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COMPACT DIAGONAL LINEAR OPERATORS ON BANACH SPACES WITH UNCONDITIONAL BASES

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ABSTRACT. Let E and F be Banach spaces with equivalent normalized unconditional bases. In this note we show that a bounded diagonal linear operator $T : E \to F$ is compact if and only if its entries tend to 0, using the concept of weak uniform continuity.

KEY WORDS and PHRASES. Weakly uniformly continuous, diagonal operators, compact operators, unconditional bases.
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1. Introduction.

Let E and F be two complex Banach spaces and E^* be the dual space of E. A function $f : E \to F$ is said to be weakly uniformly continuous on bounded subsets of E if for each bounded set $B \subset E$ and $\epsilon > 0$, there are $\phi_1, \cdots, \phi_k \in E^*$ and $\delta > 0$ such that if $x, y \in B$, $|\phi_i(x - y)| \leq \delta$ $(i = 1, \cdots, k)$, then $\|f(x) - f(y)\| \leq \epsilon$. R.M. Aron and J.B. Prolla [1] showed that a bounded linear operator $T : E \to F$ is compact if and only if T is weakly uniformly continuous on bounded subsets of E. Applying this result we generalize the following well-known Hilbert space fact to a Banach space with an unconditional basis: A diagonal bounded linear operator is compact if and only if its entries tend to 0. See, for example, [2, Proposition 4.6].

We recall some relevant definitions and results about a Banach space with an unconditional basis. Let E be a complex Banach space with an unconditional basis (e_n). For every choice of signs $\theta = (\theta_n)$, we have a bounded linear operator M_θ on E defined by

$$M_\theta(\sum a_n e_n) = \sum a_n \theta_n e_n. \tag{1.1}$$

The uniform bounded principle implies that the number $K = \sup \|M_\theta\|$ is finite, which is called the unconditional constant of (e_n). Then for every choice of a complex sequence (a_n) such that $\sum a_n e_n$ converges and every choice of a bounded complex sequence (α_n), we have

$$\|\sum \alpha_n a_n e_n\| \leq 2K(sup|\alpha_n|)\|\sum a_n e_n\|. \tag{1.2}$$

For details see [3].

2. Main Results.

THEOREM 1. Let (e_n) and (f_n) be equivalent normalized unconditional bases of E and F, respectively. Given a bounded sequence (α_n), let $T : E \to F$ be the bounded linear operator with $T(e_n) = \alpha_n f_n$ for each n. Then T is compact if and only if $\alpha_n \to 0$.

Proof. Suppose T is compact. Let (P_n) be the sequence of the natural projections associated with (f_n). Then $(P_n \circ T)$ converges uniformly to T on the closed unit ball B_E, from which it follows that $\alpha_n \to 0$.

Conversely suppose that $\alpha_n \to 0$. We will show that T is weakly uniformly continuous on bounded subsets of E. Let B_r be the closed ball of E with the radius r and the center 0 and C be the positive number with $|\alpha_n| \le C$ for all n. Given $\epsilon > 0$, $x = \sum a_n e_n$ and $y = \sum b_n e_n$ in B_r,

$$\|T(x) - T(y)\| = \|\sum \alpha_n(a_n - b_n)f_n\| \tag{2.1}$$

$$\le C \sum_{n=1}^{N-1} |a_n - b_n| + \|\sum_{n=N}^{\infty} \alpha_n(a_n - b_n)f_n\| \tag{2.2}$$

$$\le C \sum_{n=1}^{N-1} |a_n - b_n| + 2K(\sup_{n \ge N} |\alpha_n|)\|\sum_{n=N}^{\infty} (a_n - b_n)f_n\|, \tag{2.3}$$

where K is the unconditional constant of (f_n). Since (e_n) and (f_n) are equivalent, it is easy to see that

$$\|\sum_{n=N}^{\infty} (a_n - b_n)f_n\| \le 2(1 + K)r\|T\|. \tag{2.4}$$

Let (f_n^*) be the sequence of coefficient functionals associated with (f_n). Since $\alpha_n \to 0$, choosing sufficiently large N, we conclude that

$$\|T(x) - T(y)\| \le \epsilon \tag{2.5}$$

if $|f_1^*(x - y)|, \cdots, |f_{N-1}^*(x - y)|$ are sufficiently small. Hence T is weakly uniformly continuous on bounded subset of E.

From the above proof it is easy to see that the Banach space c_0 of null complex sequences is isomorphic with the Banach space of compact diagonal linear operators $T : E \to F$, where E and F are Banach spaces with equivalent unconditional bases. We would like to remark that if (e_n) and (f_n) are not equivalent, then given a bounded complex sequence (α_n), the map $T(e_n) = \alpha_n f_n$ is not necessarily extended to a bounded linear operator from E into F. For example take $E = \ell_2$, $F = \ell_1$ and $\alpha_n = 1$ for all n with respect to the canonical bases of them.

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