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ON THE SPECTRUM OF THE NEGATIVE LAPLACIAN FOR GENERAL DOUBLY-CONNECTED BOUNDED DOMAINS

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ABSTRACT. This paper is devoted to asymptotic formulas for functions related with the spectrum of the standard Laplace operator in two and three dimensional bounded doubly connected domains with impedance boundary conditions, where the impedances are assumed to be positive functions. Moreover, asymptotic expressions for the difference of eigenvalues related to impedance boundary value problems with different impedances are derived. Further results may be obtained.

KEY WORDS AND PHRASES: Eigenvalues of the negative Laplacian, doubly connected domains, impedance eigenvalue problem, asymptotic expansions of the heat kernel.
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1. INTRODUCTION

The underlying problem is to deduce the geometrical properties of a membrane from a complete knowledge of the eigenvalues $\{\mu_k(\sigma)\}_{k=1}^{\infty}$ for the negative Laplacian $-\Delta_n = -\sum_{i=1}^{n}\left(\frac{\partial}{\partial x^i}\right)^2$ in R^n, $n = 2$ or 3.

Let Ω be a simply connected bounded domain in R^n with a smooth boundary $\partial\Omega$ in the case $n = 2$, or a smooth bounding surface S in the case $n = 3$. Consider the impedance problem

$$(\Delta_n + \lambda)u = 0 \quad \text{in} \quad \Omega, \tag{1.1}$$

$$\left(\frac{\partial}{\partial n} + \sigma\right)u = 0 \quad \text{on} \quad \partial\Omega(\text{or } S), \tag{1.2}$$

where $\frac{\partial}{\partial n}$ denotes differentiation along the inward-pointing normal to $\partial\Omega$ (or S), and σ is a positive function. Denote its eigenvalues, counted according to multiplicity, by

$$0 < \mu_1(\sigma) \le \mu_2(\sigma) \le \ldots \le \mu_k(\sigma) \le \ldots \to \infty \quad \text{as} \quad k \to \infty. \tag{1.3}$$

At the beginning of this century the principal problem was that of investigating the asymptotic distribution of the eigenvalues (1.3). It is well known [1] that in the case $n = 2$

$$\mu_k(\sigma) \sim \left(\frac{4\pi}{|\Omega|}\right)k \quad \text{as} \quad k \to \infty, \tag{1.4}$$

while in the case $n = 3$

$$\mu_k(\sigma) \sim \left(\frac{6\pi^2}{V}k\right)^{2/3} \quad \text{as} \quad k \to \infty, \tag{1.5}$$

where $|\Omega|$ and V are respectively the area and the volume of the domain Ω. The problem of determining further information about the geometry of Ω has been discussed by many authors, see

for example Pleijel [2,3], Kac [4], McKean and Singer [5], Stewartson and Waechter [6], Waechter [7], Greiner [8], Smith [9], Gottlieb [10-12], Hsu [13], Sleeman and Zayed [14,15] and Zayed [16-23], using the asymptotic behavior of the spectral function

$$\Theta(t) = \sum_{k=1}^{\infty} \exp[-t\mu_k(\sigma)] \quad \text{as} \quad t \to 0. \tag{1.6}$$

Thus, if $\sigma = 0$ (Neumann problem), it is well known that in the case $n = 2$

$$\Theta(t) = \frac{|\Omega|}{4\pi t} + \frac{|\partial\Omega|}{8(\pi t)^{1/2}} + a_0 + \frac{7}{256}\left(\frac{t}{\pi}\right)^{1/2} \int_{\partial\Omega} K^2(Q)dQ + 0(t) \quad \text{as} \quad t \to 0, \tag{1.7}$$

while in the case $n = 3$

$$\Theta(t) = \frac{V}{(4\pi t)^{3/2}} + \frac{|S|}{16\pi t} + \frac{1}{12\pi^{3/2}t^{1}/2} \int_S H(Q)dQ + \frac{7}{128\pi} \int_S [H^2(Q) - N(Q)]dQ + 0(t^{1/2}) \text{ as } t \to 0. \tag{1.8}$$

If $\sigma = \infty$ (Dirichlet problem), it is well known that in the case $n = 2$

$$\Theta(t) = \frac{|\Omega|}{4\pi t} - \frac{|\partial\Omega|}{8(\pi t)^{1/2}} + a_0 + \frac{1}{256}\left(\frac{t}{\pi}\right)^{1/2} \int_{\partial\Omega} K^2(Q)dQ + 0(t) \quad \text{as} \quad t \to 0, \tag{1.9}$$

while in the case $n = 3$

$$\Theta(t) = \frac{V}{(4\pi t)^{3/2}} - \frac{|S|}{16\pi t} + \frac{1}{12\pi^{3/2}t^{1}/2} \int_S H(Q)dQ + \frac{1}{128\pi} \int_S [H^2(Q) - N(Q)]dQ + 0(t^{1/2}) \text{ as } t \to 0. \tag{1.10}$$

An examination of the results (1.7) and (1.9) shows that in the case $n = 2$ the first term of $\Theta(t)$ determines the area $|\Omega|$ of Ω, the second term determines the total length $|\partial\Omega|$ of the boundary $\partial\Omega$ and the fourth term determines the curvature $K(Q)$ of $\partial\Omega$ at the point $Q \varepsilon \partial\Omega$ while the sign $\pm$ of the second term determines whether we have a Neumann or a Dirichlet problem. The third term a_0 in (1.7) and (1.9) has geometric significance, e.g., if Ω is smooth and convex, then $a_0 = \frac{1}{6}$ while if Ω is permitted to have a finite number of smooth convex holes "h", then $a_0 = (1-h)\frac{1}{6}$.

Similarly, an examination of the results (1.8) and (1.10) shows that in the case $n = 3$ the first term of $\Theta(t)$ determines the volume V of Ω, the second term determines the surface area $|S|$ of S, the third term determines the mean curvature $H(Q) = \frac{1}{2}\left[\frac{1}{R_1(Q)} + \frac{1}{R_2(Q)}\right]$ and the fourth term determines the Gaussian curvature $N(Q) = \frac{1}{R_1(Q)R_2(Q)}$ of the surface S at the point $Q \varepsilon S$, where R_1 and R_2 are the principal radii of curvature, while the sign $\pm$ of the second term of $\Theta(t)$ determines whether we have a Neumann or a Dirichlet problem.

We merely note that aspects of the question of Kac, namely, "Can one hear the shape of a drum?" have been discussed by Sleeman and Zayed [14] when $n = 2$ and by Zayed [16] when $n = 3$ for problem (1.1)-(1.2) in the case σ is a positive constant.

Suppose that Ω is a general doubly domain bounded in R^n, $n = 2$ or 3 consisting of a simply connected bounded inner domain Ω_1 with a smooth boundary $\partial\Omega_1$ in the case $n = 2$ (or a smooth bounding surface S_1 in the case $n = 3$) and a simply connected bounded outer domain $\Omega_2 \supset \overline{\Omega}_1$ with a smooth boundary $\partial\Omega_2$ in the case $n = 2$ (or a smooth bounding surface S_2 in the case $n = 3$). Consider the impedance problem

$$(\Delta_n + \lambda)u = 0 \quad \text{in} \quad \Omega, \tag{1.11}$$

$$\left(\frac{\partial}{\partial n_1} + \sigma_1\right)u = 0 \quad \text{on} \quad \partial\Omega_1 \text{ (or } S_1), \tag{1.12}$$

and

$$\left(\frac{\partial}{\partial n_2}+\sigma_2\right)u=0 \quad \text{on} \quad \partial\Omega_2 \ (\text{or } S_2), \tag{1.13}$$

where $\frac{\partial}{\partial n_1}$ and $\frac{\partial}{\partial n_2}$ denote differentiations along the inward-pointing normals to $\partial\Omega_1$ (or S_1) and $\partial\Omega_2$ (or S_2) respectively, in which the impedances σ_1 and σ_2 are positive functions.

Denote its eigenvalues, counted according to multiplicity, by

$$0<\mu_1(\sigma_1,\sigma_2)\le\mu_2(\sigma_1,\sigma_2)\le\ldots\le\mu_k(\sigma_1,\sigma_2)\le\ldots\to\infty \quad \text{as} \quad k\to\infty. \tag{1.14}$$

The problem of determining the geometry of Ω as well as the impedances σ_1 and σ_2 from a complete knowledge of the eigenvalues (1.14) has been discussed by Zayed [21] in the case $n=2$ and by Zayed [22] in the case $n=3$ where σ_1 and σ_2 are positive constants, using the asymptotic expansion of the spectral function

$$\Theta(t)=\sum_{k=1}^{\infty}\exp[-t\mu_k(\sigma_1,\sigma_2)] \quad \text{as} \quad t\to 0. \tag{1.15}$$

Thus in the case $n=2$, Zayed [21,23] has shown that

$$\Theta(t)=\frac{|\Omega|}{4\pi t}+\frac{|\partial\Omega_1|+|\partial\Omega_2|}{8(\pi t)^{1/2}}-\frac{1}{2\pi}(\sigma_2|\partial\Omega_2|-\sigma_1|\partial\Omega_1|)$$

$$+\frac{7}{256}\left(\frac{t}{\pi}\right)^{1/2}\sum_{i=1}^{2}\int_{\partial\Omega_i}\left[K_i^2(Q)-\frac{64}{7}\left(\frac{\pi\sigma_i}{|\partial\Omega_i|}-\sigma_i^2\right)\right]dQ+0(t) \quad \text{as} \quad t\to 0, \tag{1.16}$$

where $|\partial\Omega_1|$ and $K_1(Q),(Q\varepsilon\partial\Omega_1)$ are respectively the total length and the curvature of $\partial\Omega_1$ while $|\partial\Omega_2|$ and $K_2(Q),(Q\varepsilon\partial\Omega_2)$ are respectively the total length and the curvature of $\partial\Omega_2$.

In the case $n=3$, Zayed [22] has shown that

$$\Theta(t)=\frac{V}{(4\pi t)^{3/2}}+\frac{|S_1|+|S_2|}{16\pi t}+\frac{1}{12\pi^{3/2}t^{1/2}}\sum_{i=1}^{2}\int_{S_i}[H_i(Q)-3\sigma_i]dQ$$

$$+\frac{7}{128\pi}\sum_{i=1}^{2}\int_{S_i}\left\{[H_i(Q)-3\sigma_i]^2-\left[N_i(Q)-\frac{26}{7}\sigma_iH_i(Q)+\frac{47}{7}\sigma_i^2\right]\right\}dQ+0(t^{1/2}) \quad \text{as} \quad t\to 0, \tag{1.17}$$

where $|S_1|, H_1(Q)$ and $N_1(Q),(Q\varepsilon S_1)$ are respectively the surface area, mean curvature and Gaussian curvature of the surface S_1, while $|S_2|, H_2(Q)$ and $N_2(Q),(Q\varepsilon S_2)$ are respectively the surface area, mean curvature and Gaussian curvature of the surface S_2. Further interpretations of formulae (1.16) and (1.17) can be found in Zayed [21-23].

In Theorem 1, we generalize the results (1.16) and (1.17) to the case when σ_1 and σ_2 are positive functions satisfying the Lipschitz condition, by using the expression

$$\sum_{k=1}^{\infty}\{\mu_k(\sigma_1,\sigma_2)+P\}^{-2}, \tag{1.18}$$

where P is a positive constant.

In Theorem 2, we show that this generalization plays an important role in establishing a method to study the asymptotic behavior of the difference

$$\sum_{\mu_k(\sigma_1,\sigma_2)\le\lambda}\{\mu_k(\alpha_2,\beta_2)-\mu_k(\alpha_1,\beta_1)\}, \tag{1.19}$$

for large values of λ, where the three pairs of functions (σ_1,σ_2), (α_1,β_1) and (α_2,β_2) are distinct and satisfying the Lipschitz condition and the summation is taken over all values of k for which $\mu_k(\sigma_1,\sigma_2)\le\lambda$. The method uses an interesting and important Tauberian theorem due to Hardy and Littlewood and developed by Titchmarsh [24].

Theorems 3, 4 and Corollaries 1-5 contain further results which can be considered as a generalization of the results of Theorem 2.

2. STATEMENT AND PROOFS OF RESULTS

THEOREM 1. If the functions $\sigma_1(Q), Q \varepsilon \partial\Omega_1$ (or S_1) and $\sigma_2(Q), Q \varepsilon \partial\Omega_2$ (or S_2) satisfy the Lipschitz condition and if P is a positive constant, then in the case $n = 2$

$$\sum_{k=1}^{\infty} \{\mu_k(\sigma_1, \sigma_2) + P\}^{-2} = \frac{|\Omega|}{4\pi P} + \frac{|\partial\Omega_1| + |\partial\Omega_2|}{16P^{3/2}} - \frac{1}{2\pi P^2}\left\{\int_{\partial\Omega_2} \sigma_2(Q)dQ - \int_{\partial\Omega_1} \sigma_1(Q)dQ\right\}$$

$$+ \frac{21}{1024P^{5/2}} \sum_{i=1}^{2} \int_{\partial\Omega_i} \{K_i^2(Q) - \frac{32}{7}[\sigma_i(Q)K_i(Q)$$

$$-2\sigma_i^2(Q)]\}dQ + O\left(\frac{1}{P^3}\right) \quad \text{as} \quad P \to \infty, \tag{2.1}$$

while in the case $n = 3$

$$\sum_{k=1}^{\infty} \{\mu_k(\sigma_1, \sigma_2) + P\}^{-2} = \frac{V}{8\pi P^{1/2}} + \frac{|S_1| + |S_2|}{16\pi P} + \frac{1}{24\pi P^{3/2}} \sum_{i=1}^{2} \int_{S_i} [H_i(Q) - 3\sigma_i(Q)]dQ$$

$$+ \frac{7}{128\pi P^2} \sum_{i=1}^{2} \int_{S_i} \{[H_i(Q) - 3\sigma_i(Q)]^2 - [N_i(Q)$$

$$-\frac{26}{7}\sigma_i(Q)H_i(Q) + \frac{47}{7}\sigma_i^2(Q)]\}dQ + O\left(\frac{1}{P^{5/2}}\right) \quad \text{as} \quad P \to \infty. \tag{2.2}$$

Note that the expression (1.18) is just the Laplace transform of the function $t\Theta(t)$ with respect to t and $P > 0$ is the Laplace transform parameter. With this connection we deduce that formulae (2.1) and (2.2) can be considered as a generalization of formulae (1.16) and (1.17) respectively.

THEOREM 2. If the three pairs of functions $(\sigma_1(Q), \sigma_2(Q))$, $(\alpha_1(Q), \beta_1(Q))$ and $(\alpha_2(Q), \beta_2(Q))$ are distinct and satisfying the Lipschitz condition, then we deduce for $\lambda \to \infty$ that

$$\sum_{\mu_k(\sigma_1,\sigma_2) \le \lambda} \{\mu_k(\alpha_2, \beta_2) - \mu_k(\alpha_1, \beta_1)\} = \begin{cases} \frac{a_1}{1\pi}\lambda + o(\lambda) & \text{in the case } n = 2, \quad (2.3) \\ \frac{b_1}{3\pi^2}\lambda^{3/2} + o(\lambda^{3/2}) & \text{in the case } n = 3, \quad (2.4) \end{cases}$$

where

$$a_1 = \int_{\partial\Omega_2} [\beta_2(Q) - \beta_1(Q)]dQ - \int_{\partial\Omega_1} [\alpha_2(Q) - \alpha_1(Q)]dQ,$$

and

$$b_1 = \int_{S_2} [\beta_2(Q) - \beta_1(Q)]dQ + \int_{S_1} [\alpha_2(Q) - \alpha_1(Q)]dQ.$$

Formulae (2.3) and (2.4) can be considered as a generalization of the familiar formulae of Gel'fand and Levitan [25] for the difference of traces of two Sturm-Liouville operators.

Let us now give the proofs of Theorems 1,2. To prove Theorem 1, we shall use the Laplace transform of Green's function for the heat equation $\left(\Delta_n - \frac{\partial}{\partial t}\right)u = 0, n = 2$ or 3 with respect to the time t, and use s^2 as the Laplace transform parameter.

PROOF OF THEOREM 1. With reference to [26, Sec. 2], let

$$\overline{G}\left(\underset{\sim}{x}, \underset{\sim}{x}_1; -s^2\right) = \frac{1}{2\pi} K_0\left(s\left|\underset{\sim}{x} - \underset{\sim}{x}_1\right|\right) - \overline{g}\left(\underset{\sim}{x}, \underset{\sim}{x}_1; -s^2\right), \tag{2.5}$$

be the Green's function of the expression $(\Delta_2 - s^2)u$ in the domain $\Omega \subseteq R^2$ together with the boundary conditions (1.12) and (1.13) on $\partial\Omega_2$ respectively, where s is a sufficiently large positive constant while $\underset{\sim}{x}$ and $\underset{\sim}{x}_1$ are points belong to Ω. In (2.5), K_0 is the modified Bessel function of the second kind and of zero order, while $\bar{g}$ is a regular part of the Green's function.

With reference to [2], we deduce that as $x_1 \to x$ the equality

$$\bar{G}\left(\underset{\sim}{x}, \underset{\sim}{x}_1; -s_1^2\right) - \bar{G}\left(\underset{\sim}{x}, \underset{\sim}{x}_1; -s^2\right) = (s^2 - s_1^2) \sum_{k=1}^{\infty} \frac{\phi_k\left(\underset{\sim}{x}\right)\phi_k\left(\underset{\sim}{x}_1\right)}{\{\mu_k(\sigma_1, \sigma_2) + s_1^2\}\{\mu_k(\sigma_1, \sigma_2) + s^2\}}, \tag{2.6}$$

where $\left\{\phi_k\left(\underset{\sim}{x}\right)\right\}$ are normalized eigenfunctions and $s \neq s_1$, implies

$$\frac{1}{2\pi}\log\left(\frac{s_1}{s}\right) + \bar{g}\left(\underset{\sim}{x}, \underset{\sim}{x}; -s_1^2\right) - \bar{g}\left(\underset{\sim}{x}, \underset{\sim}{x}; -s^2\right) = (s_1^2 - s^2) \sum_{k=1}^{\infty} \frac{\phi_k^2\left(\underset{\sim}{x}\right)}{\{\mu_k(\sigma_1, \sigma_2) + s_1^2\}\{\mu_k(\sigma_1, \sigma_2) + s^2\}}. \tag{2.7}$$

Thus we get the formula

$$\sum_{k=1}^{\infty} \{\mu_k(\sigma_1, \sigma_2) + s^2\}^{-2} = \frac{|\Omega|}{4\pi s^2} + \frac{1}{2s} \int\int_{\Omega} \bar{g}_s'\left(\underset{\sim}{x}, \underset{\sim}{x}; -s^2\right) dx. \tag{2.8}$$

Using methods similar to those obtained in [14], [21], [23] we can show that

$$\begin{aligned}\int\int_{\Omega} \bar{g}_s'\left(\underset{\sim}{x}, \underset{\sim}{x}; -s^2\right) dx = {} & \frac{|\partial\Omega_1| + |\partial\Omega_2|}{8s^2} - \frac{1}{\pi s^3}\left\{\int_{\partial\Omega_2} \sigma_2(Q)dQ - \int_{\partial\Omega_1} \sigma_1(Q)dQ\right\} \\ & + \frac{21}{512s^4} \sum_{i=1}^{2} \int_{\partial\Omega_i} \left\{K_i^2(Q) - \frac{32}{7}[\sigma_i(Q)K_i(Q) - 2\sigma_i^2(Q)]\right\} dQ \\ & + 0\left(\frac{1}{s^5}\right) \quad \text{as} \quad s \to \infty. \end{aligned} \tag{2.9}$$

On inserting (2.9) into (2.8) and letting $s^2 = P$ we arrive at (2.1).

Similarly, let

$$\bar{G}\left(\underset{\sim}{x}, \underset{\sim}{x}_1; -s^2\right) = \frac{\exp\left(-s\left|\underset{\sim}{x} - \underset{\sim}{x}_1\right|\right)}{4\pi\left|\underset{\sim}{x} - \underset{\sim}{x}_1\right|} - \bar{g}\left(\underset{\sim}{x}, \underset{\sim}{x}_1; -s^2\right), \tag{2.10}$$

be the Green's function of the expression $(\Delta_3 - s^2)u$ in the domain $\Omega \subseteq R^3$ together with the boundary conditions (1.12) and (1.13) on S_1 and S_2 respectively.

With reference to [3], we deduce that as $\underset{\sim}{x}_1 \to \underset{\sim}{x}$ the equality (2.6) implies

$$\frac{(s_1 - s)}{4\pi} + \bar{g}\left(\underset{\sim}{x}, \underset{\sim}{x}; -s_1^2\right) - \bar{g}\left(\underset{\sim}{x}, \underset{\sim}{x}; -s^2\right) = (s_1^2 - s^2) \sum_{k=1}^{\infty} \frac{\phi_k^2\left(\underset{\sim}{x}\right)}{\{\mu_k(\sigma_1, \sigma_2) + s_1^2\}\{\mu_k(\sigma_1, \sigma_2) + s^2\}}. \tag{2.11}$$

Thus we get the formula

$$\sum_{k=1}^{\infty} \{\mu_k(\sigma_1, \sigma_2) + s^2\}^{-2} = \frac{V}{8\pi s} + \frac{1}{2s} \int\int\int_{\Omega} \bar{g}_s'\left(\underset{\sim}{x}, \underset{\sim}{x}; -s^2\right) d\underset{\sim}{x}. \tag{2.12}$$

Using methods similar to those obtained in [16], [22], we can show that

$$\int\int\int_{\Omega} \overline{g}_s'(x,x;-s^2)dx = \frac{|S_1|+|S_2|}{8\pi s} + \frac{1}{12\pi s^2}\sum_{i=1}^{2}\int_{S_i}[H_i(Q)-3\sigma_i(Q)]dQ$$

$$+\frac{7}{64\pi s^3}\sum_{i=1}^{2}\int_{S_i}\{[H_i(Q)-3\sigma_i(Q)]^2-[N_i(Q)-\frac{26}{7}\sigma_i(Q)H_i(Q)$$

$$+\frac{47}{7}\sigma_i^2(Q)]\}dQ+0\left(\frac{1}{s^4}\right) \quad \text{as} \quad s\to\infty. \tag{2.13}$$

On inserting (2.13) into (2.12) and letting $s^2=P$ we arrive at (2.2).

Finally, we note that the proof of either (2.9) or (2.13) is omitted here since it is very similar to those obtained in [21] or [22] respectively.

PROOF OF THEOREM 2. With reference to [26], let us assume that $\alpha_2(Q)\geq\alpha_1(Q),(Q\in\partial\Omega_1)$ and $\beta_2(Q)\geq\beta_1(Q),(Q\in\partial\Omega_2)$ and introduce the non-negative and non-decreasing function

$$\Phi(\lambda)=\sum_{\mu_k(\alpha_2,\beta_2)\leq\lambda}\{\mu_k\alpha_2,\beta_2)-\mu_k(\alpha_1,\beta_1)\}, \tag{2.14}$$

moreover we let

$$\psi(P)=\sum_{k=1}^{\infty}\frac{\{\mu_k(\alpha_2,\beta_2)-\mu_k(\alpha_1,\beta_1)\}^2\{\mu_k\alpha_2,\beta_2)+2\mu_k(\alpha_1,\beta_1)+3P\}}{\{\mu_k(\alpha_2,\beta_2)+P\}^3\{\mu_k(\alpha_1,\beta_1)+P\}^2}. \tag{2.15}$$

Using formula (2.1) first for the functions $(\alpha_1(Q),\beta_1(Q))$, then for the functions $(\alpha_2(Q),\beta_2(Q))$ and subtracting the second one from the first, we find after some reduction that

$$2\sum_{k=1}^{\infty}\frac{\{\mu_k(\alpha_2,\beta_2)-\mu_k(\alpha_1,\beta_1)\}}{\{\mu_k(\alpha_2,\beta_2)+P\}^3}+\psi(P)=\frac{a_1}{2\pi P^2}+\frac{3a_2}{16P^{5/2}}+0\left(\frac{1}{P^3}\right) \quad \text{as} \quad P\to\infty, \tag{2.16}$$

where

$$a_2=\int_{\partial\Omega_2}[\beta_2(Q)-\beta_1(Q)]\left[\frac{1}{2}K_2(Q)-\beta_1(Q)-\beta_2(Q)\right]dQ+\int_{\partial\Omega_1}[\alpha_2(Q)-\alpha_1(Q)]\left[\frac{1}{2}K_1(Q)-\alpha_1(Q)-\alpha_2(Q)\right]dQ.$$

Formula (2.16) can be written for any $c<\mu_1(\alpha_2,\beta_2)$ in the equivalent form

$$2\int_c^{+\infty}\frac{d\Phi(\lambda)}{(\lambda+P)^3}+\psi(P)=\frac{a_1}{2\pi P^2}+\frac{3a_2}{16P^{5/2}}+0\left(\frac{1}{P^3}\right) \quad \text{as} \quad P\to\infty. \tag{2.17}$$

Further, noting that

$$\psi(P)=o\left\{\int_c^{+\infty}(\lambda+P)^{-3}d\Phi(\lambda)\right\} \quad \text{as} \quad P\to\infty,$$

we get

$$\int_c^{+\infty}\frac{d\Phi(\lambda)}{(\lambda+P)^3}\sim\frac{a_1}{4\pi P^2} \quad \text{as} \quad P\to\infty. \tag{2.18}$$

Applying a Tauberian Theorem of Hardy and Littlewood (see, for example [24]), we find that

$$\Phi(\lambda)\sim\frac{a_1}{2\pi}\lambda \quad \text{as} \quad \lambda\to\infty. \tag{2.19}$$

Analogously, one establishes the asymptotic formula

$$\sum_{\mu_k(\alpha_1,\beta_1)\leq\lambda}\{\mu_k(\alpha_2,\beta_2)-\mu_k(\alpha_1,\beta_1)\}\sim\frac{a_1}{2\pi}\lambda \quad \text{as} \quad \lambda\to\infty. \tag{2.20}$$

Further, noting that

$$\sum_{\mu_k(\alpha_2^*,\beta_2^*)\le\lambda}\{\mu_k(\alpha_2,\beta_2)-\mu_k(\alpha_1,\beta_1)\}\le\sum_{\mu_k(\sigma_1,\sigma_2)\le\lambda}\{\mu_k(\alpha_2,\beta_2)-\mu_k(\alpha_1,\beta_1)\}$$

$$\le\sum_{\mu_k(\alpha_1^*,\beta_1^*)\le\lambda}\{\mu_k(\alpha_2,\beta_2)-\mu_k(\alpha_1,\beta_1)\}\,, \tag{2.21}$$

where

$$\alpha_1^*(Q)=\min\{\sigma_1(Q),\alpha_1(Q)\}\,,\qquad \beta_1^*(Q)=\min\{\sigma_2(Q),\beta_1(Q)\}\,,$$

$$\alpha_2^*(Q)=\min\{\sigma_1(Q),\alpha_2(Q)\}\,,\qquad \beta_2^*(Q)=\min\{\sigma_2(Q),\beta_2(Q)\}\,,$$

and the fact that as $\lambda\to\infty$ the functions

$$\sum_{\mu_k(\alpha_2^*,\beta_2^*)\le\lambda}\{\mu_k(\alpha_2,\beta_2)-\mu_k(\alpha_1,\beta_1)\}=\sum_{\mu_k(\alpha_2^*,\beta_2^*)\le\lambda}\{\mu_k(\alpha_2^*,\beta_2^*)-\mu_k(\alpha_1,\beta_1)\}$$

$$-\sum_{\mu_k(\alpha_2^*,\beta_2^*)\le\lambda}\{\mu_k(\alpha_2^*,\beta_2^*)-\mu_k(\alpha_2,\beta_2)\}\,, \tag{2.22}$$

and likewise for (α_1^*,β_1^*) are asymptotically equal to $\frac{a_1}{2\pi}\lambda$, we obtain (2.3) for the special case $\alpha_2(Q)\ge\alpha_1(Q),(Q\in\partial\Omega_1)$ and $\beta_2(Q)\ge\beta_1(Q),(Q\in\partial\Omega_2)$.

Similarly, we derive (2.4) for the special case $\alpha_2(Q)\ge\alpha_1(Q)$, $(Q\in S_1)$ and $\beta_2(Q)\ge\beta_1(Q),(Q\in S_2)$ as follows: Using formula (2.2) first for the functions $(\alpha_1(Q),\beta_1(Q))$, then for the functions $(\alpha_2(Q),\beta_2(Q))$ and subtracting the second one from the first, we find for any $c<\mu_1(\alpha_2,\beta_2)$ that

$$2\int_c^{+\infty}\frac{d\Phi(\lambda)}{(\lambda+P)^3}+\psi(P)=\frac{b_1}{8\pi P^{3/2}}+\frac{b_2}{8\pi P^2}+0\left(\frac{1}{P^{5/2}}\right)\quad\text{as}\quad P\to\infty\,, \tag{2.23}$$

where

$$b_2=\int_{S_2}[\beta_2(Q)-\beta_1(Q)][H_2(Q)-\beta_1(Q)-\beta_2(Q)]dQ+\int_{S_1}[\alpha_2(Q)-\alpha_1(Q)][H_1(Q)-\alpha_1(Q)-\alpha_2(Q)]dQ\,.$$

On using the same nature of $\psi(P)$, we write the integral in (2.23) in the asymptotic form

$$\int_c^{+\infty}\frac{d\Phi(\lambda)}{(\lambda+P)^3}\sim\frac{b_1}{16\pi P^{3/2}}\quad\text{as}\quad P\to\infty\,. \tag{2.24}$$

Consequently, we deduce that

$$\Phi(\lambda)\sim\frac{b_1}{3\pi^2}\lambda^{3/2}\quad\text{as}\quad\lambda\to\infty\,. \tag{2.25}$$

Analogously, one establishes the asymptotic formula

$$\sum_{\mu_k(\alpha_1,\beta_1)\le\lambda}\{\mu_k(\alpha_2,\beta_2)-\mu_k(\alpha_1,\beta_1)\}\sim\frac{b_1}{3\pi^2}\lambda^{3/2}\quad\text{as}\quad\lambda\to\infty\,. \tag{3.26}$$

On using (2.21) and the fact that as $\lambda\to\infty$ the functions (2.22) for (α_2^*,β_2^*) and likewise for (α_1^*,β_1^*) are asymptotically equal to $\frac{b_1}{3\pi^2}\lambda^{3/2}$, we obtain (2.4) for the special case $\alpha_2(Q)\ge\alpha_1(Q),(Q\in S_1)$ and $\beta_2(Q)\ge\beta_1(Q),(Q\in S_2)$.

In order to prove the theorem in the general case it is sufficient to apply the equality

$$\sum_{\mu_k(\sigma_1,\sigma_2)\le\lambda}\{\mu_k(\alpha_2,\beta_2)-\mu_k(\alpha_1,\beta_1)\}=\sum_{\mu_k(\sigma_1,\sigma_2)\le\lambda}\{\mu_k(\sigma_0,\sigma_0^*)-\mu_k(\alpha_1,\beta_1)\}$$

$$-\sum_{\mu_k(\sigma_1,\sigma_2)\le\lambda}\{\mu_k(\sigma_0,\sigma_0^*)-\mu_k(\alpha_2,\beta_2)\}\,, \tag{2.27}$$

where

$$\sigma_0(Q) = \max\{\alpha_1(Q), \alpha_2(Q)\}, \quad \sigma_0^*(Q) = \max\{\beta_1(Q), \beta_2(Q)\},$$

and apply the special case of the theorem which we just proved.

3. FURTHER RESULTS

COROLLARY 1. On using formulae (1.4) and (1.5) we deduce as $m \to \infty$ that

$$\sum_{k=1}^{m} \{\mu_k(\alpha_2, \beta_2) - \mu_k(\alpha_1, \beta_1)\} = \begin{cases} \left(\frac{2a_1}{|\Omega|}\right) m + o(m) & \text{in the case } n = 2, \quad (3.1) \\ \left(\frac{2b_1}{V}\right) m + o(m) & \text{in the case } n = 3. \quad (3.2) \end{cases}$$

Using Theorem 2 we easily prove the following Theorems:

THEOREM 3. Let the three pairs of functions $(\sigma_1(Q), \sigma_2(Q)), (\alpha_1(Q), \beta_1(Q)), (\alpha_2(Q), \beta_2(Q))$ and the quantity $a_1 \neq 0$ be the same as in (2.3). Furthermore, on the half-axis $[c, +\infty)$ let a function $f(\lambda)$ of constant sign be given which is absolutely continuous on each interval $[c, d], d < \infty$; further we assume that the expression $\frac{\lambda f'(\lambda)}{f(\lambda)}$ is bounded almost everywhere and $\int_c^{+\infty} f(\lambda) d\lambda = \infty$. Then as $\lambda \to \infty$ we get

$$\sum_{0 < \mu_k(\sigma_1, \sigma_2) \leq \lambda} f[\mu_k(\sigma_1, \sigma_2)] \{\mu_k(\alpha_2, \beta_2) - \mu_k(\alpha_1, \beta_1)\} = \left[\frac{a_1}{2\pi} + o(1)\right] \int_c^{\lambda} f(t) dt . \quad (3.3)$$

THEOREM 4. Let the three pairs of functions $(\sigma_1(Q), \sigma_2(Q)), (\alpha_1(Q), \beta_1(Q)), (\alpha_2(Q), \beta_2(Q))$ and the quantity $b_1 \neq 0$ be the same as in (2.4). Furthermore, on the half-axis $[c, +\infty)$ let a function $f(\lambda)$ of constant sign be given which is absolutely continuous on each interval $[c, d], d < \infty$, further we assume that the expression $\frac{\lambda f'(\lambda)}{f(\lambda)}$ is bounded almost everywhere and $\int_c^{+\infty} \lambda^{1/2} f(\lambda) d\lambda = \infty$. Then as $\lambda \to \infty$ we get

$$\sum_{0 < \mu_k(\sigma_1, \sigma_2) \leq \lambda} f[\mu_k(\sigma_1, \sigma_2)] \{\mu_k(\alpha_2, \beta_2) - \mu_k(\alpha_1, \beta_1)\} = \left[\frac{b_1}{2\pi^2} + o(1)\right] \int_c^{\lambda} |t|^{1/2} f(t) dt . \quad (3.4)$$

PROOF. On setting

$$Z(\lambda) = \sum_{0 < \mu_k(\sigma_1, \sigma_2) \leq \lambda} \{\mu_k(\alpha_2, \beta_2) - \mu_k(\alpha_1, \beta_1)\},$$

where the summation is taken over all values of k, for which $\mu_k(\sigma_1, \sigma_2) \leq \lambda$, we deduce for any $c < \mu_1(\sigma_1, \sigma_2)$ that

$$\sum_{0 < \mu_k(\sigma_1, \sigma_2) \leq \lambda} f[\mu_k(\sigma_1, \sigma_2)] \{\mu_k(\alpha_2, \beta_2) - \mu_k(\alpha_1, \beta_1)\} = \int_c^{+\infty} f(\lambda) dZ(\lambda) . \quad (3.5)$$

On inserting (2.3) and (2.4) into (3.5) we get easily (3.3) and (3.4) respectively.

COROLLARY 2. On using the mean value theorem, we deduce for any $c < \mu_1(\sigma_1, \sigma_2)$ that

$$\sum_{k=1}^{\infty} \{f[\mu_k(\sigma_2, \beta_2)] - f[\mu_k(\alpha_1, \beta_1)]\} =$$

$$\sum_{0 < \mu_k(\sigma_1, \sigma_2) \leq \lambda} f'[\mu_k(\sigma_1, \sigma_2)] \{\mu_k(\alpha_2, \beta_2) - \mu_k(\alpha_1, \beta_1)\} = \int_c^{+\infty} f'(\lambda) dZ(\lambda), \quad (3.6)$$

where $\mu_k(\alpha_1, \beta_1) \leq \mu_k(\sigma_1, \sigma_2) \leq \mu_k(\alpha_2, \beta_2)$ and the summation is taken over all values of k, for which $\mu_k(\sigma_1, \sigma_2) \leq \lambda$.

Consequently, if $f(\lambda) = \lambda^i, i > 0$ we deduce for $m \to \infty$ that in the case $n = 2$

$$\sum_{k=1}^{m}\{\mu_k^i(\alpha_2,\beta_2)-\mu_k^i(\alpha_1,\beta_1)\}=\frac{a_1}{2\pi}\left(\frac{4\pi}{|\Omega|}\right)^i m^i+o(m^i), \tag{3.7}$$

while if $f(\lambda)=\lambda^i, i\geq -\frac{1}{2}$ we deduce for $m\to\infty$ that in the case $n=3$

$$\sum_{k=1}^{m}\{\mu_k^i(\alpha_2,\beta_2)-\mu_k^i(\alpha_1,\beta_1)\}=\begin{cases}\dfrac{ib_1}{\pi^2(2i+1)}\left(\dfrac{6\pi^2}{V}m\right)^{(2i+1)/3}+o(m^{(2i+1)/3}) & \text{if } i>-\dfrac{1}{2}\\ \dfrac{-b_1}{6\pi^2}ln\left(\dfrac{6\pi^2}{V}m\right)+o\left(ln\left(\dfrac{6\pi^2}{\nu}m\right)\right) & \text{if } i=-\dfrac{1}{2}.\end{cases} \tag{3.8}$$

COROLLARY 3. Assuming that the function $f(\lambda)$ of Theorem 3 has the form: $f(\lambda)=\lambda^i, i\geq -1$ then we deduce as $\lambda\to\infty$ that

$$\sum_{0<\mu_k(\sigma_1,\sigma_2)\leq\lambda}\mu_k^i(\sigma_1,\sigma_2)\{\mu_k(\alpha_2,\beta_2)-\mu_k(\alpha_1,\beta_1)\}=\begin{cases}\dfrac{a_1}{2\pi(i+1)}\lambda^{i+1}+o(\lambda^{i+1}) & \text{if } i>-1,\\ \dfrac{a_1}{2\pi}ln\lambda+o(ln\lambda) & \text{if } i=-1.\end{cases} \tag{3.9}$$

COROLLARY 4. Assuming that the function $f(\lambda)$ of Theorem 4 has the form $f(\lambda)=\lambda^i, i\geq -3/2$ we deduce for $\lambda\to\infty$ that

$$\sum_{0<\mu_k(\sigma_1,\sigma_2)\leq\lambda}\mu_k^i(\sigma_1,\sigma_2)\{\mu_k(\alpha_2,\beta_2)-\mu_k(\alpha_1,\beta_1)\}=\begin{cases}\dfrac{b_1}{\pi^2(2i+3)}\lambda^{(2i+3)/2}+o(\lambda^{(2i+3)/2}) & \text{if } i>-3/2\\ \dfrac{b_1}{2\pi^2}ln\lambda+o(ln\lambda) & \text{if } i=-3/2.\end{cases} \tag{3.10}$$

COROLLARY 5. If $\mu_k(\alpha_1,\beta_1)\neq 0$ we deduce for $m\to\infty$ that in the case $n=2$

$$\sum_{k=1}^{m}\frac{\mu_k(\alpha_2,\beta_2)}{\mu_k(\alpha_1,\beta_1)}=m+\frac{a_1}{2\pi}ln\left(\frac{4\pi}{|\Omega|}m\right)+o\left(ln\left(\frac{4\pi}{|\Omega|}m\right)\right), \tag{3.11}$$

while in the case $n=3$

$$\sum_{k=1}^{m}\frac{\mu_k(\alpha_2,\beta_2)}{\mu_k(\alpha_1,\beta_1)}=m+b_1\left(\frac{6}{\pi^4V}\right)^{1/3}m^{1/3}+o(m^{1/3}). \tag{3.12}$$

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