COORDINATE d-DIMENSION PRINTS

HUNG HWAN LEE and IN SOO BAEK

Department of Mathematics

Kyungpook National University

Taegu 702-701, Korea

ABSTRACT. We define the coordinate d-dimension print to distinguish sets of same fractal dimension, and investigate its geometrical properties.

KEY WORDS AND PHRASES. Hausdorff dimension, Hausdorff dimension print, d-dimension, coordinate d-dimension print.

1991 AMS SUBJECT CLASSIFICATION CODES. Primary 28A80, Secondary 28A78.

1. INTRODUCTION

Sets of very different geometric characteristics may have same Hausdorff dimension. In [6], C. A. Rogers introduced the notion of Hausdorff dimension print to distinguish such sets. It is not easy to obtain the Hausdorff dimension print of a given set, even though it may be highly regular. In particular, we recognized that it is extremely difficult to find the Hausdorff dimension print of a nowhere differentiable continuous function. We introduce the coordinate d-dimension print, evolved from the d-dimension to deal with above difficulties. The coordinate d-dimension print of a given set informs us of its geometric characteristics, its d-dimension, and the d-dimension of its projection to x-axis.

We investigate coordinate d-dimension prints for the graphs of nowhere differentiable continuous functions and for regular sets.

At the end, we shows how the coordinate d-dimension print can be used for calculating the d-dimension of a set related to the aforementioned graph.

2. PRELIMINARIES

We restrict our attention to subsets of $\mathbf{R}^2$ for simplicity.

We mean a (b, a) coordinate rectangle by the set of the form $[mb, (m + 1)b] \times [na, (n + 1)a]$ where m and n are integers.

For $s, t \geq 0$, we define a pre-measure for a set $E \subset \mathbf{R}^2$ by

$$CD^{(s,t)}(E) = \lim_{\delta \to 0} \inf_{\delta > 0} \{N_E(a, b)a^s b^t : 0 < b \leq a \leq \delta\}$$

where $N_E(a, b)$ is the number of (b, a) coordinate rectangles that interest E.

We obtain an outer measure using Method I by Munroe [4]

$$cd^{(s,t)}(E) = \inf\{\sum_{n=1}^{\infty} CD^{(s,t)}(E_n) : E = \cup_{n=1}^{\infty} E_n\}$$

for a set $E \subset \mathbf{R}^2$.

Now, we recall lower box dimension ([2]) (lower capacity ([5])), $\underline{\dim}_B(E)(= \underline{\text{Cap}}\ (E)) = \liminf_{a \to 0} \frac{\log N(E,a)}{-\log a}$, and modified lower box dimension ([2]) (d-dimension ([3])), $\underline{\dim}_{MB}(E)(= d - \dim(E)) = \inf\{\sup_n \underline{\dim}_B(E_n) : E = \cup_{n=1}^{\infty} E_n\}$, where $N(E,a) = N_E(a,a)$ for $E \subset \mathbf{R}^2$ and $N(E,a)$ is the number of intervals of the form $[ma, (m+1)a]$ that interest E if $E \subset \mathbf{R}^1$.

It is not difficult to show that

$$\sup\{s > 0 : CD^{(s,0)}(E) > 0\} = \underline{\text{Cap}}(E)$$

and

$$\sup\{s > 0 : cd^{(s,0)}(E) > 0\} = d - \dim(E)$$

for $E \subset \mathbf{R}^2$ (cf. [2], [3]). (Note that the smallest number of squares of side a that cover $E \le \inf_{0 < b \le a} N_E(a,b) \le N_E(a,a)$.) Note that $CD^{(s,0)}$ and $cd^{(s,0)}$ are only the variations of D^s-premeasure and d^s-measure in [3] respectively.

Hence we just write $CD^{(s,0)}(E) = D^s(E)$ and

$$cd^{(s,0)}(E) = d^s(E) \text{ for } E \subset \mathbf{R}^2.$$

For $E \subset \mathbf{R}^1$, similarly we define

$$D^s(E) = \lim_{a \to 0} \inf N(E,a)a^s$$

and

$$d^s(E) = \inf\{\sum_{n=1}^{\infty} D^s(E_n) : E = \cup E_n\}.$$

Then we have

$$\sup\{s > 0 : D^s(E) > 0\} = \underline{\text{Cap}}(E)$$

and

$$\sup\{s > 0 : d^s(E) > 0\} = d - \dim(E).$$

We define *the coordinate d-dimension print* of $E \subset \mathbf{R}^2$ by

$$cd\text{ - Print }(E) = \{(s,t) : cd^{(s,t)}(E) > 0\}.$$

Plainly the coordinate d-dimension print is monotonic; cd-Print $(E_1) \subset$ cd- Print (E_2) for $E_1 \subset E_2$. Further it is additive, in the sense that

$$cd\text{ - Print }(\cup_{n=1}^{\infty} E_n) = \cup_{n=1}^{\infty} cd\text{ - Print}(E_n).$$

3. PROPERTIES OF cd-PRINTS

By the definition, we have the following straightforward propositions.

PROPOSITION 1. $\mathcal{H}$-Print $(E) \subset$ cd-Print (E) for $E \subset \mathbf{R}^2$. (Here, $\mathcal{H}$-Print means Hausdorff dimension print [6])

PROOF. Comparing the families of rectangles that cover E in the definitions, we have $\mathcal{H}^{(s,t)}(E) \leq cd^{(s,t)}(E)$ for $E \subset \mathbf{R}^2$.

If a point (s, t) is in the coordinate d-dimension print of E, then $\{(s', t') : s' + t' < s + t, t' < t\}$ is contained in the coordinate d-dimension print of E

PROPOSITION 2. If $cd^{(s,t)}(E) > 0$, then $cd^{(s',t')}(E) = \infty$ for $s' + t' < s + t$ and $t' \leq t$.

PROOF. Suppose $cd^{(s,t)}(E) > 0$. Then for every sequence $\{E_n\}$ of subsets that cover E, $\sum_{n=1}^{\infty} CD^{(s,t)}(E_n) > 0$. So there exists E_{n_0} such that $CD^{(s,t)}(E_{n_0}) > \alpha > 0$. Thus $\inf\{N_{E_{n_0}}(a,b)a^s b^t : 0 < b \leq a \leq \delta_0\} > \alpha$ for some δ_0.

For $s' + t' < s + t$, $t' \leq t$, and $b \leq a \leq \delta \leq \delta_0$

$$N_{E_{n_0}}(a,b)a^{s'}b^{t'} = N_{E_{n_0}}(a,b)a^s b^t a^{s'-s} b^{t'-t}$$

$$\geq N_{E_{n_0}}(a,b)a^s b^t a^{s'-s} a^{t'-t}$$

$$\geq N_{E_{n_0}}(a,b)a^s b^t \delta^{(s'+t')-(s+t)}$$

$$> \alpha \delta^{(s'+t')-(s+t)}$$

Therefore,

$$CD^{(s',t')}(E_{n_0}) = \infty \quad \text{for } s' + t' < s + t \text{ and } t' \leq t.$$

Hence $cd^{(s',t')}(E) = \infty$.

REMARK 3. Let $C_{\frac{1}{4}} = \left\{ \sum_{n=1}^{\infty} a_n 4^{-n} : a_n \in \{0,3\} \right\}$.

By Example 2 in [6] and Proposition 1

$$cd^{(\frac{1}{2},\frac{1}{2})}(C_{\frac{1}{4}} \times C_{\frac{1}{4}}) > 0.$$

It follows from Proposition 2 that $cd^{(0,\frac{1}{2})}(C_{\frac{1}{4}} \times C_{\frac{1}{4}}) = \infty$.

However we note that $d^{\frac{1}{2}}(C_{\frac{1}{4}}) = 1$. (Compare this with Theorem 8.)

PROPOSITION 4. $cd^{(s',t')}(E) \geq cd^{(s,t)}(E)$ for $s' + t' \leq s + t$ and $t' \leq t$.

PROOF. It follows from similar argument with the proof of Proposition 2.

COROLLARY 5. If $cd^{(0,t)}(E) < \infty$, then $cd^{(s,t)}(E) = 0$ for any $s > 0$, and $cd^{(0,t')}(E) = 0$ for $t' > t$.

PROOF. It follows immediately from Proposition 2.

To deal with the geometric characteristics of cd-Print, we need the following two simple but interesting lemmas.

LEMMA 6. If $d - \dim(A) = \alpha$ for $A \subset \mathbf{R}^1$, then $cd^{(0,\beta)}(A \times \mathbf{R}) = 0$ for $\beta > \alpha$. In particular, if $d - \dim(\mathrm{Proj}_x E) = \alpha$ for $E \subset \mathbf{R}^2$, then $cd^{(0,\beta)}(E) = 0$ for $\beta > \alpha$. (Here $\mathrm{Proj}_x E$ denotes the projection of E to x-axis.)

PROOF. Since $d - \dim(A) = \alpha$, $d^{\beta'}(A) = 0$ for $\beta > \beta' > \alpha$. By the definition of d-measure, there exists a sequence $\{A_n^0\}_{n=1}^\infty$ such that $\cup_{n=1}^\infty A_n^0 = A$ and $\sum_{n=1}^\infty D^{\beta'}(A_n^0) < 1$. Thus, for any integer m and n and $\beta > \beta'$,

$$CD^{(0,\beta)}(A_n^0 \times [m, m+1]) = \lim_{\delta \to 0} \inf\{N_{A_n^0 \times [m,m+1]}(a, b)a^0 b^\beta : 0 < b \leq a \leq \delta\}$$

$$\leq \lim_{\delta \to 0}(\frac{1}{\delta} + 2)\inf\{N(A_n^0, b)b^\beta : 0 < b \leq \delta\}$$

$$\leq \lim_{\delta \to 0}(\frac{1}{\delta} + 2)\lim_{\rho \to 0}\inf_{0 < b \leq \rho} N(A_n^0, b)b^\beta$$

$$= \lim_{\delta \to 0}(\frac{1}{\delta} + 2)D^\beta(A_n^0) = 0.$$

Therefore

$$cd^{(0,\beta)}(A \times \mathbf{R}) \leq \inf\{\sum_{m=-\infty}^\infty \sum_{n=1}^\infty CD^{(0,\beta)}(A_n \times [m, m+1]) : A = \cup_{n=1}^\infty A_n\}$$

$$\leq \sum_{m=-\infty}^\infty \sum_{n=1}^\infty CD^{(0,\beta)}(A_n^0 \times [m, m+1]) = 0.$$

LEMMA 7. If $d - \dim (\text{Proj}_x E) = \alpha$ for $E \subset \mathbf{R}^2$, then $cd^{(o,\gamma)}(E) = \infty$ for $\gamma < \alpha$.

PROOF. Since $d^\gamma (\text{Proj}_x E) = \infty$ for $\gamma < \alpha$, for any cover $\{E_n\}$ of E,

$$\sum_{n=1}^\infty CD^{(0,\gamma)}(E_n)$$

$$\geq \sum_{n=1}^\infty D^\gamma(Proj_x E_n)$$

$$\geq d^\gamma(Proj_x E) = \infty.$$

As $d - \dim(E) = \alpha$ doesn't always mean $d^\alpha(E) > 0$, it is natural to consider $cl[$ cd-Print $(E)]$, the closure of cd-Print (E) in the space $\{(s, t) : s \geq 0, t \geq 0\}$.

There is very interesting connection between d-dimension of the projection of a set to x-axis and its cd-Print.

THEOREM 8. For $E \subset \mathbf{R}^2$, $d - \dim (\text{Proj}_x E) = \alpha$ iff $cl[$ cd - Print $(E)] \cap y - axis = [0, \alpha]$.

PROOF. If $d - \dim (\text{Proj}_x E) = \alpha$, then cl [cd- Print $(E)] \cap y - axis = [0, \alpha]$ by Lemma 6 and 7.

Suppose that cl [cd - Print $(E)] \cap y - axis = [0, \alpha]$. Then $cd^{(0,\gamma)}(E) = \infty$ for $\gamma < \alpha$, and $cd^{(0,\beta)}(E) = 0$ for $\beta > \alpha$ by Proposition 4 and Corollary 5.

It follows from Lemma 6 that $d - \dim (\text{Proj}_x E) \geq \alpha$. And $d - \dim (\text{Proj}_x E) \leq \alpha$ follows from Lemma 7.

Next theorem tells us a geometrical connection between d-dimension and cd-Print.

THEOREM 9. For $E \subset \mathbf{R}^2$, $d - \dim(E) = \beta$ iff $cl[$cd - Print $(E)] \cap x - axis = [0, \beta]$.

PROOF. It follows from $\sup\{s > 0 : cd^{(s,0)}(E) > 0\} = d - \dim(E)$.

COROLLARY 10. If $d - \dim(\text{Proj}_x E) = \alpha$ and $d - \dim(E) = \beta$ for $E \subset \mathbf{R}^2$, then cd-Print $(E) \subset \{(s,t) : s + t \leq \beta, t \leq \alpha\}$. In particular, cd-Print $(E) \subset \{(s,t) : s + t \leq 2, t \leq 1\}$ for $E \subset \mathbf{R}^2$.

PROOF. It is immediate from Proposition 4, Theorems 8 and 9.

4. CALCULATION OF cd-PRINTS

Now, we calculate the cd-Print of some specific subsets of $\mathbf{R}^2$. Let $|J|$ denote the length of interval J.

THEOREM 11. Let $\varphi : [0,1] \to \mathbf{R}$ be a function such that $C_1|J|^\alpha \leq \sup_{x,y \in J}|\varphi(x) - \varphi(y)| \leq C_2|J|^\alpha$ for any interval $J \subset [0,1]$, where some constants $C_1, C_2 > 0$ and $0 < \alpha \leq 1$. Let $A(\subset [0,1])$ be a compact set such that $\underline{\text{Cap}}(A \cap (a - \delta, a + \delta)) \geq \underline{\text{Cap}}(A) = \beta$ for any $a \in A$ and $\delta > 0$, with $0 \leq \beta \leq 1$. Then for the graph of φ on A, $\mathcal{G}_A$

$$cl[\text{cd - Print}(\mathcal{G}_A)] = \{(s,t) \cdot s + t \leq 1 + \beta - \alpha, t \leq \beta - \alpha s\}.$$

PROOF. Since $\mathcal{G}_A$ is a closed subset of $\mathbf{R}^2$, $\cup_{n=1}^\infty \overline{G_n} = \mathcal{G}_A$ for each sequence $\{G_n\}$ such that $\cup_{n=1}^\infty G_n = \mathcal{G}_A$, where $\overline{G_n}$ is the closure of G_n in $\mathbf{R}^2$. Further, by Baire Category theorem, there exists integer n_0 such that $\overline{G_{n_0}}$ contains $\mathcal{G}_A \cap B_r(x)$ for some $x \in \overline{G_{n_0}} \subset \mathcal{G}_A$ and some $r > 0$.

By continuity of φ, we can choose $\delta > 0$ such that $\{(b, (\varphi(b)) : b \in (a - \delta, a + \delta), (a, \varphi(a)) = x\} \subset B_r(x)$.

Therefore we have $\{(b, \varphi(b)) : b \in (a - \delta, a + \delta) \cap A, (a, \varphi(a)) = x\} \subset \mathcal{G}_A \cap B_r(x)$. And, we note that $\underline{\text{Cap}}((a - \delta, a + \delta) \cap A) = \beta$ and $d - \dim(A) = \beta$ ([7]).

Hence we only need to show that

(1) $CD^{(s,t)}(\mathcal{G}_A) = \infty$ for $s + t < 1 + \beta - \alpha$ and $t < \beta - \alpha s$.

(2) $CD^{(s,t)}(\mathcal{G}_A) = 0$ for $s + t < 1 + \beta - \alpha$ or $t > \beta - \alpha s$.

(1): Suppose that $s + t < 1 + \beta - \alpha$ and $t < -\alpha s + \beta$. Then we can find $\varepsilon > 0$ satisfying $s + t < 1 + \beta - \alpha - \varepsilon$ and $t < -\alpha s + \beta - \varepsilon$. Since $\underline{\text{Cap}}(A) = \beta$, there is b_0 such that for all positive $b \leq b_0$,

$$N(A, b) \geq b^{-\beta + \varepsilon}.$$

Now, consider $\delta \leq b_0$ and b, a such that $0 < b \leq a \leq \delta$.

In case that $C_1 b^\alpha > a$, we have

$$N_{\mathcal{G}_A}(a,b)a^s b^t \geq (C_1 b^\alpha / a) b^{-\beta + \varepsilon} a^s b^t$$

$$\geq C_1 b^{\alpha - \beta + \varepsilon + t} a^{s-1}.$$

If $\alpha - \beta + \varepsilon - t < 0$, then

$$N_{\mathcal{G}_A}(a,b)a^s b^t \geq C_1 a^{\alpha - \beta + \varepsilon - 1 + s + t}.$$

Otherwise,

$$N_{\mathcal{G}_A}(a,b)a^s b^t \geq C_1[(a/C_1)^{\frac{1}{\alpha}}]^{\alpha - \beta + \varepsilon + t} a^{s-1}$$

$$= (\frac{1}{C_1})^{\frac{-\beta+\epsilon+t}{\alpha}} a^{\frac{-\beta+\epsilon+\alpha s+t}{\alpha}}$$

In case that $C_1 b^\alpha \leq a$, we have

$$N_{\mathcal{G}_A}(a,b)a^s b^t \geq b^{-\beta+\epsilon} a^s b^t.$$

Since $t < \beta - \epsilon$,

$$N_{\mathcal{G}_A}(a,b)a^s b^t \geq a^s[(\frac{a}{C_1})^{\frac{1}{\alpha}}]^{-\beta+\epsilon+t}$$

$$= (\frac{1}{C_1})^{\frac{-\beta+\epsilon+t}{\alpha}} a^{\frac{-\beta+\epsilon+\alpha s+t}{\alpha}}$$

Thus

$$\lim_{\delta\to 0}\inf\{N_{\mathcal{G}_A}(a,b)a^s b^t : 0 < b \leq a \leq \delta\} = \infty.$$

(2) : Suppose that $s + t > 1 + \beta - \alpha$ or $t > -\alpha s + \beta$. Then we can find $\epsilon > 0$ such that $s + t > 1 + \beta - \alpha + \epsilon$ or $t > -\alpha s + \beta + \epsilon$. Since $\underline{Cap}(A) = \beta$, for such $\epsilon > 0$, there exist infinitely many n such that $N(A, \frac{1}{n}) \leq (\frac{1}{n})^{-\beta-\epsilon}$. Hence,

$$N_{\mathcal{G}_A}(C_2(\frac{1}{n})^\alpha, \frac{1}{n})[C_2(\frac{1}{n})^\alpha]^s(\frac{1}{n})^t$$

$$\leq 2(\frac{1}{n})^{-\beta-\epsilon}[C_2(\frac{1}{n})^\alpha]^s(\frac{1}{n})^t$$

$$= 2C_2^s(\frac{1}{n})^{-\beta-\epsilon+\alpha s+t}.$$

(Here, we may assume $C_2 > 1$). And,

$$N_{\mathcal{G}_A}(\frac{1}{n}, \frac{1}{n})(\frac{1}{n})^s(\frac{1}{n})^t$$

$$\leq (\frac{1}{n})^{-\beta-\epsilon}[C_2(\frac{1}{n})^\alpha/(\frac{1}{n}) + 2](\frac{1}{n})^{s+t}$$

$$\leq C_2(\frac{1}{n})^{\alpha-\beta-\epsilon-1+s+t} + 2(\frac{1}{n})^{-\beta-\epsilon+s+t}$$

Hence

$$\lim_{\delta\to 0}\inf\{N_{\mathcal{G}_A}(a,b)a^s b^t : 0 < b \leq a \leq \delta\} = 0.$$

COROLLARY 12. Let $\varphi : [0,1] \to \mathbf{R}^1$ be a function satisfying $C_1|J|^\alpha \leq \sup_{x,y\in J}|\varphi(x) - \varphi(y)| \leq C_2|J|^\alpha$ for any interval $J \subset [0,1]$, where some $0 < \alpha \leq 1$ and some constants C_1 and $C_2 > 0$. Let $A(\subset [0,1])$ be a symmetric Cantor set with a sequence of contracting ratios $\{a_n\}([7])$. Then

$$cl[\text{cd-Print}(\{(x,\varphi(x) : x \in A\})]$$

$$= \{(s,t) : s + t \leq 1 + \liminf\frac{-n\log 2}{\log a_n} - \alpha$$

$$\text{and}\quad t \leq \max\{\liminf\frac{-n\log 2}{\log a_n} - \alpha s, 0\}\}.$$

PROOF. It is easy to show

$$\underline{Cap}(A \cap (a - \delta, a + \delta)) = \liminf\frac{-n\log 2}{\log a_n} = \underline{Cap}(A)$$

for any $a \in A$, $\delta > 0$, and A is a compact set ([7]). It follows immediately from Theorem 11 and the fact above.

REMARK 13. There are many examples of Theorem 11. Kiesswetter's curve([1]) is one of them and the closure of its cd-Print is $\{(s,t) : s + t \leq \frac{3}{2} \text{ and } t \leq 1 - \frac{1}{2}s\}$.

Now, we introduce a technique to gain the closure of cd-Print of some particular sets.

PROPOSITION 14. Let $E \subset \mathbf{R}^2$ and $d - \dim(\mathrm{Proj}_x E) = \alpha$, $d - \dim(E) = \alpha + \beta$ and $cd^{(s,t)}(E) = \infty$ for $s + t < \alpha + \beta$ and $t < \alpha$. Then $cl\,[\mathrm{cd}\text{-Print}\,(E)] = \{(s,t) : s + t \leq \alpha + \beta, t \leq \alpha\}$.

PROOF. If follows from Proposition 4, Theorem 8 and 9.

COROLLARY 15. Let A, B be compact subsets of $\mathbf{R}^1$ with $d - \dim(A \times B) = \alpha + \beta$, satisfying

$$\overline{\mathrm{Cap}}(A \cap (a - \delta, a + \delta)) \geq \overline{\mathrm{Cap}}(A) = \alpha \text{ for any } a \in A \text{ and any } \delta > 0$$

and

$$\underline{\mathrm{Cap}}(B \cap (b - \delta, b + \delta)) \geq \underline{\mathrm{Cap}}(B) = \beta \text{ for any } b \in B \text{ and any } \delta > 0.$$

Then

$$cl[\mathrm{cd}\text{-Print}(A \times B)] = \{(s,t) : s + t \leq \alpha + \beta \text{ and } t \leq \alpha\}.$$

PROOF. Recalling the argument in the proof of Theorem 11, we only need to show $CD^{(s,t)}(A \times B) = \infty$ to prove $cd^{(s,t)}(A \times B) = \infty$ for $s + t < \alpha + \beta$ and $t < \alpha$.

Suppose that $s + t < \alpha + \beta$ and $t < \alpha$. Then there is $\varepsilon > 0$ such that $s + t < \alpha + \beta - 2\varepsilon$ and $t < \alpha - \varepsilon$. Since $\underline{\mathrm{Cap}}(A) = \alpha$ and $\underline{\mathrm{Cap}}(B) = \beta$, there exists b_0 such that for all positive $b \leq b_0$, $N(A, b) \geq b^{-\alpha + \varepsilon}$ and there is a_0 such that for all positive $a \leq a_0$, $N(B, a) \geq a^{-\beta + \varepsilon}$. For $0 < b \leq a \leq \min\{a_0, b_0\}$,

$$N_{A \times B}(a, b)a^s b^t \geq b^{-\alpha + \varepsilon}a^{-\beta + \varepsilon}a^s b^t$$

$$= b^{-\alpha + \varepsilon + t}a^{-\beta + \varepsilon + s}.$$

Since $-\alpha + \varepsilon + t < 0$,

$$N_{A \times B}(a, b)a^s b^t \geq a^{-\alpha - \beta + 2\varepsilon + s + t}.$$

Since $-\alpha - \beta + 2\varepsilon + s + t < 0$,

$$\liminf_{\delta \to 0}\{N_{A \times B}(a, b)a^s b^t : 0 < b \leq a \leq \delta \leq \min\{a_0, b_0\}\} = \infty.$$

Noting $d - \dim(\mathrm{Proj}_x A \times B) = d - \dim(A) = \alpha$ ([7]) and $d - \dim(A \times B) = \alpha + \beta$, we have our conclusion from Proposition 14.

EXAMPLES 16. Let A and B be symmetric Cantor sets in $\mathbf{R}$ with sequences of contracting ratios $\{a_n\}$ and $\{b_n\}$ respectively. Let $\liminf_{n \to \infty} \frac{-n \log 2}{\log a_n} = \alpha$, and $\lim_{n \to \infty} \frac{-n \log 2}{\log b_n} = \beta$.

Then $d - \dim(A \times B) = \alpha + \beta = d - \dim(B \times A)$ ([7]).

Clearly A and B satisfy the assumptions of Corollary 15.

Hence

$$cl[\mathrm{cd}\text{ -Print}(A \times B)] = \{(s,t) : s + t \leq \alpha + \beta \text{ and } t \leq \alpha\}.$$

Also

$$cl[\mathrm{cd}\text{ - Print}(B \times A)] = \{(s,t) : s + t \leq \alpha + \beta \text{ and } t \leq \beta\}.$$

EXAMPLE 17. Using Proposition 14 or Corollary 15, we easily see the following fact.

(a) If E is a smooth curve and $\text{Proj}_x E$ is not a singleton, then $cl\,[\text{cd-Print}\,(E)] = \{(s,t) : s+t \leq 1\}$.

(b) If E is any set in $\mathbf{R}^2$ with non-empty interior, then

$$cl[\text{cd-Print}(E)] = \{(s,t) : s+t \leq 2, t \leq 1\}.$$

(c) Let A be a symmetric Cantor set with a sequence of contracting ratios $\{a_n\}$ and $\liminf_{n\to\infty} \frac{-n\log 2}{\log a_n} = \alpha$. Then

$$cl[\text{cd - Print}(A \times \mathbf{R})] = \{(s,t) : s+t \leq 1+\alpha, t \leq \alpha\}.$$

and

$$cl[\text{cd -Print}(\mathbf{R} \times A)] = \{(s,t) : s+t \leq 1+\alpha, t \leq 1\}.$$

EXAMPLE 18. Let E_θ denote the rotation of E with the angle θ with respect to origin and let $C_{\frac{1}{4}}$ be as in Remark 3. Then for almost all $\theta \in (0,\pi)$,

$$cl[\text{cd - Print}(C_{\frac{1}{4}} \times C_{\frac{1}{4}})_\theta] = \{(s,t) : s+t \leq 1\}$$

since Hausdorff dimension of $\text{Proj}_x[(C_{\frac{1}{4}} \times C_{\frac{1}{4}})_\theta]$ is the Hausdorff dimension of $C_{\frac{1}{4}} \times C_{\frac{1}{4}}$, 1, almost all $\theta \in (0,\pi)$ ([2] Theorem 6.1).

We note that

$$cl[\text{cd - Print}(C_{\frac{1}{4}} \times C_{\frac{1}{4}})] = \{(s,t) : s+t \leq 1 \text{ and } t \leq \frac{1}{4}\},$$

and

$$cl[\text{cd-Print}(C_{\frac{1}{4}} \times C_{\frac{1}{4}})_{\frac{\pi}{4}}] = \{(s,t) : s+t \leq 1 \text{ and } t \leq \frac{\log 3}{\log 4}\}.$$

Now, we introduce an application of cd-Print.

THEOREM 19. Let $\varphi : [0.1] \to \mathbf{R}$ be a function such that

$$C_1|J|^\alpha \leq \sup_{x,y\in J} |\varphi(x) - \varphi(y)| \leq C_2|J|^\alpha$$

for any interval $J \subset [0,1]$, where some constants $C_1, C_2 > 0$ and $0 < \alpha \leq 1$. Then $d-\dim(\{x \in [0,1] : \varphi(x) \text{ is not an algebraic number }\}) = 1$.

PROOF. Let $K = \{x \in [0,1] : \varphi(x) \notin \mathcal{A}\}$, where $\mathcal{A}$ is the set of algebraic numbers. Suppose that $d - \dim(K) < 1$. Then $d - \dim(K) < \beta < 1$ for some $\beta > 0$. Now,

$$\{(x,y) : y - \varphi(x) \in \mathcal{A}\} = \{(x,y) : \varphi(x) \in \mathcal{A}, y - \varphi(x) \in \mathcal{A}\}$$

$$\cup \{(x,y) : \varphi \notin \mathcal{A}, y - \varphi(x) \in \mathcal{A}\}$$

$$= \{(x,y) : y \in \mathcal{A}\} \cup \{(x,y) : x \in K, y - \varphi(x) \in \mathcal{A}\}.$$

Since $\mathcal{A}$ is a countable set, we can enumerate $\mathcal{A} = \{a_n\}_{n=1}^\infty$. Therefore

$$\{(x,y): y - \varphi(x) \in \mathcal{A}\} = \cup_{n=1}^{\infty}\{(x,\varphi(x) + a_n): x \in [0,1]\}.$$

By the additivity of cd-Print and Theorem 11,

$$cl[cd \text{ - } \mathrm{Print}(\{(x,y): y - \varphi(x) \in \mathcal{A}\})] = \{(s,t): s + t \le 2 - \alpha, t \le 1 - \alpha s\}.$$

But

$$cl[cd \text{ - } \mathrm{Print}(\{(x,y): x \in K, y - \varphi(x) \in \mathcal{A}\})] \subset \{(s,t): s + t \le 2 - \alpha, t \le \beta\}$$

and

$$cl[cd \text{ - } \mathrm{Print}(\{(x,y): y \in \mathcal{A})]$$

$$\subset cl[cd \text{ - } \mathrm{Print}(\cup_{n=1}^{\infty}\{(x,a_n): x \in [0,1]\})]$$

$$= \{(s,t): s + t \le 1\}.$$

A contradiction arises by the monotonicity of cd-Print.

REMARK 20. Let $A \subset \mathbf{R}$ be a compact set such that $\underline{\mathrm{Cap}}(A \cap (a - \varepsilon, a + \varepsilon)) = \underline{\mathrm{Cap}}(A) = \alpha$ for any $a \in A$ and any $\varepsilon > 0$. Then $d - \dim(\{x \in A: \varphi(x) \notin \mathcal{A}\}) = \alpha$, where $\mathcal{A}$ is the set of algebraic numbers.

REMARK 21. There might be several different way to get cl [cd-Print $(\mathbf{R} \times \mathbf{R})$] = $\{(s,t): s + t \le 2, t \le 1\}$.

For example, we could consider countable graphs of φ_n in Theorem 11 with $\alpha = \frac{1}{n}$.

Another method is to use countable sets of $[0,1] \times K_n$, where K_n is a symmetric Cantor set with a sequence of contracting ratios $\{a_{n,k}\}_{k=1}^{\infty}$ satisfying

$$\lim_{k \to \infty} \frac{-k \log 2}{\log a_{n,k}} = 1 - \frac{1}{n}.$$

CONJECTURE 22. Let $E \subset \mathbf{R}^2$ with $d - \dim(E) = \alpha > 1$. We conjecture

$$cl[cd \text{ -Print}(E_\theta)] = \{(s,t): s + t \le \alpha, t \le 1\}$$

for almost all $\theta \in [0, \pi)$.

REMARK 23. While the Hausdorff dimension print is invariant under linear transformations, our coordinate d-dimension print is not so. For example, a straight line parallel to y-axis has a smaller coordinate d-dimension print than a line at 45° to y-axis. Nevertheless our coordinate d-dimension print is particularly useful for the study of sets, such as the graphs of functions or Cartesian products which naturally have special relationship to the coordinate axes. (We thank our referee for pointing out the previous fact.) It would be interesting to investigate how the coordinate d-dimension print changes according to linear transformations.

REFERENCES

[1] G. A. EDGAR, Kiesswetter's fractal has Hausdorff dimension $\frac{3}{2}$, <u>Real Analysis Exchange</u>, Vol. 14 (1988-1989), 215-223.

[2] K. J. FALCONER, Fractal geometry (John Wiley & Sons, 1990).

[3] H. H. LEE and I. S. BAEK, On d-measure and d-dimension, Real Analysis Exchange, Vol.17(1991-92), 590-596.

[4] M. MUNROE, Measure and Integration (Addison-Wesley, 1971).

[5] F. PRZYTYCKI and M. URBANSKI, On the Hausdorff dimension of some fractal sets, Studia Math (1989), 155-186.

[6] C. A. ROGERS, Dimension prints, Mathematika, Vol.35, Part 1, No.69 (1988), 1-27.

[7] C. TRICOT, Two definitions of fractal dimension, Math. Proc. Camb. Phil. Soc. 91(1982), 57-74.

Special Issue on
Time-Dependent Billiards

Call for Papers

This subject has been extensively studied in the past years for one-, two-, and three-dimensional space. Additionally, such dynamical systems can exhibit a very important and still unexplained phenomenon, called as the Fermi acceleration phenomenon. Basically, the phenomenon of Fermi acceleration (FA) is a process in which a classical particle can acquire unbounded energy from collisions with a heavy moving wall. This phenomenon was originally proposed by Enrico Fermi in 1949 as a possible explanation of the origin of the large energies of the cosmic particles. His original model was then modified and considered under different approaches and using many versions. Moreover, applications of FA have been of a large broad interest in many different fields of science including plasma physics, astrophysics, atomic physics, optics, and time-dependent billiard problems and they are useful for controlling chaos in Engineering and dynamical systems exhibiting chaos (both conservative and dissipative chaos).

We intend to publish in this special issue papers reporting research on time-dependent billiards. The topic includes both conservative and dissipative dynamics. Papers discussing dynamical properties, statistical and mathematical results, stability investigation of the phase space structure, the phenomenon of Fermi acceleration, conditions for having suppression of Fermi acceleration, and computational and numerical methods for exploring these structures and applications are welcome.

To be acceptable for publication in the special issue of Mathematical Problems in Engineering, papers must make significant, original, and correct contributions to one or more of the topics above mentioned. Mathematical papers regarding the topics above are also welcome.

Authors should follow the Mathematical Problems in Engineering manuscript format described at http://www .hindawi.com/journals/mpe/. Prospective authors should submit an electronic copy of their complete manuscript through the journal Manuscript Tracking System at http:// mts.hindawi.com/ according to the following timetable:

Manuscript Due	December 1, 2008
First Round of Reviews	March 1, 2009
Publication Date	June 1, 2009

Guest Editors

Edson Denis Leonel, Departamento de Estatística, Matemática Aplicada e Computação, Instituto de Geociências e Ciências Exatas, Universidade Estadual Paulista, Avenida 24A, 1515 Bela Vista, 13506-700 Rio Claro, SP, Brazil ; edleonel@rc.unesp.br

Alexander Loskutov, Physics Faculty, Moscow State University, Vorob'evy Gory, Moscow 119992, Russia; loskutov@chaos.phys.msu.ru