

**ABOUT THE EXISTENCE AND UNIQUENESS
THEOREM FOR HYPERBOLIC EQUATION**

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ABSTRACT. In this paper we prove the existence and uniqueness theorem for almost everywhere solution of the hyperbolic equation using the method of successive approximations [1].

KEY WORDS AND PHRASES. Hyperbolic equation, existence and uniqueness.

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1. INTRODUCTION.

Mixed problems for partial differential equations have been investigated by a number of authors [2], [3], [4], [5]. In this case we investigate the almost everywhere solution for the hyperbolic equation that have been studied in [6]. Namely, the solution for the hyperbolic equation in the space $B_{2,2,T}^{2,1}$ with a nonlinear operator at the right hand side.

2. STATEMENT OF THE PROBLEM.

Consider the following system

$$u_{tt}(t, x) - Lu(t, x) = F(u(t, x)) \quad \text{in } Q_T \quad (2.1)$$

subject to the initial conditions

$$u(0, x) = \phi(x) \quad u_t(0, x) = \psi(x) \quad x \in \Omega, \quad (2.2)$$

and the boundary condition

$$u(t, x) | \Gamma = 0 \quad t \in [0, T] \quad (2.3)$$

where $Q_T = [0, T] \times \Omega$, $0 < T < \infty$, Ω is a bounded domain in R^n and G is the boundary of Ω ;

$$L(u) = \sum_{i,j=1}^n \frac{\partial}{\partial x_i} \left(a_{ij}(x) \frac{\partial u}{\partial x_j} \right) - a(x)u, \quad (2.4)$$

and moreover the functions $a_{ij}(x)$ have continuous $\bar{\Omega}$ and $\frac{\partial a_{ij}(x)}{\partial x_k}$, $a(x)$ are measurable and bounded in Ω and satisfy the following conditions in Ω :

$$a_{ij}(x) = a_{ji}(x), \quad a(x) \geq 0, \quad \sum_{i,j=1}^n a_{ij}(x) \xi_i \xi_j \geq a \sum_{i=1}^n \xi_i^2, \quad (2.5)$$

ξ_i are any real number; $\psi(\xi)$, $\phi(x)$ are given functions in Ω ; F is a nonlinear operator.

3. PRELIMINARIES.

DEFINITION 1. The almost everywhere solution for the problem (2.1)-(2.3) is the function $u(x, t)$, element of $W_2^2(Q_T)$, belongs to $D_1^0(Q_T)$ and satisfies (2.1) almost everywhere in Q_T and $t \rightarrow +0$ satisfies the following

$$\int_{\Omega} [u(t, x) - \phi(x)]^2 dx = 0, \quad \int_{\Omega} \left[\frac{\partial u(t, x)}{\partial t} - \psi(x) \right]^2 dx = 0 \quad (2.6)$$

DEFINITION 2. We define the space $B_{\beta_0, \dots, \beta_\ell, T}^{\alpha_0, \dots, \alpha_\ell}$ of all functions $u(t, x) = \sum_{s=1}^{\infty} u_s(t) \vartheta_s(x)$ in $Q_T = [0, T] \times \Omega$, where $\vartheta_s(x)$ are eigenfunctions for the operator L with the boundary condition (2.3) corresponding to the eigenvalues λ_s

$$(0 < \lambda_s \rightarrow \text{as } s \rightarrow \infty) [7], \quad u_s(t) \text{ are } \ell \geq 0$$

times continuously differentiable in $[0, T]$ and

$$\sum_{i=1}^{\ell} \left\{ \sum_{s=1}^{\infty} \left[\lambda_s^{\alpha_i} \max_{0 \leq t \leq T} |u_s^{(i)}(t)| \right]^{\beta_i} \right\}^{1/\beta_i} < +\infty \quad (2.7)$$

and has the norm

$$\|u\|_{B_{\beta_0, \dots, \beta_\ell, T}^{\alpha_0, \dots, \alpha_\ell}} = \sum_{i=1}^{\ell} \left\{ \sum_{s=1}^{\infty} \left[\lambda_s^{\alpha_i} \max_{0 \leq t \leq T} |u_s^{(i)}(t)| \right]^{\beta_i} \right\}^{1/\beta_i} \quad (2.8)$$

where $\alpha_i \geq 0, 1 \leq \beta_i \leq 2, (i = 0, \dots, \ell)$.

DEFINITION 3. The function $u_s(t)$ is called the s -component of the function

$$u_s(t, x) = \sum_{s=1}^{\infty} u_s(t) \vartheta_s(x)$$

and $\mu_s (s = 1, 2, \dots)$ is the set of all s -components of elements of μ where $\mu \subset B_{\beta_0, \dots, \beta_\ell, T}^{\alpha_0, \dots, \alpha_\ell}$.

THEOREM 2.1. The necessary and sufficient conditions for μ to be compact in $B_{\beta_0, \dots, \beta_\ell, T}^{\alpha_0, \dots, \alpha_\ell}$ are

- (a) for every $s (s = 1, 2, \dots)$ the set μ_s is compact in $C^0[0, T]$; and
- (b) for any given $\epsilon > 0$ there exists a natural number n_ϵ so that for all $u(t, x) = \sum_{s=1}^{\infty} u_s \vartheta_s(x) \in \mu$,

$$\sum_{i=1}^{\ell} \left\{ \sum_{s=n}^{\infty} \left[\lambda_s^{\alpha_i} \max_{0 \leq t \leq T} |u_s^{(i)}(t)| \right]^{\beta_i} \right\}^{1/\beta_i} < \epsilon.$$

This theorem can be proved analogously as in ([9] page 277-278).

LEMMA 1. For any almost everywhere solution $u(t, x)$ of (2.1) - (2.3) functions $u_s(t) = \int_{\Omega} u(t, x) \vartheta_s(x) dx$ satisfy the following system ([7], [8])

$$\begin{aligned} u_s(t) &= \phi_s \cos \lambda_s t + \frac{\psi_s}{\lambda_s} \sin \lambda_s t + \\ &\quad \frac{1}{\lambda_s} \int_0^t \int_{\Omega} F(u(\tau, x)) \vartheta_s(x) \sin \lambda_s(t-\tau) dx d\tau, (s = 1, 2, \dots), \end{aligned} \quad (2.9)$$

where

$$\phi_s = \int_{\Omega} \psi(x) \vartheta_s(x) dx, \quad \psi_s = \int_{\Omega} \psi(x) \vartheta_s(x) dx.$$

3. ASSUMPTION AND RESULTS.

THEOREM 3.1. Let

1. $a_{ij}(x)$ are continuously differentiable on $\bar{\Omega}$ and $a(x)$ continuous on $\bar{\Omega}$;
2. The eigenfunctions ϑ_s are twice continuously differentiable on $\bar{\Omega}$;
3. $\phi(x) \in W_2^2(\Omega) \cap D^o(\Omega)$, $\psi(x) \in D^o(\Omega)$;
4. $F: B_{2,2,T}^{2,1} \cup (W_2^2(Q_T) \cap B_{2,2,T}^{1,0}) \rightarrow W_{x,t,2}^{1,0}(Q_T)$ and satisfies

$$\| F(u(t,x)) \|_{W_2^1(\Omega)} \leq c(t) + d(t) \| u \|_{B_{2,2,t}^{2,1}} \quad (3.1)$$

for all $u \in B_{2,2,T}^{2,1}$; where $c(t), d(t) \in L_2(0, T)$.

5. For any $u, v \in \mathbb{K}_o$ (where \mathbb{K}_o is the sphere $\| u \|_{B_{2,2,T}^{2,1}} \leq C_o$)

$$\| F(u, t, x) - F(v, t, x) \|_{W_2^1(\Omega)} \leq g(t) \| u - v \|_{B_{2,2,t}^{2,1}}, \quad g(t) \in L_2(0, T), \quad (3.2)$$

where

$$C_o = \left\{ \left[2 \| W(t, x) \|_{B_{2,2,T}^{2,1}}^2 + 16 T a_o^2 \| c(t) \|_{L_2(0, T)}^2 \right] \exp \left[16 T a_o^2 \| d(t) \|_{L_2(0, T)}^2 \right] \right\}^{1/2} \quad (3.3)$$

and

$$a_o^2 = \max \left\{ n \cdot \max_{ij} \left\{ \| a_{ij}(x) \|_{C(\bar{\Omega})} \right\}, \| a(x) \|_{C(\bar{\Omega})} \right\}$$

6. For any $u \in B_{2,2,T}^{2,1} \cup (W_2^2(Q_T) \cap B_{2,2,T}^{1,0})$ and $t \in [0, T]$ $F(u(t, x)) \in D^o(\Omega)$.

Then the problem (2.1) - (2.3) has a unique solution,

PROOF. Let

$$W(x, t) = \sum_{s=1}^{\infty} (\phi_s \cos \lambda_s t + \frac{\psi_s}{\lambda_s} \sin \lambda_s t) \vartheta_s(x), \quad (3.4)$$

and

$$PF(u) = \sum_{s=1}^{\infty} \frac{1}{\lambda_s} \int_0^t \int_{\Omega} F(u(\tau, x)) \cdot \vartheta_s(x) \sin \lambda_s(t - \tau) dx d\tau \cdot \vartheta_s(x) \quad (3.5)$$

From (3.4) and (3.5) let us assume that

$$Q(u) = W + PF(u) \quad (3.6)$$

Then it is easy to see that the operator Q acts in $B_{2,2,T}^{2,1}$ and satisfies Lipschitz condition

$$\| Q(u) - Q(v) \|_{B_{2,2,t}^{2,1}} \leq 2\sqrt{T} a_o \| g(t) \|_{L_2(0, T)} \| u - v \|_{B_{2,2,t}^{2,1}} \quad (3.7)$$

in the sphere \mathbb{K}_o .

Consider the sequence $u_k(t, x) = Q(u_{k-1}(t, x))$ in $B_{2,2,T}^{2,1}$ where $u_0(t, x) = 0$. Using (3.1) and

the mathematical induction we get for any $k(k = 1, 2, 3, \dots)$ and $t \in [0, T]$:

$$\begin{aligned}
 \|u_k\|_{B_{2,2,t}^{2,1}}^2 &\leq 2\|W\|_{B_{2,2,T}^{2,1}}^2 + 8Ta_o^2 \int_0^t \|F(u_{k-1}(\tau, x))\|_{W_2^1(\Omega)}^2 d\tau \\
 &\leq 2\|W\|_{B_{2,2,T}^{2,1}}^2 + 16Ta_o^2 \left\{ \int_0^T c^2(\tau) d\tau + \int_0^t d^2(\tau) \|u_{k-1}\|_{B_{2,2,t}^{2,1}}^2 d\tau \right\} \\
 &= A^2 + \int_0^t \mathbb{B}^2(\tau) \|u_{k-1}\|_{B_{2,2,t}^{2,1}}^2 d\tau \\
 &\leq A^2 + \mathcal{A}^2 \int_0^t \mathbb{B}^2(\tau) d\tau + \dots + \mathcal{A}^2 \frac{\left\{ \int_0^t \mathbb{B}^2(\tau) d\tau \right\}^{k-1}}{(k-1)!},
 \end{aligned} \tag{3.8}$$

where

$$\mathcal{A}^2 = 2\|W\|_{B_{2,2,T}^{2,1}}^2 + 16Ta_o^2 \|c(t)\|_{L_2(0,T)}^2, \tag{3.9}$$

and

$$\mathbb{B}^2(t) = 16Ta_o^2 d^2(t)$$

From (3.8) for any $k(k = 1, 2, \dots)$, we get

$$\|u_k\|_{B_{2,2,t}^{2,1}}^2 \leq \mathcal{A}^2 \cdot \exp \left\{ \int_0^T \mathbb{B}^2(\tau) d\tau \right\} = C_o^2 \tag{3.10}$$

i.e., all $u_k(t, x)$ are contained in the sphere \mathfrak{K}_o . Further, using (3.2) and (3.3) we get for any $t \in [0, T]$ and $k(k = 1, 2, 3, \dots)$

$$\begin{aligned}
 \|u_{k+1} - u_k\|_{B_{2,2,t}^{2,1}}^2 &\leq 4Ta_o^2 \|F(u_k(\tau, x)) - F(u_{k-1}(\tau, x))\|_{L_2(\Omega)}^2 d\tau \\
 &\leq 4Ta_o^2 \int_0^t g^2(\tau) \|u_k - u_{k-1}\|_{B_{2,2,t}^{2,1}}^2 d\tau \\
 &\leq \|u_1 - u_o\|_{B_{2,2,T}^{2,1}}^2 \frac{\left\{ 4Ta_o^2 \int_0^t g^2(\tau) d\tau \right\}^k}{k!} \\
 &= \|u_1\|_{B_{2,2,T}^{2,1}}^2 \frac{\left\{ 4Ta_o^2 \int_0^t g^2(\tau) d\tau \right\}^k}{k!} \leq C_o^2 \frac{\left\{ 4Ta_o^2 \int_0^t g^2(\tau) d\tau \right\}^k}{k!}
 \end{aligned} \tag{3.11}$$

Therefore,

$$\|u_{k+1} - u_k\|_{B_{2,2,T}^{2,1}}^2 \leq C_o^2 \frac{\left\{ 4Ta_o^2 \|g(t)\|_{L_2(0,T)}^2 \right\}^k}{k!}, (k = 1, 2, \dots) \tag{3.12}$$

Then $\{u_k(t, x)\}$ is a fundamental sequence in $B_{2,2,T}^{2,1}$. Since $B_{2,2,T}^{2,1}$ complete, then

$$u_k(t, x) \underset{\rightharpoonup}{\longrightarrow} B_{2,2,T}^{2,1} \quad u(t, x) \in \mathfrak{K}_o \quad \text{as } k \rightarrow \infty \quad (3.13)$$

Since Q is continuous in \mathfrak{K}_o , then from the relation $u_k(t, x) = Q(u_{k-1}(t, x))$

we have

$$u_o(t, x) = Q(u(t, x))$$

Therefore, as in (3.11), (3.12) the speed of convergence is governed by the following inequality

$$\begin{aligned} \|u_k - u\|_{B_{2,2,T}^{2,1}}^2 &\leq \|u_o - u\|_{B_{2,2,T}^{1,0}}^2 \frac{\left\{4Ta_o^2 \|g(t)\|_{L_2(0,T)}^2\right\}^k}{k!} \\ &\leq C_o^2 \frac{\left\{4Ta_o^2 \|g(t)\|_{L_2(0,T)}^2\right\}^k}{k!}, \quad (k = 1, 2, \dots). \end{aligned} \quad (3.14)$$

Now to prove the uniqueness let us assume the $u(t, x) = \sum_{s=1}^{\infty} u_s(t) \ell_2(x)$ solution to (2.1) - (2.3) then $F(u(t, x)) \in L_2(Q_T)$. By Lemma (1) $u_s(t)$ satisfy (2.9); from (2.9) we get

$$\|u(t, x)\|_{B_{2,2,t}^{1,0}} \leq \|W(t, x)\|_{B_{2,2,T}^{1,0}} + 2\sqrt{T} \|F(u(t, x))\|_{L_2(Q_T)} < +\infty \quad (3.15)$$

Therefore $u \in B_{2,2,t}^{1,0}$. Since $u(t, x) \in W_2^2(Q_T) \cap B_{2,2,T}^{1,0}$ then by (3.1) $F(u(t, x)) \in W_{x,t,2}^{1,0}(Q_T)$, but by condition 6 Theorem 2 for all $t \in [0, T]$, $F(u(t, x)) \in \dot{D}(\Omega)$. Thus using (2.9) with some manipulation

$$\|u(t, x)\|_{B_{2,2,t}^{2,1}} \leq \|W(t, x)\|_{B_{2,2,T}^{2,1}} + 2\sqrt{T} a_o \|F(u(t, x))\|_{W_{x,t,2}^{1,0}(Q_T)} < +\infty \quad (3.16)$$

Therefore, $u \in B_{2,2,T}^{2,1}$. Then, using (3.1), (3.8), (3.10), we get $\|u(t, x)\|_{B_{2,2,t}^{2,1}} \leq C_o$. Thus, all almost everywhere solutions (2.1)-(2.3) belong to the sphere K_o and they are fixed points in $B_{2,2,T}^{2,1}$ for operator Q . Let u, v be two solutions to (2.1)-(2.3), then by (3.2) we get

$$\begin{aligned} \|u - v\|_{B_{2,2,t}^{2,1}}^2 &\leq 4Ta_o^2 \int_0^t \|F(u(\tau, x)) - F(v(\tau, x))\|_{W_2^1(\Omega)}^2 d\tau \\ &\leq 4Ta_o^2 \int_0^t g^2(\tau) \|u - v\|_{B_{2,2,t}^{2,1}}^2 d\tau \end{aligned} \quad (3.17)$$

Therefore, using Belmann's inequality [10] we have

$$\|u - v\|_{B_{2,2,t}^{2,1}}^2 = 0 \text{ in } [0, T]. \text{ Therefore, } u = v.$$

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