

# ABOUT THE EXISTENCE AND UNIQUENESS THEOREM FOR HYPERBOLIC EQUATION

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**ABSTRACT.** In this paper we prove the existence and uniqueness theorem for almost everywhere solution of the hyperbolic equation using the method of successive approximations [1].

**KEY WORDS AND PHRASES.** Hyperbolic equation, existence and uniqueness.

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## 1. INTRODUCTION.

Mixed problems for partial differential equations have been investigated by a number of authors [2], [3], [4], [5]. In this case we investigate the almost everywhere solution for the hyperbolic equation that have been studied in [6]. Namely, the solution for the hyperbolic equation in the space  $B_{2,2,T}^{2,1}$  with a nonlinear operator at the right hand side.

## 2. STATEMENT OF THE PROBLEM.

Consider the following system

$$u_{tt}(t, x) - Lu(t, x) = F(u(t, x)) \quad \text{in } Q_T \quad (2.1)$$

subject to the initial conditions

$$u(0, x) = \phi(x) \quad u_t(0, x) = \psi(x) \quad x \in \Omega, \quad (2.2)$$

and the boundary condition

$$u(t, x) | \Gamma = 0 \quad t \in [0, T] \quad (2.3)$$

where  $Q_T = [0, T] \times \Omega$ ,  $0 < T < \infty$ ,  $\Omega$  is a bounded domain in  $R^n$  and  $G$  is the boundary of  $\Omega$ ;

$$L(u) = \sum_{i,j=1}^n \frac{\partial}{\partial x_i} \left( a_{ij}(x) \frac{\partial u}{\partial x_j} \right) - a(x)u, \quad (2.4)$$

and moreover the functions  $a_{ij}(x)$  have continuous  $\bar{\Omega}$  and  $\frac{\partial a_{ij}(x)}{\partial x_k}$ ,  $a(x)$  are measurable and bounded in  $\Omega$  and satisfy the following conditions in  $\Omega$  :

$$a_{ij}(x) = a_{ji}(x), \quad a(x) \geq 0, \quad \sum_{i,j=1}^n a_{ij}(x) \xi_i \xi_j \geq a \sum_{i=1}^n \xi_i^2, \quad (2.5)$$

$\xi_i$  are any real number;  $\psi(\xi)$ ,  $\phi(x)$  are given functions in  $\Omega$ ;  $F$  is a nonlinear operator.

## 3. PRELIMINARIES.

DEFINITION 1. The almost everywhere solution for the problem (2.1)-(2.3) is the function  $u(x, t)$ , element of  $W_2^2(Q_T)$ , belongs to  $D_1^0(Q_T)$  and satisfies (2.1) almost everywhere in  $Q_T$  and  $t \rightarrow +0$  satisfies the following

$$\int_{\Omega} [u(t, x) - \phi(x)]^2 dx = 0, \quad \int_{\Omega} \left[ \frac{\partial u(t, x)}{\partial t} - \psi(x) \right]^2 dx = 0 \quad (2.6)$$

DEFINITION 2. We define the space  $B_{\beta_0, \dots, \beta_\ell, T}^{\alpha_0, \dots, \alpha_\ell}$  of all functions  $u(t, x) = \sum_{s=1}^{\infty} u_s(t) \vartheta_s(x)$  in  $Q_T = [0, T] \times \Omega$ , where  $\vartheta_s(x)$  are eigenfunctions for the operator  $L$  with the boundary condition (2.3) corresponding to the eigenvalues  $\lambda_s$

$$(0 < \lambda_s \rightarrow \infty \text{ as } s \rightarrow \infty) [7], \quad u_s(t) \text{ are } \ell \geq 0$$

times continuously differentiable in  $[0, T]$  and

$$\sum_{i=1}^{\ell} \left\{ \sum_{s=1}^{\infty} \left[ \lambda_s^{\alpha_i} \max_{0 \leq t \leq T} |u_s^{(i)}(t)| \right]^{\beta_i} \right\}^{1/\beta_i} < +\infty \quad (2.7)$$

and has the norm

$$\|u\|_{B_{\beta_0, \dots, \beta_\ell, T}^{\alpha_0, \dots, \alpha_\ell}} = \sum_{i=1}^{\ell} \left\{ \sum_{s=1}^{\infty} \left[ \lambda_s^{\alpha_i} \max_{0 \leq t \leq T} |u_s^{(i)}(t)| \right]^{\beta_i} \right\}^{1/\beta_i} \quad (2.8)$$

where  $\alpha_i \geq 0, 1 \leq \beta_i \leq 2, (i = 0, \dots, \ell)$ .

DEFINITION 3. The function  $u_s(t)$  is called the  $s$ -component of the function

$$u(t, x) = \sum_{s=1}^{\infty} u_s(t) \vartheta_s(x)$$

and  $\mu_x(s = 1, 2, \dots)$  is the set of all  $s$ -components of elements of  $\mu$  where  $\mu \in B_{\beta_0, \dots, \beta_\ell, T}^{\alpha_0, \dots, \alpha_\ell}$ .

THEOREM 2.1. The necessary and sufficient conditions for  $\mu$  to be compact in  $B_{\beta_0, \dots, \beta_\ell, T}^{\alpha_0, \dots, \alpha_\ell}$  are

- (a) for every  $s(s = 1, 2, \dots)$  the set  $\mu$  is compact in  $C^\ell[0, T]$ ; and  
 (b) for any given  $\epsilon > 0$  there exists a natural number  $n_\epsilon$  so that for all  $u(t, x) = \sum_{s=1}^{\infty} u_s(t) \vartheta_s(x) \in \mu$ ,

$$\sum_{i=1}^{\ell} \left\{ \sum_{s=n_\epsilon}^{\infty} \left[ \lambda_s^{\alpha_i} \max_{0 \leq t \leq T} |u_s^{(i)}(t)| \right]^{\beta_i} \right\}^{1/\beta_i} < \epsilon.$$

This theorem can be proved analogously as in ([9] page 277-278).

LEMMA 1. For any almost everywhere solution  $u(t, x)$  of (2.1) - (2.3) functions  $u_s(t) = \int_{\Omega} u(t, x) \vartheta_s(x) dx$  satisfy the following system ([7], [8])

$$u_s(t) = \phi_s \cos \lambda_s t + \frac{\psi_s}{\lambda_s} \sin \lambda_s t + \frac{1}{\lambda_s} \int_0^t \int_{\Omega} F(u(\tau, x)) \cdot \vartheta_s(x) \sin \lambda_s(t - \tau) dx d\tau, (s = 1, 2, \dots), \quad (2.9)$$

where

$$\phi_s = \int_{\Omega} \psi(x) \vartheta_s(x) dx, \quad \psi_s = \int_{\Omega} \psi(x) \vartheta_s(x) dx.$$

### 3. ASSUMPTION AND RESULTS.

THEOREM 3.1. Let

1.  $a_{ij}(x)$  are continuously differentiable on  $\bar{\Omega}$  and  $a(x)$  continuous on  $\bar{\Omega}$ ;
2. The eigenfunctions  $\vartheta_s$  are twice continuously differentiable on  $\bar{\Omega}$ ;
3.  $\phi(x) \in W_2^2(\Omega) \cap D^o(\Omega)$ ,  $\psi(x) \in D^o(\Omega)$ ;
4.  $F: B_{2,2,T}^{2,1} \cup (W_2^2(Q_T) \cap B_{2,2,T}^{1,0}) \rightarrow W_{x,t,2}^{1,0}(Q_T)$  and satisfies

$$\|F(u(t,x))\|_{W_2^1(\Omega)} \leq c(t) + d(t) \|u\|_{B_{2,2,t}^{2,1}} \quad (3.1)$$

for all  $u \in B_{2,2,T}^{2,1}$ ; where  $c(t), d(t) \in L_2(0, T)$ .

5. For any  $u, v \in \mathfrak{K}_o$  (where  $\mathfrak{K}_o$  is the sphere  $\|u\|_{B_{2,2,T}^{2,1}} \leq C_o$ )

$$\|F(u, t, x) - F(v, t, x)\|_{W_2^1(\Omega)} \leq g(t) \|u - v\|_{B_{2,2,t}^{2,1}}, \quad g(t) \in L_2(0, T), \quad (3.2)$$

where

$$C_o = \left\{ \left[ 2 \|W(t, x)\|_{B_{2,2,T}^{2,1}}^2 + 16Ta_o^2 \|c(t)\|_{L_2(0,T)}^2 \right] \exp \left[ 16Ta_o^2 \|d(t)\|_{L_2(0,T)}^2 \right] \right\}^{1/2} \quad (3.3)$$

and

$$a_o^2 = \max \left\{ n \cdot \max_{ij} \left\{ \|a_{ij}(x)\|_{C(\bar{\Omega})} \right\}, \|a(x)\|_{C(\bar{\Omega})} \right\}$$

6. For any  $u \in B_{2,2,T}^{2,1} \cup (W_2^2(Q_T) \cap B_{2,2,T}^{1,0})$  and  $t \in [0, T]$   $F(u(t, x)) \in D^o(\Omega)$ .  
Then the problem (2.1) - (2.3) has a unique solution,

PROOF. Let

$$W(x, t) = \sum_{s=1}^{\infty} (\phi_s \cos \lambda_s t + \frac{\psi_s}{\lambda_s} \sin \lambda_s t) \vartheta_s(x), \quad (3.4)$$

and

$$PF(u) = \sum_{s=1}^{\infty} \frac{1}{\lambda_s} \int_0^t \int_{\Omega} F(u(\tau, x)) \cdot \vartheta_s(x) \sin \lambda_s(t - \tau) dx d\tau \cdot \vartheta_s(x) \quad (3.5)$$

From (3.4) and (3.5) let us assume that

$$Q(u) = W + PF(u) \quad (3.6)$$

Then it is easy to see that the operator  $Q$  acts in  $B_{2,2,T}^{2,1}$  and satisfies Lipschitz condition

$$\|Q(u) - Q(v)\|_{B_{2,2,t}^{2,1}} \leq 2\sqrt{T}a_o \|g(t)\|_{L_2(0,T)} \|u - v\|_{B_{2,2,t}^{2,1}} \quad (3.7)$$

in the sphere  $\mathfrak{K}_o$ .

Consider the sequence  $u_k(t, x) = Q(u_{k-1}(t, x))$  in  $B_{2,2,T}^{2,1}$  where  $u_o(t, x) = 0$ . Using (3.1) and

the mathematical induction we get for any  $k(k = 1, 2, 3, \dots)$  and  $t \in [0, T]$ :

$$\begin{aligned}
 \|u_k\|_{B_{2,2,t}^2}^2 &\leq 2\|W\|_{B_{2,2,T}^2}^2 + 8Ta_o^2 \int_0^t \|F(u_{k-1}(\tau, x))\|_{W_2^1(\Omega)}^2 d\tau \\
 &\leq 2\|W\|_{B_{2,2,T}^2}^2 + 16Ta_o^2 \left\{ \int_0^T c^2(\tau) d\tau + \int_0^t d^2(\tau) \|u_{k-1}\|_{B_{2,2,t}^2}^2 d\tau \right\} \\
 &= A^2 + \int_0^t \mathfrak{B}^2(\tau) \|u_{k-1}\|_{B_{2,2,t}^2}^2 d\tau \\
 &\leq A^2 + A^2 \int_0^t \mathfrak{B}^2(\tau) d\tau + \dots + A^2 \frac{\left\{ \int_0^t \mathfrak{B}^2(\tau) d\tau \right\}^{k-1}}{(k-1)!},
 \end{aligned} \tag{3.8}$$

where

$$A^2 = 2\|W\|_{B_{2,2,T}^2}^2 + 16Ta_o^2 \|c(t)\|_{L_2(0,T)}^2, \tag{3.9}$$

and

$$\mathfrak{B}^2(t) = 16Ta_o^2 d^2(t)$$

From (3.8) for any  $k(k = 1, 2, \dots)$ , we get

$$\|u_k\|_{B_{2,2,t}^2}^2 \leq A^2 \cdot \exp \left\{ \int_0^t \mathfrak{B}^2(\tau) d\tau \right\} = C_o^2 \tag{3.10}$$

i.e., all  $u_k(t, x)$  are contained in the sphere  $\mathfrak{K}_o$ . Further, using (3.2) and (3.3) we get for any  $t \in [0, T]$  and  $k(k = 1, 2, 3, \dots)$

$$\begin{aligned}
 \|u_{k+1} - u_k\|_{B_{2,2,t}^2}^2 &\leq 4Ta_o^2 \|F(u_k(\tau, x)) - F(u_{k-1}(\tau, x))\|_{L_2(\Omega)}^2 d\tau \\
 &\leq 4Ta_o^2 \int_0^t g^2(\tau) \|u_k - u_{k-1}\|_{B_{2,2,t}^2}^2 d\tau \\
 &\leq \|u_1 - u_o\|_{B_{2,2,T}^2}^2 \frac{\left\{ 4Ta_o^2 \int_0^t g^2(\tau) d\tau \right\}^k}{k!} \\
 &= \|u_1\|_{B_{2,2,T}^2}^2 \frac{\left\{ 4Ta_o^2 \int_0^t g^2(\tau) d\tau \right\}^k}{k!} \leq C_o^2 \frac{\left\{ 4Ta_o^2 \int_0^t g^2(\tau) d\tau \right\}^k}{k!}
 \end{aligned} \tag{3.11}$$

Therefore,

$$\|u_{k+1} - u_k\|_{B_{2,2,T}^2}^2 \leq C_o^2 \frac{\left\{ 4Ta_o^2 \|g(t)\|_{L_2(0,T)}^2 \right\}^k}{k!}, (k = 1, 2, \dots) \tag{3.12}$$

Then  $\{u_k(t, x)\}$  is a fundamental sequence in  $B_{2,2,T}^{2,1}$ . Since  $B_{2,2,T}^{2,1}$  complete, then

$$u_k(t, x) \xrightarrow{B_{2,2,T}^{2,1}} u(t, x) \in \mathfrak{K}_o \quad \text{as } k \rightarrow \infty \quad (3.13)$$

Since  $Q$  is continuous in  $\mathfrak{K}_o$ , then from the relation  $u_k(t, x) = Q(u_{k-1}(t, x))$

we have

$$u(t, x) = Q(u(t, x))$$

Therefore, as in (3.11), (3.12) the speed of convergence is governed by the following inequality

$$\begin{aligned} \|u_k - u\|_{B_{2,2,T}^{2,1}}^2 &\leq \|u_o - u\|_{B_{2,2,T}^{1,0}}^2 \frac{\left\{4Ta_o^2 \|g(t)\|_{L_2(0,T)}^2\right\}^k}{k!} \\ &\leq C_o^2 \frac{\left\{4Ta_o^2 \|g(t)\|_{L_2(0,T)}^2\right\}^k}{k!}, \quad (k = 1, 2, \dots). \end{aligned} \quad (3.14)$$

Now to prove the uniqueness let us assume the  $u(t, x) = \sum_{s=1}^{\infty} u_s(t) \ell_2(x)$  solution to (2.1) - (2.3) then  $F(u(t, x)) \in L_2(Q_T)$ . By Lemma (1)  $u_s(t)$  satisfy (2.9); from (2.9) we get

$$\|u(t, x)\|_{B_{2,2,t}^{1,0}} \leq \|W(t, x)\|_{B_{2,2,T}^{1,0}} + 2\sqrt{T} \|F(u(t, x))\|_{L_2(Q_T)} < +\infty \quad (3.15)$$

Therefore  $u \in B_{2,2,t}^{1,0}$ . Since  $u(t, x) \in W_{2,2,T}^{2,0} \cap B_{2,2,T}^{1,0}$  then by (3.1)  $F(u(t, x)) \in W_{x,t,2}^{1,0}(Q_T)$ , but by condition 6 Theorem 2 for all  $t \in [0, T]$ ,  $F(u(t, x)) \in \dot{D}(\Omega)$ . Thus using (2.9) with some manipulation

$$\|u(t, x)\|_{B_{2,2,t}^{2,1}} \leq \|W(t, x)\|_{B_{2,2,T}^{2,1}} + 2\sqrt{T} a_o \|F(u(t, x))\|_{W_{x,t,2}^{1,0}(Q_T)} < +\infty \quad (3.16)$$

Therefore,  $u \in B_{2,2,T}^{2,1}$ . Then, using (3.1), (3.8), (3.10), we get  $\|u(t, x)\|_{B_{2,2,t}^{2,1}} \leq C_o$ . Thus, all almost everywhere solutions (2.1)-(2.3) belong to the sphere  $K_o$  and they are fixed points in  $B_{2,2,T}^{2,1}$  for operator  $Q$ . Let  $u, v$  be two solutions to (2.1)-(2.3), then by (3.2) we get

$$\begin{aligned} \|u - v\|_{B_{2,2,t}^{2,1}}^2 &\leq 4Ta_o^2 \int_0^t \|F(u(\tau, x)) - F(v(\tau, x))\|_{W_2^1(\Omega)}^2 d\tau \\ &\leq 4Ta_o^2 \int_0^t g^2(\tau) \|u - v\|_{B_{2,2,t}^{2,1}}^2 d\tau \end{aligned} \quad (3.17)$$

Therefore, using Belmann's inequality [10] we have

$$\|u - v\|_{2,2,t}^2 = 0 \text{ in } [0, T]. \text{ Therefore, } u = v.$$

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