \dots, a_n), b = (b_1, b_2, \dots, b_n)$ in R^n such that $a \leq b$; if $x(t)$ is a solution of $(E_\lambda)(B)$ such that $a \leq \bar{x} \leq b$ and $\|\bar{x}\|_\infty \leq M_1$, then

$$\|x\|_\infty \leq \left[\sum_{i=1}^n [\max(|a_i|, |b_i|)]^2 \right]^{1/2} + M_1.$$

Taking L^2 -inner product by $x''(t)$ on both sides of (E_λ) , we have

$$\begin{aligned} & \sum_{i=1}^n \int_0^{2\pi} [x_i''(t)]^2 dt + \lambda \sum_{i=1}^n \int_0^{2\pi} \frac{\partial^2 F(x)}{\partial x_i^2} x_i'(t) x_i''(t) dt \\ & + \lambda \sum_{i=1}^n \int_0^{2\pi} g_i(x_i(t)) x_i''(t) dt + \lambda \sum_{i=1}^n \int_0^{2\pi} h_i(t, x(t)) x_i''(t) dt \\ & = \lambda \sum_{i=1}^n \int_0^{2\pi} \bar{e}_i(t) x_i''(t) dt. \end{aligned}$$

Since F is a C^2 -function, for each $i = 1, 2, \dots, n$, there exists $i > 0$ such that

$$\left| \frac{\partial^2 F(x)}{\partial x_i^2} \right| \leq D_i,$$

and also since g is continuous, for each $i = 1, 2, \dots, n$, there exists $L_i > 0$ such that

$$|g_i(x_i)| \leq L_i.$$

Hence

$$\begin{aligned} \sum_{i=1}^n \int_0^{2\pi} [x_i''(t)]^2 dt & \leq \left(\max_{1 \leq i \leq n} D_i \right) \left[\sum_{i=1}^n \int_0^{2\pi} |x_i'(t)|^2 dt \right]^{1/2} \left[\sum_{i=1}^n \int_0^{2\pi} |x_i''(t)|^2 dt \right]^{1/2} \\ & + \sqrt{2\pi} \left[\sum_{i=1}^n L_i^2 \right]^{1/2} + \left[\sum_{i=1}^n K_i^2 \right]^{1/2} \left[\sum_{i=1}^n \int_0^{2\pi} |x_i''(t)|^2 dt \right]^{1/2} \\ & + \left[\sum_{i=1}^n \int_0^{2\pi} |\bar{e}_i(t)|^2 dt \right]^{1/2} \left[\sum_{i=1}^n \int_0^{2\pi} |x_i''(t)|^2 dt \right]^{1/2}. \end{aligned}$$

and thus we have

$$\|x''\|_{L^2} \leq \left(\max_{1 \leq i \leq n} D_i \right) M_0 + \sqrt{2\pi} \left[\sum_{i=1}^n L_i^2 \right]^{1/2} + \left[\sum_{i=1}^n K_i^2 \right]^{1/2} + \|\bar{e}\|_{L^2} = M_2.$$

By the Sobolev inequality

$$\|x'\|_\infty \leq \sqrt{\frac{\pi}{6}} M_2$$

for every solution of the problem $(E_\lambda)(B)$ where M_2 depends on a, b, M_0 and M_1 .

3. OPERATOR FORMULATION

Define

$$L: D(L) \subseteq C^1([0, 2\pi], R^n) \rightarrow L^2([0, 2\pi], R^n)$$

by

$$(x_1(t), x_2(t), \dots, x_n(t)) \rightarrow (x_1''(t), x_2''(t), \dots, x_n''(t))$$

where $D(L) = C^2([0, 2\pi], R^n)$. Then $\text{Ker } L = R^2$ and

$$ImL = \left\{ e \in L^2([0, 2\pi], R^n) \mid \int_0^{2\pi} e(t)dt = 0 \right\}.$$

Consider two continuous projections

$$P: C^1([0, 2\pi], R^n) \rightarrow C^1([0, 2\pi], R^n)$$

such that

$$ImP = KerL$$

and

$$Q: L^2([0, 2\pi], R^n) \rightarrow L^2([0, 2\pi], R^n)$$

defined by

$$(Qe)(t) = \frac{1}{2\pi} \int_0^{2\pi} e(t)dt.$$

Then

$$KerQ = ImL, C([0, 2\pi], R^n) = KerL \oplus KerP$$

and $L^2([0, 2\pi], R^n) = ImL \oplus ImQ$ as a topological sum. Since

$$\dim[L^2([0, 2\pi], R^n)/ImL] = \dim[ImQ] = \dim[KerL] = n,$$

L is a Fredholm mapping of index zero and hence there exists an isomorphism $J: ImQ \rightarrow KerL$. The operator L is not bijective but the restriction of L on $DomL \cap KerP$ is one-to-one and onto ImL , so it has its algebraic right inverse K_R and, as well known, it is compact. Define

$$N: C^1([0, 2\pi], R^n) \rightarrow L^2([0, 2\pi], R^n)$$

by

$$x(t) \rightarrow -\frac{d}{dt}[\nabla F(x(t))] - g(x(t)) - h(t, x(t)) + e(t)$$

where $x(t) = (x_1(t), x_2(t), \dots, x_n(t))$. Then N is continuous and maps bounded sets into bounded sets. Let G be any open bounded subset of $C^1([0, 2\pi], R^n)$, then $QN: G \rightarrow L^2([0, 2\pi], R^n)$ is bounded and $K_R(I - Q): \overline{G} \rightarrow L^2([0, 2\pi], R^n)$ is compact and continuous. Hence N is L -compact on G . Now we see $x \in D(L)$ is a solution to the problem $(E_\lambda)(B)$ if and only if

$$Lx = \lambda Nx.$$

4. MAIN RESULTS

THEOREM 4.1. Besides conditions on F, g, e , and $(H_1), (H_2)$, we assume

(H_3) there exists $r = (r_1, r_2, \dots, r_n), s = (s_1, s_2, \dots, s_n), A = (A_1, A_2, \dots, A_n)$ and $B = (B_1, B_2, \dots, B_n)$ in R^n such that $r < s$ and $A \leq B$

$$\frac{1}{2\pi} \int_0^{2\pi} g(r + \bar{x}(t))dt + \frac{1}{2\pi} \int_0^{2\pi} h(t, \bar{x} + \bar{x}(t))dt \leq A$$

and

$$\frac{1}{2\pi} \int_0^{2\pi} g(s + \bar{x}(t))dt + \frac{1}{2\pi} \int_0^{2\pi} h(t, \bar{x} + \bar{x}(t))dt \geq B$$

for every $\bar{x} \in R^n$ such that

$$\|\bar{x}\| \leq \left[\sum_{i=1}^n [\max(|r_i|, |s_i|)^2]^{1/2} \right]$$

and for every $\bar{x} \in C^1([0, 2\pi], \mathbb{R}^n)$ having mean value zero, satisfying the boundary condition (B) and such that

$$\|\bar{x}\|_\infty \leq \sqrt{\frac{\pi}{6}} \left(\frac{1}{\min_{1 \leq i \leq n} C_i} \right) \left[\sqrt{2\pi} \left[\sum_{i=1}^n K_i^2 \right]^{1/2} + \|\bar{e}\|_{L^2} \right].$$

Then (E)(B) has at least one solution if

$$A < \frac{1}{2\pi} \int_0^{2\pi} e(t) dt < B.$$

PROOF. We construct a bounded open set Ω in $C^1([0, 2\pi], \mathbb{R}^n)$ to apply Mawhin's continuation theorem in [2]. Using a'priori estimate, we have

$$\|x'\|_{L^2} \leq \left(\frac{1}{\min_{1 \leq i \leq n} C_i} \right) \left[\sqrt{2\pi} \left[\sum_{i=1}^n K_i^2 \right]^{1/2} + \|\bar{e}\|_{L^2} \right] = M_0$$

for any solution $x(t)$ of $(E_\lambda)(B)$, $\lambda \in (0, 1)$. Hence $\|\bar{x}\|_\infty \leq \sqrt{\frac{\pi}{6}} M_0 = M_1$. Define a bounded set Ω^0 by

$$\Omega^0 = \{x \in C^1([0, 2\pi], \mathbb{R}^n) \mid r \leq \bar{x} \leq s, \|\bar{x}\|_\infty \leq M_1\}.$$

Then, for any solution $x(t)$ of $(E_\lambda)(B)$ lying in Ω^0 , we have

$$\|x\|_\infty \leq \left[\sum_{i=1}^n [\max(|r_i|, |s_i|)]^2 \right]^{1/2} + M_1$$

and

$$\|x'\|_{L^2} \leq \left(\max_{1 \leq i \leq n} D_i \right) M_0 + \sqrt{2\pi} \left[\sum_{i=1}^n L_i^2 \right]^{1/2} + \left[\sum_{i=1}^n K_i^2 \right]^{1/2} + \|\bar{e}\|_{L^2} = M_2,$$

where L_i depends on r, s and M_1 . Thus $\|x'\|_\infty \leq \sqrt{\frac{\pi}{6}} M_2$. Define a bounded open set Ω by

$$\Omega = \left\{ x \in C^1([0, 2\pi], \mathbb{R}^n) \mid r < \bar{x} < s, \|\bar{x}\|_\infty < 2M_1, \|x'\|_\infty < \sqrt{\frac{2\pi}{6}} M_2 \right\}.$$

Let $(x, \lambda) \in [D(L) \cap \partial\Omega] \times (0, 1)$ and if (x, λ) is any solution to $Lx = \lambda Nx$, then (x, λ) is a solution to the problem $(E_\lambda)(B)$,

$$\|\bar{x}\| \leq \left[\sum_{i=1}^n [\max(|r_i|, |s_i|)]^2 \right]^{1/2}, \quad \|\bar{x}\| \leq M_1$$

and there exists some $i \in \{1, 2, \dots, n\}$ such that $\bar{x}_i = r_i$ or s_i . Take L^2 -inner product with $e_i = (0, 0, \dots, 0, 1, 0, \dots, 0)$ on both sides of (E_λ) , we have

$$\lambda \int_0^{2\pi} g_i(x_i(t)) dt + \lambda \int_0^{2\pi} h_i(t, x(t)) dt = \lambda \int_0^{2\pi} e_i(t) dt,$$

or

$$\int_0^{2\pi} g_i(x_i(t)) dt + \int_0^{2\pi} h_i(t, x(t)) dt - \int_0^{2\pi} e_i(t) dt = 0$$

if $\bar{x}_i = r_i$, then, by assumption

$$\int_0^{2\pi} g_i(r_i + \bar{x}_i(t)) dt + \int_0^{2\pi} h_i(t, \bar{x}_1 + \bar{x}_i(t), \dots, r_i + \bar{x}_i(t), \dots, \bar{x}_n + \bar{x}_i(t)) dt - \int_0^{2\pi} e_i(t) dt < 0.$$

If $\bar{x}_i = s_i$, then again by assumption,

$$\int_0^{2\pi} g_i(s_i + \bar{x}_i(t))dt + \int_0^{2\pi} h_i(t, \bar{x}_1 + \bar{x}_1(t), \dots, s_i + \bar{x}_i(t), \dots, \bar{x}_n + \bar{x}_n(t))dt - \int_0^{2\pi} e_i(t)dt < 0.$$

Thus, for each $\lambda \in (0, 1)$, for every solution of

$$Lx = \lambda Nx$$

is such that $x \notin \partial\Omega$.

Next, we will show that $QNx \neq 0$ for each $x \in KerL \cap \partial\Omega$ and $d_B[JQN, \Omega \cap KerL, 0] \neq 0$ where d_B is the Brouwer topological degree. Since $J: ImQ \rightarrow KerL$ is an isomorphism and $\dim[ImQ] = \dim[KerL] = n$, we may take J to be the identity on R^n and hence

$$(JQN)(x)(t) = -\frac{1}{2\pi} \int_0^{2\pi} g(x(t))dt - \frac{1}{2\pi} \int_0^{2\pi} h(t, x(t))dt + \frac{1}{2\pi} \int_0^{2\pi} e(t)dt$$

with, for $i = 1, 2, \dots, n$,

$$(JQN)_i(x)(t) = -\frac{1}{2\pi} \int_0^{2\pi} g_i(x_i(t))dt - \frac{1}{2\pi} \int_0^{2\pi} h_i(t, x_i(t))dt + \frac{1}{2\pi} \int_0^{2\pi} e_i(t)dt$$

where $x(t) = (x_1(t), x_2(t), \dots, x_n(t))$.

Let $x \in KerL \cap \partial\Omega$, then $x = \bar{x}$ is constant in R^n ,

$$\|\bar{x}\| \leq \left[\sum_{i=1}^n [\max(|r_i|, |s_i|)]^2 \right]^{1/2},$$

and there exists $i \in \{1, 2, \dots, n\}$ such that $x_i = \bar{x}_i = r_i$ or s_i . In a similar manner we have $(QN)_i(x) \neq 0$.

Thus $QNx \neq 0$ for each $x \in KerL \cap \partial\Omega$. It is easy to see that $P = \overline{\Omega \cap KerL} = \prod_{i=1}^n [r_i, s_i]$. Let $P_i = \{x \in P \mid x_i = r_i\}$, $P'_i = \{x \in P \mid x_i = s_i\}$ and $x \in P_i, x' \in P'_i, i = 1, 2, \dots, n$.

Then $x = \bar{x}, x' = \bar{x}'$ are constant with

$$\|\bar{x}\|, \quad \text{and} \quad \|\bar{x}'\| \leq \left[\sum_{i=1}^n [\max(|r_i|, |s_i|)]^2 \right]^{1/2},$$

and $x_i = \bar{x}_i = r_i, x'_i = \bar{x}'_i = s_i$. Hence

$$(JQN)_i(x) = -\frac{1}{2\pi} \int_0^{2\pi} g_i(r_i)dt - \frac{1}{2\pi} \int_0^{2\pi} h_i(t, x_i, \dots, r_i, \dots, x_n)dt + \frac{1}{2\pi} \int_0^{2\pi} e_i(t)dt > 0$$

and

$$(JQN)_i(x') = -\frac{1}{2\pi} \int_0^{2\pi} g_i(s_i)dt - \frac{1}{2\pi} \int_0^{2\pi} h_i(t, x'_i, \dots, s_i, \dots, x'_n)dt + \frac{1}{2\pi} \int_0^{2\pi} e_i(t)dt < 0.$$

Thus $(JQN)_i(x)(JQN)_i(x') < 0$ for $i = 1, 2, \dots, n$. Therefore, by the generalized intermediate value theorem, $d_B[JQN, \Omega \cap KerL, 0] \neq 0$. Hence, by Mawhin's continuation theorem, the problem $(E)(B)$ has at least one solution in $D(L) \cap \overline{\Omega}$.

THEOREM 4.2. Besides conditions on F, g, e , and (H_1) and (H_2) , we assume

(H_4) there exists $q = (q_1, q_2, \dots, q_n)$, $r = (r_1, r_2, \dots, r_n)$, $s = (s_1, s_2, \dots, s_n)$, $A = (A_1, A_2, \dots, A_n)$ and $B = (B_1, B_2, \dots, B_n)$ in R^n such that $q < r < s$ and $A \leq B$ such that

$$\frac{1}{2\pi} \int_0^{2\pi} g(q + \bar{x}(t))dt + \frac{1}{2\pi} \int_0^{2\pi} h(t, \bar{x} + \bar{x}(t))dt \geq B,$$

$$\frac{1}{2\pi} \int_0^{2\pi} g(r + \bar{x}(t))dt + \frac{1}{2\pi} \int_0^{2\pi} h(t, \bar{x} + \bar{x}(t))dt \leq A,$$

and

$$\frac{1}{2\pi} \int_0^{2\pi} g(s + \bar{x}(t))dt + \frac{1}{2\pi} \int_0^{2\pi} h(t, \bar{x} + \bar{x}(t))dt \geq B$$

for every $\bar{x} \in R^n$ such that

$$\|\bar{x}\| \leq \left[\sum_{i=1}^n \max(|q_i|, |r_i|, |s_i|)^2 \right]^{1/2}$$

and for every $\bar{x} \in C^1([0, 2\pi], R^n)$ having mean value zero, satisfying the boundary condition (B) such that

$$\|\bar{x}\|_\infty \leq \sqrt{\frac{\pi}{6}} \left(\frac{1}{\min_{1 \leq i \leq n} C_i} \right) \left[\sqrt{2\pi} \left[\sum_{i=1}^n K_i^2 \right]^{1/2} + \|\bar{\epsilon}\|_{L^2} \right]$$

Then (E)(B) has at least 2^n solutions if

$$A < 1/2\pi \int_0^{2\pi} e(t)dt < B.$$

PROOF. We construct 2^n bounded open sets in $C^1([0, 2\pi], R^n)$ to apply Mawhin's continuation theorem in [3]. Using a priori estimate, we have

$$\|x'\|_{L^2} \leq \left(\frac{1}{\min_{1 \leq i \leq n} C_i} \right) \left[\sqrt{2\pi} \left[\sum_{i=1}^n K_i^2 \right]^{1/2} + \|\bar{\epsilon}\|_{L^2} \right] = M_0$$

for any solution $x(t)$ of $(E_\lambda)(B)$, $\lambda \in (0, 1)$. Hence $\|\bar{x}\|_\infty \leq \sqrt{\frac{\pi}{6}} M_0 = M_1$. Let I, J be two disjoint subsets of $\{1, 2, \dots, n\}$ such that $I \cup J = \{1, 2, \dots, n\}$ and define Ω_{IJ}^0 by $\Omega_{IJ}^0 = \{x \in C^1([0, 2\pi], R^n) \mid q_i \leq \bar{x}_i \leq r_i \text{ for } i \in I, r_j \leq \bar{x}_j \leq s_j \text{ for } j \in J, \|\bar{x}\|_\infty \leq M_1\}$; then the number of such sets is 2^n and for any solution, $x(t)$ of $(E_\lambda)(B)$ lying in Ω_{IJ}^0 , we have

$$\|x'\|_\infty \leq \left[\sum_{i \in I} [\max(|q_i|, |r_i|)]^2 + \sum_{j \in J} [\max(|r_j|, |s_j|)]^2 \right]^{1/2} + M_1$$

and

$$\|x''\|_{L^2} \leq \left(\max_{1 \leq i \leq n} D_i \right) M_0 + \sqrt{2\pi} \left[\sum_{i=1}^n L_i^2 \right]^{1/2} + \left[\sum_{i=1}^n K_i^2 \right]^{1/2} + \|\bar{\epsilon}\|_{L^2} = M_2$$

where L_i depends on q, r, s and M_1 . Thus $\|x'\|_\infty \leq \sqrt{\frac{\pi}{6}} M_2$. Define a bounded open set Ω_{IJ} by

$$\Omega_{IJ} = \{x \in C^1([0, 2\pi], R^n) \mid q_i \leq \bar{x}_i \leq r_i \text{ for } i \in I, r_j < \bar{x}_j < s_j$$

$$\text{for } j \in J, \|\bar{x}\|_\infty < 2M_1, \|x''\|_\infty < \sqrt{\frac{2\pi}{3}} M_2.$$

Let $(x, \lambda) \in [D(L) \cap \partial\Omega_{IJ}] \times (0, 1)$ and if (x, λ) is any solution to

$$Lx = \lambda Nx,$$

then (x, λ) is a solution to the problem $(E_\lambda)(B)$,

$$\|\bar{x}\| \leq \left[\sum_{i \in I} [\max(|q_i|, |r_i|)]^2 + \sum_{j \in J} [\max(|r_j|, |s_j|)]^2 \right]^{1/2}, \|\bar{x}\| \leq M_1$$

and there exists some $i \in \{1, 2, \dots, n\}$, such that $\bar{x}_i = q_i, r_i$ or s_i . By (H_4) and assumption we can see for each $\lambda \in (0, 1)$, for every solution of $Lx = \lambda Nx$ is such that $x \notin \partial\Omega_{IJ}$. And similarly, we can also see $QNx \neq 0$ for each $x \in KerL \cap \partial\Omega_{IJ}$. It is easy to see $P = \Omega_{IJ} \cap KerL = \Pi_{i \in I} [q_i, r_i] \times \Pi_{j \in J} [r_j, s_j]$. Let

$$\begin{aligned}
P_i &= \{x \in p \mid x_i = q_i\} \quad \text{if } i \in I, \\
P_j &= \{x \in p \mid x_j = r_j\} \quad \text{if } j \in J, \\
P'_i &= \{x \in p \mid x_i = r_i\} \quad \text{if } i \in I, \\
P'_j &= \{x \in p \mid x_j = s_j\} \quad \text{if } j \in J,
\end{aligned}$$

and let $x \in P_i$, $x' \in P'_i$ with $i \in I \cup J$. Then, for $i \in I$, we have $x_i = q_i$, $x_i = r_i$. Hence $(JQN)_i(x)(JQN)_i(x') < 0$ for $i \in I$. For $j \in J$, we have $x_j = r_j$, $x'_j = s_j$. Thus $(JQN)_j(x)(JQN)_j(x') < 0$ for $j \in J$. Therefore, we have $d_B[JQN, \Omega_{IJ} \cap \text{Ker } L, 0] \neq 0$. Thus, by Mawhin's continuation theorem, the problem $(E_\lambda)(B)$ has at least one solution in $D(L) \cap \bar{\Omega}_{IJ}$. Thus $(E_\lambda)(B)$ has at least 2^n solutions.

Corollary 4.3. Besides the conditions on F , g and e , and (H_1) and (H_2) , we assume

(H_5) there exists $T = (T_1, T_2, \dots, T_n) > 0$ in R^n such that

$$g(T+x) = g(x) \quad \text{and} \quad h(t, T+x) = h(t, x)$$

for all $(t, x) \in [0, 2\pi] \times R^n$.

(H_6) there exists $r = (r_1, r_2, \dots, r_n)$, $s = (s_1, s_2, \dots, s_n)$, $A = (A_1, A_2, \dots, A_n)$ and $B = (B_1, B_2, \dots, B_n)$ in R^n such that $0 < s - r < T$, $r < s$, $A \leq B$

$$\begin{aligned}
\frac{1}{2\pi} \int_0^{2\pi} g(r + \bar{x}(t)) dt + \frac{1}{2\pi} \int_0^{2\pi} h(t, \bar{x} + \bar{x}(t)) dt &\leq A, \\
\frac{1}{2\pi} \int_0^{2\pi} g(s + \bar{x}(t)) dt + \frac{1}{2\pi} \int_0^{2\pi} h(t, \bar{x} + \bar{x}(t)) dt &\geq B
\end{aligned}$$

for every $\bar{x} \in R^n$ such that

$$\|\bar{x}\| \left[\sum_{i=1}^n [\max(|s_i - T_i|, |r_i|, |s_i|)]^2 \right]^{1/2}$$

and for every $\bar{x} \in C^1([0, 2\pi], R^n)$ having mean value zero, satisfying the boundary condition (B) and such that

$$\|\bar{x}\|_\infty \leq \sqrt{\frac{\pi}{6}} \left(\frac{1}{\min_{1 \leq i \leq n} C_i} \right) \left[\sqrt{2\pi} \left[\sum_{i=1}^n K_i^2 \right]^{1/2} + \|\bar{e}\|_{L^2} \right].$$

Then $(E)(B)$ has at least 2^n solutions if

$$A < \frac{1}{2\pi} \int_0^{2\pi} e(t) dt < B.$$

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