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SPECIAL MEASURES AND REPLETENESS

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ABSTRACT. Let X be an abstract set and $\mathcal{L}$ a lattice of subsets of X. To each lattice-regular measure μ, we associate two induced measures $\hat{\mu}$ and $\tilde{\mu}$ on suitable lattices of the Wallman space $I_R(\mathcal{L})$ and another measure μ' on the space $I_R^\sigma(\mathcal{L})$. We will investigate the reflection of smoothness properties of μ onto $\hat{\mu}, \tilde{\mu}$ and μ'; and try to set some new criterion for repleteness and measure repleteness.

KEY WORDS AND PHRASES. Replete and measure replete lattices, Lattice regular measure, Wallman space and remainder, σ-smooth, τ-smooth and tight measures, purely finitely additive measures, purely σ-additive measures, purely τ-additive measures.

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1. INTRODUCTION:

Let X be an abstract set and $\mathcal{L}$ a lattice subsets of X. To each lattice regular measure μ, we associate following Bachman and Szeto [1], two induced measures $\hat{\mu}$ and $\tilde{\mu}$ on suitable lattices of subsets of the Wallman space $I_R(\mathcal{L})$ of $(X,\mathcal{L})$; we also associate to μ a measure μ' on the space $I_R^\sigma(\mathcal{L})$ (see below for definitions). We give in section 2, a brief review of the lattice notation and terminology relevant to the paper. We will be consistent with the standard terminology as used, for example, in Alexandroff [2], Frolik [3], Grassi [4], and Nöbeling [5]. We also give a brief review of the principal Theorems of [1] that we need in order to make the paper reasonably self-contained.

2. DEFINITIONS AND NOTATIONS

Let X be an abstract set, then $\mathcal{L}$ is a lattice of subsets of X if for $A,B \subset X$ then $A \cup B \in \mathcal{L}$ and $A \cap B \in \mathcal{L}$. Throughout this work we will always assume that $\varnothing$ and X are in $\mathcal{L}$. If $A \subset X$ then we will denote the complement of A by A' i.e. $A' = X - A$. If $\mathcal{L}$ is a Lattice of subsets of X then $\mathcal{L}'$ is defined $\mathcal{L}' = \{L' \mid L \in \mathcal{L}\}$.

Lattice Terminology

DEFINITIONS 2.1. Let $\mathcal{L}$ be a Lattice of subsets of X. We say that:

1- $\mathcal{L}$ is a δ-Lattice if it is closed under countable intersections.

2- $\mathcal{L}$ is separating or T_1 if for $x,y \in X; x \neq y$ then $\exists L \in \mathcal{L}$ such that $x \in L$ and $y \notin L$.

3- $\mathcal{L}$ is Hausdorff or T_2 if for $x,y \in X; x \neq y$ then $\exists A,B \in \mathcal{L}$ such that $x \in A', y \in B'$ and $A' \cap B' = \varnothing$.

4- $\mathcal{L}$ is disjunctive if for $x \in X$ and $L \in \mathcal{L}$ where $x \notin L; \exists A,B \in \mathcal{L}$ such that $x \in A, L \subset B$ and $A \cap B = \varnothing$.

5- $\mathcal{L}$ is regular if for $x \in X, L \in \mathcal{L}$ and $x \notin L; \exists A,B \in \mathcal{L}$ such that $x \in A', L \subset B'$ and $A' \cap B' = \varnothing$.

6- $\mathcal{L}$ is normal if for $A, B \in \mathcal{L}$ where $A \cap B = \emptyset \exists \tilde{A}, \tilde{B} \in \mathcal{L}$ such that $A \subset \tilde{A}', B \subset \tilde{B}'$ and $\tilde{A}' \cap \tilde{B}' = \emptyset$.

7- $\mathcal{L}$ is compact if $X = \bigcup_{\alpha} L'_{\alpha}$ where $L_{\alpha} \in \mathcal{L}$ then there exists a finite number of L_{α} that cover X i.e.

$X = \bigcup_{i=1}^{n} L'_{\alpha_i}$ where $L'_{\alpha_i} \in \mathcal{L}$.

$A(\mathcal{L})$ = the algebra generated by $\mathcal{L}$.
$\sigma(\mathcal{L})$ = the σ-algebra generated by $\mathcal{L}$.
$\delta(\mathcal{L})$ = the Lattice of countable intersections of sets of $\mathcal{L}$.
$\tau(\mathcal{L})$ = the Lattice of arbitrary intersection of sets of $\mathcal{L}$.
$\rho(\mathcal{L})$ = the smallest class containing $\mathcal{L}$ and closed under countable unions and intersections.

If $A \in \mathcal{A}(\mathcal{L})$ then $A = \bigcup_{i=1}^{n} (L_i - \tilde{L}'_i)$ where the union is disjoint and $L_i, \tilde{L}_i \in \mathcal{L}$.

Measure Terminology

Let $\mathcal{L}$ be a lattice of subsets of X. $M(\mathcal{L})$ will denote the set of finite valued bounded finitely additive measures on $\mathcal{A}(\mathcal{L})$. Clearly since any measure in $M(\mathcal{L})$ can be written as a difference of two non-negative measures there is no loss of generality in assuming that the measures are non-negative, and we will assume so throughout this paper.

DEFINITIONS 2.2.

1- A measure $\mu \in M(\mathcal{L})$ is said to be σ-smooth on $\mathcal{L}$ if for $L_n \in \mathcal{L}$ and $L_n \downarrow \emptyset$ then $\mu(L_n) \to 0$.

2- A measure $\mu \in M(\mathcal{L})$ is said to be σ-smooth on $\mathcal{A}(\mathcal{L})$ if for $A_n \in \mathcal{A}(\mathcal{L}), A_n \downarrow \emptyset$ then $\mu(A_n) \to 0$.

3- A measure $\mu \in M(\mathcal{L})$ is said to be τ-smooth on $\mathcal{L}$ if for $L_\alpha \in \mathcal{L}, \alpha \in \Lambda, L_\alpha \downarrow \emptyset$ then $\mu(L_\alpha) \to 0$.

4- A measure $\mu \in M(\mathcal{L})$ is said to be $\mathcal{L}$-regular if for any $A \in \mathcal{A}(\mathcal{L})$

$$\mu(A) = \sup_{\substack{L \subset A \\ L \in \mathcal{L}}} \mu(L)$$

If $\mathcal{L}$ is a lattice of subsets of X, then we will denote by:

$M_R(\mathcal{L})$ = the set of $\mathcal{L}$-regular measures of $M(\mathcal{L})$

$M_\sigma(\mathcal{L})$ = the set of σ-smooth measures on $\mathcal{L}$ of $M(\mathcal{L})$
$M^\sigma(L)$= the set of σ-smooth measures on $\mathcal{A}(\mathcal{L})$ of $M(\mathcal{L})$
$M_R^\sigma(\mathcal{L})$ = the set of regular measures of $M^\sigma(\mathcal{L})$
$M_R^\tau(\mathcal{L})$ = the set of τ-smooth measures on $\mathcal{L}$ of $M_R(\mathcal{L})$
$M_r^t(\mathcal{L})$ = the set of tight measures on $\mathcal{L}$ of $M_R(\mathcal{L})$.

Clearly

$$M_R^T(\mathcal{L}) \subset M_R^\sigma(\mathcal{L}) \subset M_R(\mathcal{L})$$

DEFINITION 2.3. If $A \in \mathcal{A}(\mathcal{L})$ then μ_x is the measure concentrated at $x \in X$.

$$\mu_x(A) = \begin{cases} 1 \text{ if } x \in A \\ 0 \text{ if } x \notin A \end{cases}$$

$I(\mathcal{L})$ is the subset of $M(\mathcal{L})$ which consists of non-trivial zero-one measures which are finitely additive on $\mathcal{A}(\mathcal{L})$.

$I_R(\mathcal{L})$ = the set of $\mathcal{L}$-regular measures of $I(\mathcal{L})$

$I_\sigma(\mathcal{L})$ = the set of σ-smooth measures on $\mathcal{L}$ of $I(\mathcal{L})$
$I^\sigma(L)$= the set of σ-smooth measures on $\mathcal{A}(\mathcal{L})$ of $I(\mathcal{L})$

$I_\tau(\mathcal{L})$ = the set of τ-smooth measures on $\mathcal{L}$ of $I(\mathcal{L})$

$I_R^\sigma(\mathcal{L})$ = the set of $\mathcal{L}$-regular measures of $I^\sigma(\mathcal{L})$

$I_R^\tau(\mathcal{L})$ = the set of $\mathcal{L}$-regular measures of $I_\tau(\mathcal{L})$

DEFINITION 2.4: If $\mu \in M(\mathcal{L})$ then we define the support of μ to be:

$$S(\mu) = \cap\{L \in \mathcal{L}/\mu(L) = \mu(X)\}.$$

Consequently if $\mu \in I(\mathcal{L})$

$$S(\mu) = \cap\{L \in \mathcal{L}/\mu(L) = 1\}$$

DEFINITION 2.5. If $\mathcal{L}$ is a Lattice of subsets of X, we say that $\mathcal{L}$ is replete if for any $\mu \in I_R^\sigma(\mathcal{L})$ then $S(\mu) \neq \varnothing$.

DEFINITION 2.6. Let $\mathcal{L}$ be a lattice of subsets of X. We say that $\mathcal{L}$ is measure replete if $S(\mu) \neq \varnothing$ for all $\mu \in M_R^\sigma(\mathcal{L}), \mu \neq 0$.

Separation Terminology

Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be two Lattices of subsets of X.

DEFINITION 2.7. $\mathcal{L}_1$ separates $\mathcal{L}_2$ if for $A_2, B_2 \in \mathcal{L}_2$ and $A_2 \cap B_2 = \varnothing$ then there exists $A_1, B_1 \in \mathcal{L}_1$ such that $A_2 \subset A_1, B_2 \subset B_1$ and $A_1 \cap B_2 = \varnothing$.

REMARK 2.1. We now list few known facts found in [1] which will enable us to characterize some previously defined properties in a measure theoretic fashion.

1. $\mathcal{L}$ is disjunctive if and only if $\mu_x \in I_R(\mathcal{L}), \forall x \in X$.
2. $\mathcal{L}$ is regular if and only if for any $\mu_1, \mu_2 \in I(\mathcal{L})$ such that $\mu_1 \leq \mu_2$ on $\mathcal{L}$ we have $S(\mu_1) = S(\mu_2)$.
3. $\mathcal{L}$ is T_2 if and only if $S(\mu) = \varnothing$ or a singleton for any $\mu \in I(\mathcal{L})$.
4. $\mathcal{L}$ is compact if and only if $S(\mu) \neq \varnothing$ for any $\mu \in I_R(\mathcal{L})$.

3. THE INDUCED MEASURES

If $\mathcal{L}$ is a disjunctive lattice of subsets of an abstract set X then there is a Wallman space associated with it. We will briefly review the fundamental properties of this Wallman space, and then associate with a regular lattice measure μ, two measures $\tilde{\mu}$ and $\hat{\mu}$ on certain algebras in the Wallman space (see [1]). We then investigate how properties of μ reflect to those of $\hat{\mu}$ and $\tilde{\mu}$, and conversely, then give a variety of applications of these results. Let X be an abstract set and $\mathcal{L}$ a disjunctive lattice of subsets of X such that $\varnothing$ and X are in $\mathcal{L}$. For any A in $\mathcal{A}(\mathcal{L})$, define $W(A) = \{\mu \in I_R(\mathcal{L}) : \mu(A) = 1\}$. If $A, B \in \mathcal{A}(\mathcal{L})$ then

1) $W(A \cup B) = W(A) \cup W(B)$.
2) $W(A \cap B) = W(A) \cap W(B)$.
3) $W(A') = W(A)'$.
4) $W(A) \subset W(B)$ if and only if $A \subset B$.
5) $W(A) = W(B)$ if and only if $A = B$.
6) $W[\mathcal{A}(\mathcal{L})] = \mathcal{A}[W(\mathcal{L})]$.

Let $W(\mathcal{L}) = \{W(L), L \in \mathcal{L}\}$. Then $W(\mathcal{L})$ is a compact lattice of $I_R(\mathcal{L})$, and $I_R(\mathcal{L})$ with $\tau W(\mathcal{L})$ as the topology of closed sets is a compact T_1 space (the Wallman space) associated with the pair $X, \mathcal{L}$. It is a T_2-space if and only if $\mathcal{L}$ is normal. For $\mu \in M(\mathcal{L})$ we define $\hat{\mu}$ on $\mathcal{A}(W(\mathcal{L}))$ by: $\hat{\mu}(W(A)) = \mu(A)$ where $A \in \mathcal{A}(\mathcal{L})$. Then $\hat{\mu} \in M(W(\mathcal{L}))$, and $\hat{\mu} \in M_R(W(\mathcal{L}))$ if and only if $\mu \in M_R(\mathcal{L})$.

Finally, since $\tau W(\mathcal{L})$ and $W(\mathcal{L})$ are compact lattices, and $W(\mathcal{L})$ separates $\tau W(\mathcal{L})$, then $\hat{\mu}$ has a unique extension to $\tilde{\mu} \in M_R(\tau W(\mathcal{L}))$. We note that by compactness $\hat{\mu}$ and $\tilde{\mu}$ are in $M_R^\tau(W(\mathcal{L}))$ and $M_R^\tau(\tau W(\mathcal{L}))$ respectively, where they are certainly τ-smooth and of course σ-smooth. $\hat{\mu}$ can be extended to $\sigma(W(\mathcal{L}))$ where it is $\delta W(\mathcal{L})$-regular; while $\tilde{\mu}$ can be extended to $\sigma(\tau(W(\mathcal{L})))$, the Borel sets of $I_R(\mathcal{L})$, and is $\tau W(\mathcal{L})$-regular on it. One is now concerned with how further properties of μ reflect over to $\hat{\mu}$ and $\tilde{\mu}$ respectively. The following are known to be true (see [1]) and we list them for the reader's convenience.

THEOREM 3.1. Let $\mathcal{L}$ be a separating and disjunctive lattice of subsets of X, and let $\mu \in M_R(\mathcal{L})$. Then

1. $\mu \in M_R^\sigma(\mathcal{L})$ if and only if $\hat{\mu}^*(X) = \hat{\mu}(I_R(\mathcal{L}))$.

2. $\mu \in M_R^\tau(\mathcal{L})$ if and only if $\tilde{\mu}^*(X) = \tilde{\mu}(I_R(\mathcal{L}))$.

3. If $\mathcal{L}$ is also normal (or T_2) then $\mu \in M_R^\tau(\mathcal{L})$ if and only if X is $\tilde{\mu}^*$-measurable and $\tilde{\mu}^*(X) = \tilde{\mu}(I_R(\mathcal{L}))$.

We now give some further results related to the induced measures $\hat{\mu}$ and $\tilde{\mu}$.

THEOREM 3.2. Let $\mathcal{L}$ be a separating and disjunctive lattice, and $\mu \in M_R(\mathcal{L})$ then $\tilde{\mu}$ is $W(\mathcal{L})$ regular on $(\tau W(\mathcal{L}))'$.

PROOF. We know that $W(\mathcal{L})$ and $\tau W(\mathcal{L})$ are compact lattices and that $W(\mathcal{L})$ separates $\tau W(\mathcal{L})$. Since $\mu \in M_R(\mathcal{L})$ then $\hat{\mu} \in M_R[W(\mathcal{L})]$. Extend $\hat{\mu}$ to $\tau W(\mathcal{L})$. The extension is

$$\tilde{\mu} \in M_R[\tau W(\mathcal{L})] = M_R^\sigma[\tau W(\mathcal{L})] = M_R^\tau[\tau W(\mathcal{L})] = M_R^\tau[\tau W(\mathcal{L})].$$

Let $0 \in [\tau W(\mathcal{L})]'$ then since $\tilde{\mu} \in M_R[\tau W(\mathcal{L})]$ there exists $F \in \tau W(\mathcal{L}), F \subset 0$ and

$$|\tilde{\mu}(0) - \tilde{\mu}(F)| < \varepsilon;\ \varepsilon > 0.$$

Since $F \in \tau W(\mathcal{L})$, $F = \bigcap_{\alpha \in \Lambda} W(L_\alpha), L_\alpha \in \mathcal{L}$. Also since $F \subset 0$ then $F \cap 0' = \varnothing$ i.e. $\bigcap_\alpha W(L_\alpha) \cap 0' = \varnothing$ by compactness there must exist $\alpha_0 \in \Lambda$ such that $W(L_{\alpha 0}) \cap 0' = \varnothing$ thus $\overset{\alpha}{F} \subset W(L_{\alpha 0}) \subset 0'' = 0$ so

$$|\tilde{\mu}(0) - \tilde{\mu}(W(L_{\alpha 0}))| < \in$$

i.e. $\tilde{\mu}$ is $W(\mathcal{L})$ regular on $(\tau W(\mathcal{L}))'$.

THEOREM 3.3. Let $\mu \in M_R(\mathcal{L})$ then $\hat{\mu}^* = \tilde{\mu}$ on $\tau W(\mathcal{L})$.

PROOF. Since $\mu \in M_R(\mathcal{L})$ and $W(\mathcal{L})$ is compact then $\hat{\mu} \in M_R[W(\mathcal{L})] = M_R^\tau[W(\mathcal{L})]$ and since $W(\mathcal{L})$ separates $\tau W(\mathcal{L})$ and $\tau W(\mathcal{L})$ is compact then $\tilde{\mu} \in M_R[\tau W(\mathcal{L})] = M_R^\tau[\tau W(\mathcal{L})]$ furthermore $\tilde{\mu}$ extends $\hat{\mu}$ to $\tau W(\mathcal{L})$ uniquely. Let $F \in \tau W(\mathcal{L})$ then

$$\hat{\mu}^*(F) = \inf \sum_{i=1}^{\infty} \hat{\mu}(A_i), F \subset \bigcup_{i=1}^{\infty} A_i \text{ and } A_i \in \mathcal{A}[W(\mathcal{L})]$$

and since $\hat{\mu} \in M_R^\tau[W(\mathcal{L})]$ then

$$\hat{\mu}(A_i) = \inf \hat{\mu}[W(L'_i)], A_i \subset W(L'_i), L_i \in \mathcal{L}$$

thus $F \subset \bigcup_{i=1}^{\infty} W(L'_i)$ but since $W(\mathcal{L})$ is compact then $F \subset \bigcup_{i=1}^{n} W(L'_i) = W(L')$ where $L \in \mathcal{L}$ and

$$\hat{\mu}^*(F) = \inf \hat{\mu}[W(L')];\ F \subset W(L') \text{ and } L \in \mathcal{L}$$

Now $F \subset W(L') \Rightarrow F \cap W(L) = \varnothing$ then since $W(\mathcal{L})$ separates $\tau W(\mathcal{L}) \exists \tilde{L} \in \mathcal{L}$ such that $F \subset W(\tilde{L})$ and $W(\tilde{L}) \cap W(L) = \varnothing$. Therefore $W(\tilde{L}) \subset W(L')$ and hence

$$\hat{\mu}^*(F) = \inf \hat{\mu}[W(\tilde{L})]: \text{where } F \subset W(\tilde{L}); \tilde{L} \in \mathcal{L}$$

i.e. that $\hat{\mu}^*$ is regular on $\tau W(\mathcal{L})$. On the other hand since $\tau W(\mathcal{L})$ is δ then

$$F = \bigcap_\alpha W(L_\alpha) \text{ and } \tilde{\mu}\left[\bigcap_\alpha W(L_\alpha)\right] = \inf_\alpha \tilde{\mu}(W(L_\alpha)) = \inf \hat{\mu}(W(L_\alpha))$$

where $F \subset W(L_\alpha), L_\alpha \in \mathcal{L}$. Therefore $\hat{\mu}^* = \tilde{\mu}$ on $\tau W(\mathcal{L})$.

THEOREM 3.4. Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be two lattices of subsets of X such that $\mathcal{L}_1 \subset \mathcal{L}_2$ and $\mathcal{L}_1$ separates $\mathcal{L}_2$ If $\nu \in M_R^\sigma(\mathcal{L}_2)$ then $\nu = \mu^*$ on $\mathcal{L}'_2$ and $\nu = \mu_*$ on $\mathcal{L}'_2$ where $\mu = \nu|_{\mathcal{L}_1}$.

PROOF. Let $\nu \in M_R^\sigma(\mathcal{L}_2)$ then since $\mathcal{L}_1$ separates $\mathcal{L}_2, \mu \in M_R^\sigma(\mathcal{L}_1)$. Since $\mathcal{L}_1 \subset \mathcal{L}_2$ then $\sigma(\mathcal{L}_2) \subset \sigma(\mathcal{L}_2)$; Let $\in \subset X$ then

$$\nu^*(E) = \inf_{E \subset B, B \in \sigma(\mathcal{L}_2)} \nu(B) \leq \inf_{E \subset A, A \in \sigma(\mathcal{L}_1)} \nu(A) = \mu^*(E)$$

therefore, $\nu^* \leq \mu^*$. Now on $\mathcal{L}_2, \nu^* =\leq \mu^*$. Suppose $\exists L_2 \in \mathcal{L}_2$ such that $\nu(L_2) < \mu^*(L_2)$ then since

$$\nu \in M_R^\sigma(\mathcal{L}_2), \nu(L_2) = \inf \nu(\tilde{L}'_2), L_2 \subset \tilde{L}'_2 \text{ and } \tilde{L}_2 \in \mathcal{L}_2$$

then $L_2 \cap \tilde{L}_2 = \emptyset$ and by separation $\exists L_1, \tilde{L}_2 \in \mathcal{L}_1$ such that $L_2 \subset L_1, \subset \tilde{L}'_1 \subset \tilde{L}'_2$ and therefore

$$\begin{aligned}\nu(L_2) &= \inf_\alpha \mu(L_{1\alpha}) \text{ where } L_2 \subset L_{1\alpha} \\ &= \inf_\beta \nu(\tilde{L}'_{2\beta}) \text{ where } L_2 \subset \tilde{L}'_{2\beta} \\ &< \mu^*(L_2)\end{aligned}$$

$\forall \varepsilon > 0 \exists L_1 \in \mathcal{L}_1$ such that $L_2 \subset L_1$ and $\mu(L_1) - \varepsilon < \nu(L_2) < \mu(L_1)$ but since $L_2 \subset L_1$ then $\mu^*(L_2) \leq \mu(L_1) < \nu(L_2) + \varepsilon$ which is a contradiction to our assumption. Therefore $\nu = \mu^*$ on $\mathcal{L}_2$ and thus $\nu = \mu_*$ on $\mathcal{L}'_2$ This theorem is a generalization of the previous one in which we used the compactness of $W(\mathcal{L})$ to have a regular restriction of the measure. Next consider the space $I_R^\sigma(\mathcal{L})$ and the induced measure μ'.

DEFINITION 3.1. Let $\mathcal{L}$ be a disjunctive lattice of subsets of X.

1) $W_\sigma(L) = \{\mu \in I_R^\sigma(\mathcal{L}) \mid \mu(L) = 1\}; L \in \mathcal{L}$
2) $W_\sigma(\mathcal{L}) = \{W_\sigma(L), L \in \mathcal{L}\}$
3) $W_\sigma(A) = \{\mu \in I_R^\sigma(\mathcal{L}) \mid \mu(A) = 1\},\ A \in \mathcal{A}(\mathcal{L})$
4) $W_\sigma(\mathcal{L}) = W(\mathcal{L}) \cap I_R^\sigma(\mathcal{L})$

The following properties hold:

PROPOSITION 3.1. Let $\mathcal{L}$ be a disjunctive lattice then for $A, B \in \mathcal{A}(\mathcal{L})$

1) $W_\sigma(A \cap B) = W_\sigma(A) \cup W_\sigma(B)$
2) $W_\sigma(A \cup B) = W_\sigma(A) \cap W_\sigma(B)$
3) $W_\sigma(A') = W_\sigma(A)'$
4) $W_\sigma(A) \subset W_\sigma(B)$ if and only if $A \subset B$
5) $\mathcal{A}[W_\sigma(\mathcal{L})] = W_\sigma[\mathcal{A}(\mathcal{L})]$

The proof is the same as for $W(\mathcal{L})$ by simply using the properties of $W(\mathcal{L})$ and the fact that $W_\sigma(A) = W(A) \cap I_R^\sigma(\mathcal{L})$ and $W_\sigma(B) = W(B) \cap I_R^\sigma(\mathcal{L})$.

REMARK 3.1. It is not difficult to show that $\sigma[W_\sigma(\mathcal{L})] = W_\sigma[\sigma(\mathcal{L})]$. Also, for each $\mu \in M(\mathcal{L})$ we define μ' on $\mathcal{A}[W'_\sigma(\mathcal{L})]$ as follows:

$$\mu'[W_\sigma(A)] = \mu(A) \text{ where } A \in \mathcal{A}(\mathcal{L})$$

μ' is defined and the map $\mu \to \mu'$ from $M(\mathcal{L})$ to $M(W_\sigma(\mathcal{L}))$ is onto. In addition, it can readily be checked that,

THEOREM 3.5. Let $\mathcal{L}$ be disjunctive then

1) $\mu \in M(\mathcal{L})$ if and only if $\mu' \in M[W_\sigma(\mathcal{L})]$

2) $\mu \in M_R(\mathcal{L})$ if and only if $\mu' \in M_R[W_\sigma(\mathcal{L})]$

3) $\mu \in M_\sigma(\mathcal{L})$ if and only if $\mu' \in M_\sigma[W_\sigma(\mathcal{L})]$

4) $\mu \in M^\sigma(\mathcal{L})$ if and only if $\mu' \in M^\sigma[W_\sigma(\mathcal{L})]$

5) $\mu \in M_R^\sigma(\mathcal{L})$ if and only if $\mu' \in M_R^\sigma[W_\sigma(\mathcal{L})]$

THEOREM 3.6. Let $\mathcal{L}$ be a separating and disjunctive lattice of subsets of X, and let $\mu \in M_R^\sigma(\mathcal{L})$. Then

1. $\mu' \in M_R^\tau(W_\sigma(\mathcal{L}))$ if and only if $\hat{\mu}^*(I_R^\sigma(\mathcal{L})) = \hat{\mu}(I_R(\mathcal{L}))$.

2. If $\mathcal{L}$ is also normal or T_2 then $\mu' \in M_R^\tau(W_\sigma(\mathcal{L}))$ if and only if $I_R^\sigma(\mathcal{L})$ is $\tilde{\mu}^*$-measurable and $\tilde{\mu}^*(I_R^\sigma(\mathcal{L})) = \tilde{\mu}(I_R(\mathcal{L}))$.

We note some consequences.

COROLLARY 3.1. If $\mathcal{L}$ is a separating, disjunctive and replete lattice of subsets of X, then $\mu' \in [M_R^\tau(\mathcal{L})]$ implies $\mu \in M_R^\tau(\mathcal{L})$.

PROOF. Since $\mathcal{L}$ is replete then $X = I_R^\sigma(\mathcal{L})$ then from the previous theorem we have

$$\tilde{\mu}(I_R(\mathcal{L})) = \tilde{\mu}^*(I_R^\sigma(\mathcal{L})) = \tilde{\mu}^*(X)$$

i.e. $\mu \in M_R^\tau(\mathcal{L})$ from theorem 3.1.

COROLLARY 3.2. Let $\mathcal{L}$ be separating and disjunctive. If $\mu' \in M_R^\tau(W_\sigma(\mathcal{L})) \Rightarrow \mu \in M_R^\tau(\mathcal{L})$ then $\mathcal{L}$ is replete.

PROOF. Let $\mu \in I_R^\sigma(\mathcal{L})$ then since $W_\sigma(\mathcal{L})$ is replete $\mu' \in I_R^\tau[W_\sigma(\mathcal{L})]$ then by hypothesis $\mu \in I_R^\tau(\mathcal{L})$ therefore $I_R^\sigma(\mathcal{L}) = I_R^\tau(\mathcal{L})$ or $\mathcal{L}$ is replete. If we combine the two corollaries we get the following:

THEOREM 3.7. Let $\mathcal{L}$ be separating and disjunctive. Then $\mathcal{L}'$ is replete if and only if $\mu' \in M_R^\tau(W_\sigma(\mathcal{L})) \Rightarrow \mu \in M_R^\tau(\mathcal{L})$.

THEOREM 3.8. Let $\mathcal{L}$ be a separating, disjunctive, normal and replete lattice. Then

$$\mu' \in M_R^\tau[W_\sigma(\mathcal{L})] \text{ if and only if } \mu \in M_R^\tau(\mathcal{L}).$$

PROOF.

1. Let $\mu' \in M_R^\tau[W_\sigma(\mathcal{L})]$ then since $\mathcal{L}$ is replete then $X = I_R^\sigma(\mathcal{L})$ and X is $\tilde{\mu}^*$-measurable and

$$\tilde{\mu}^*(I_R^\sigma(\mathcal{L})) = \tilde{\mu}(I_R(\mathcal{L})) = \tilde{\mu}(X)$$

then by theorem 3.1 we get that $\mu \in M_R^\tau(\mathcal{L})$.

2. Conversely suppose $\mu \in M_R^\tau(\mathcal{L})$ then from theorem 3.1 we get that

$$\tilde{\mu}^*(X) = \tilde{\mu}(I_R(\mathcal{L}))$$

and X is $\tilde{\mu}^*$-measurable but $X \subset I_R^\sigma(\mathcal{L}) \subset I_R(\mathcal{L})$ therefore $\tilde{\mu}^*(I_R^\sigma(\mathcal{L})) = \tilde{\mu}(I_R(\mathcal{L}))$, then since $\mathcal{L}$ is replete $X = I_R^\sigma(\mathcal{L})$ so $\tilde{\mu}^*(X) = \tilde{\mu}^*(I_R^\sigma(\mathcal{L})) = \tilde{\mu}(I_R(\mathcal{L}))$ then from theorems 3.1 and 3.7 $\mu' \in M_R^\tau[W_\sigma(\mathcal{L})]$.

4. SPECIAL MEASURES AND REPLETENESS

In this section we define a purely finitely additive measure (p. f. a.), a purely σ-additive measure (p. σ. a.) and a purely τ-additive measure (p. τ. a.) and for each type we give a characterization theorem. Then we will define strong σ-additive measures (s. σ. a.) and (s. τ. a.) measures and give for each a characterization theorem. Finally we will investigate relationships among these measures under repleteness.

LEMMA 4.1. Let $\mathcal{L}$ be a lattice of subsets of X and $\mu \in M_R(\mathcal{L})$.

1. Consider $\hat{\mu}$ on $\sigma[W(\mathcal{L})]$; we saw in earlier work that $\hat{\mu}$ is $\delta(W(\mathcal{L}))$ regular on $\sigma[W(\mathcal{L})]$.

   Let $H \subset I_R(\mathcal{L})$ such that $\hat{\mu}^*(H) = a \neq 0$ then $\exists\rho$ countably additive on $\sigma[\tau W(\mathcal{L})]$ and $\tau - W(\mathcal{L})$ regular such that $0 \le \rho \le \hat{\mu}$ and $\rho^*(H) = \rho(I_R(\mathcal{L})) = a \neq 0$.

2. Consider $\tilde{\mu}$ in $\sigma[\tau W(\mathcal{L})]$; we say that $\tilde{\mu}$ is $\tau W(\mathcal{L})$ regular on $\sigma[\tau W(\mathcal{L})]$.

   Let $H \subset I_R(\mathcal{L})$ such that $\tilde{\mu}^*(H) = a \neq 0$ then $\exists\rho$ countably additive on $\tau W(\mathcal{L})$ regular on $\sigma[\tau W(\mathcal{L})]$ such that $0 \le \rho \le \tilde{\mu}$ and $\rho^*(H) = \rho(I_R^\sigma(\mathcal{L})) = a$

DEFINITION 4.1.

1. Let $\mu \in M_R(\mathcal{L})$; we say that μ is p. f. a. if for $\gamma \in M_\sigma(\mathcal{L})$ and $0 \le \gamma \le \mu$ on $\mathcal{A}(\mathcal{L})$ then $\gamma = 0$.
2. Let $\mu \in M_R(\mathcal{L})$; we say that μ is p. σ. a. if for $\gamma \in M_\sigma(\mathcal{L}), \gamma$ τ-smooth on $\mathcal{L}$ and $0 \le \gamma \le \mu$ then $\gamma = 0$.

THEOREM 4.1. Let $\mathcal{L}$ be a separating and disjunctive lattice and $\mu \in M_R(\mathcal{L})$ then:

1. μ is p. f. a. $\Rightarrow \hat{\mu}^*(X) = 0$.
2. μ is p. σ. a. $\Rightarrow \tilde{\mu}^*(X) = 0$.

If we further assume that $\mathcal{L}$ is δ and $\sigma(\mathcal{L}) = \rho(\mathcal{L})$ then the converses are true.

PROOF. The proof will be given only for part (1) and is similar for the second one.

1. Suppose μ is purely finitely additive. If $\hat{\mu}^*(X) = a \neq 0$ then from previous Lemma 4.1 there exists $\rho \in M_R[W(\mathcal{L})] = M_R^\tau[W(\mathcal{L})]$ such that

$$0 \le \rho \le \hat{\mu} \text{ and } \rho^*(X) = \rho(I_R(\mathcal{L})) = a; \text{ then}$$

$$\rho = \gamma \text{ and } \gamma \in M_R^\sigma(\mathcal{L}) \text{ so}$$

$$0 \le \rho = \hat{\gamma} \le \hat{\mu} \Rightarrow 0 \le \gamma \le \mu \Rightarrow \gamma = 0$$

   from the definition of purely finitely additive which is a contradiction because

$$\hat{\gamma}[I_R(\mathcal{L})] = a \neq 0 \text{ and therefore } \hat{\mu}^*(X) = 0.$$

2. Conversely if $\hat{\mu}^*(X) = 0$ and $0 \le \gamma \le \mu$ on $\mathcal{A}(\mathcal{L})$ where $\gamma \in M_\sigma(\mathcal{L})$ and $\mathcal{L}$ is δ and $\rho(\mathcal{L}) = \sigma(\mathcal{L})$ then $\gamma \in M_R^\sigma(\mathcal{L})$ and $0 \le \hat{\gamma} \le \hat{\mu}$ on $\mathcal{A}[W(\mathcal{L})]$ then $0 \le \gamma \le \hat{\mu}$ on $\mathcal{A}(\mathcal{L})$; and therefore

$$0 \le \gamma^* \le \mu^* \text{ and since } \hat{\mu}^*(X) = 0 \Rightarrow \hat{\gamma}^*(X) = 0 = \hat{\gamma}[I_R(\mathcal{L})]$$

   hence $\gamma = 0$ i.e. μ is purely finitely additive.

DEFINITIONS 4.2. Let $\mathcal{L}$ be any lattice of subsets of X.

1. Let $\mu \in M_R(\mathcal{L})$, we say that μ is σ. f. a. if for γ such that $0 \le \gamma \le \mu$ on $\mathcal{A}(\mathcal{L})$ and $\gamma' \in M^{\sigma}[W_{\sigma}(\mathcal{L})]$ then $\gamma = 0$.

2. Let $\mu \in M_R^{\sigma}(\mathcal{L})$, we say that μ is s. σ. a. if for γ such that $0 \le \gamma \le \mu$ on $\mathcal{A}(\mathcal{L})$ and $\gamma' \in M^{\sigma}[W_{\sigma}(\mathcal{L})], \gamma'$ τ-smooth on $W_{\sigma}(\mathcal{L})$ then $\gamma = 0$.

LEMMA 4.2. Let $\mathcal{L}$ be a disjunctive lattice of subsets of X. If $\lambda \in M_R(\tau W(\mathcal{L})) - M_R^{\tau}(\tau W(\mathcal{L}))$ and $\lambda^*(I_R^{\sigma}(\mathcal{L})) = \lambda(I_R(\mathcal{L}))$ then $\exists \mu \in M_R(\mathcal{L})$ such that $\lambda = \tilde{\mu}$ and $\mu' \in M_R^{\tau}[W_{\sigma}(\mathcal{L})]$. The proof is not difficult.

THEOREM 4.2. Let $\mathcal{L}$ be a disjunctive lattice of subsets of X. Let $\mu \in M_R^{\sigma}(\mathcal{L})$ then:

1. If μ is s. σ. a. then $\tilde{\mu}^*(I_R^{\sigma}(\mathcal{L})) = 0$.

2. If $W_{\sigma}(\mathcal{L})$ is δ, $\sigma[W_{\sigma}(\mathcal{L})] = \rho[W_{\sigma}(\mathcal{L})]$ and $\tilde{\mu}^*(I_R^{\sigma}(\mathcal{L})) = 0$ then μ is s. σ. a.

PROOF.

1. Suppose μ is strong σ additive but $\tilde{\mu}^*(I_R^{\sigma}(\mathcal{L})) = a \neq 0$ then from lemma (4.1) $\exists \rho$ countably additive on $\sigma[\tau W(\mathcal{L})]$ and $\tau W(\mathcal{L})$ regular such that $0 \le \rho \le \tilde{\mu}$ and $\rho^*(I_R^{\sigma}(\mathcal{L})) = \rho(I_R(\mathcal{L})) = a$ from previous lemma 4.2 $\rho = \tilde{\gamma}$ where $\gamma' \in M_R^{\tau}(W_{\sigma}(\mathcal{L}))$ then

$$0 \le \rho = \tilde{\gamma} \le \tilde{\mu} \Rightarrow 0 \le \tilde{\gamma} \le \tilde{\mu} \Rightarrow 0 \le \gamma \le \mu$$

and since μ is s. σ. a. then $\gamma = 0$ which is a contradiction to the fact that

$$\rho(I_R(\mathcal{L})) = \tilde{\gamma}(I_R(\mathcal{L})) = a \neq 0$$

and hence $\tilde{\mu}^*(I_R^{\sigma}(\mathcal{L})) = 0$.

2. Suppose $W_{\sigma}(\mathcal{L})$ is δ, $\sigma[W_{\sigma}(\mathcal{L})] = \rho[W_{\sigma}(\mathcal{L})]$ and $\tilde{\mu}^*(I_R(\mathcal{L})) = 0$. Let $\gamma \in M(\mathcal{L}), 0 \le \gamma \le \mu$ and $\gamma' \in M^{\sigma}[W_{\sigma}(\mathcal{L})]$ and τ-smooth on $W_{\sigma}(\mathcal{L})$ then $\gamma' \in M_R^{\sigma}[W_{\sigma}(\mathcal{L})]$ and even $\gamma' \in M_R^{\tau}[W_{\sigma}(\mathcal{L})]$. So

$$0 \le \tilde{\gamma} \le \tilde{\mu} \text{ on } \mathcal{A}[W(\mathcal{L})]$$

and therefore $0 \le \gamma' \le \mu'$ on $\mathcal{A}[W_{\sigma}(\mathcal{L})]$. Furthermore $0 \le \tilde{\gamma}^* \le \tilde{\mu}^*$ and since $\tilde{\mu}^*(I_R^{\sigma}(\mathcal{L})) = 0$ then $\tilde{\gamma}^*(I_R^{\sigma}(\mathcal{L})) = \tilde{\gamma}(I_R^{\sigma}(\mathcal{L})) = 0$ i.e. $\gamma = 0$ i.e. μ is s. σ. a.

NOTE. If $\mathcal{L}$ is δ and $\sigma(\mathcal{L}) = \rho(\mathcal{L})$ then $W_{\sigma}(\mathcal{L})$ is δ and $\sigma[W_{\sigma}(\mathcal{L})] = \rho(W_{\sigma}(\mathcal{L}))$ will hold.

PROPOSITION 4.1. Let $\mathcal{L}$ be separating and disjunctive if $\mathcal{L}$ is also δ and $\sigma(\mathcal{L}) = \rho(\mathcal{L})$ then μ is s. σ. a. $\Rightarrow \mu$ is p. σ. a.

PROOF. μ is s. σ. a. $\Rightarrow \tilde{\mu}^*(I_R^{\sigma}(\mathcal{L})) = 0 \Rightarrow \tilde{\mu}^*(X) = 0 : \tilde{\mu}^*(X) = 0$ and $\mathcal{L}$ is δ and $\rho(\mathcal{L}) = \sigma(\mathcal{L}) \Rightarrow \mu$ is p. σ. a.

PROPOSITION 4.2. If $\mathcal{L}$ is disjunctive then μ is s. f. a. if and only if μ is p. f. a.

PROOF.

1. Suppose μ is s. f. a. and $\gamma' \in M_{\sigma}(\mathcal{L}); 0 \le \gamma \le \mu$ then $\gamma' \in M^{\sigma}[W_{\sigma}(\mathcal{L})]$ and $0 \le \gamma \le \mu \Rightarrow \gamma = 0$ by s. f. a. Therefore μ is p. f. a.

2. Suppose μ is p. f. a. and $\gamma' \in M^{\sigma}[W_{\sigma}(\mathcal{L})]; 0 \le \gamma \le \mu$ then $\gamma \in M^{\sigma}(\mathcal{L})$ and $0 \le \gamma \le \mu \Rightarrow \gamma = 0$ by purely finitely additive. Therefore μ is s. f. a.

PROPOSITION 4.3. If $\mathcal{L}$ is replete then μ is s. σ. a. if and only if μ is p. σ. a.

PROOF. $\mathcal{L}$ replete $\Rightarrow X = I_R^{\tau}(\mathcal{L}) = I_r^{\sigma}(\mathcal{L})$ then $\mathcal{L} = W_\sigma(\mathcal{L})$ and so $\gamma \in M^{\sigma}(\mathcal{L})$ and τ-smooth on $\mathcal{L} \Leftrightarrow \gamma' \in M^{\sigma}(W_\sigma(\mathcal{L}))$ and τ-smooth or W_σ therefore the definitions are equivalent.

THEOREM 4.3. Suppose $\mathcal{L}$ is separating, disjunctive and δ and $\sigma(\mathcal{L}) = \rho(\mathcal{L})$ then $\mathcal{L}$ is replete if and only if for any $\mu \in M_R^{\sigma}(\mathcal{L})$, μ is p. σ. a. $\Rightarrow \mu$ is s. σ. a.

PROOF.

1. We saw in proposition 4.3 that if $\mathcal{L}$ is replete then p. σ. a. $\Leftrightarrow$ s. σ. a.
2. Conversely suppose that μ is p. σ. a. $\Rightarrow \mu$ is s. σ. a. for any $\mu \in M_R^{\sigma}(\mathcal{L})$ but $X \neq I_R^{\sigma}(\mathcal{L})$. Let $\mu \in I_R^{\sigma}(\mathcal{L})$ then $\tilde{\mu}$ is $\tau W(\mathcal{L})$ regular and $S(\tilde{\mu}) = \{\mu\}, \tilde{\mu}^*(X) = 0$. Now since $\tilde{\mu}^*(X) = 0, \mathcal{L}$ is δ and $\sigma(\mathcal{L}) = \rho(\mathcal{L})$ then from theorem 4.1 μ is purely σ additive by assumption; but μ is s. σ. a. $\Rightarrow \tilde{\mu}^*(I_R^{\sigma}(\mathcal{L})) = 0$ from proposition 4.2; which is a contradiction because $\mu \in M_R^{\sigma}()$ and $\tilde{\mu}[\{\mu\}] = 1$. Therefore $X = I_R^{\sigma}(\mathcal{L})$.

DEFINITION 4.3. Let $\mu \in M_R^{\tau}(\mathcal{L})$.

1. We say that μ is p. τ a. if for $\gamma \in M_\sigma(\mathcal{L})$, $0 \le \gamma \le \mu$, and $\gamma\mathcal{L}$-tight then $\gamma = 0$.
2. We say that μ is s. τ. a. if for $\gamma' \in M^{\sigma}[W_\sigma(\mathcal{L})], 0 \le \gamma \le \mu$ on $\mathcal{A}(\mathcal{L})$ and γ' is $W_\sigma(\mathcal{L})$-tight then $\gamma = 0$.

THEOREM 4.4. Let $\mathcal{L}$ be a separating, disjunctive and normal lattice. If $\mu \in M_R^{\tau}(\mathcal{L})$ then:

1. μ is p. τ. a. $\Rightarrow \tilde{\mu}^*(I_R(\mathcal{L}) - X) = \tilde{\mu}(I_R(\mathcal{L}))$.
2. μ is s. τ. a. $\Rightarrow \tilde{\mu}^*(I_R(\mathcal{L}) - I_R^{\sigma}(\mathcal{L})) = \tilde{\mu}(I_R(\mathcal{L}))$.

If we further assume that $\mathcal{L}$ is δ and $\sigma(\mathcal{L}) = \rho(\mathcal{L})$ then the converses are true.

PROOF. We will prove only the second proposition and the proof of the first is similar.

2.a) Suppose μ is s. τ. a. but $\tilde{\mu}^*(I_R(\mathcal{L}) - I_R^{\sigma}(\mathcal{L})) < \tilde{\mu}(I_R(\mathcal{L}))$, then there exists $G \in [\tau W(\mathcal{L})]'$ such that $I_R(\mathcal{L}) - I_R^{\sigma}(\mathcal{L}) \subset G$ and $\tilde{\mu}(G) < \tilde{\mu}(I_R(\mathcal{L}))$. Let $F = I_R(\mathcal{L}) - G, F \in \tau W(\mathcal{L})$ then $F \subset I_R^{\sigma}(\mathcal{L})$ and F is $W_\sigma(\mathcal{L})$ compact, for if $F \subset \bigcup_\alpha W_\sigma(L_\alpha)' \Rightarrow F \subset \bigcup_\alpha W(L_\alpha)$. Therefore

$$F \subset \bigcup_{fin} W(L_\alpha) = W(\hat{L})'\hat{L} \in \mathcal{L}$$

thus $F \subset W_\sigma(\hat{L})'$ since $F \subset I_R^{\sigma}(\mathcal{L})$ and $\tilde{\mu}(F) > 0$ since $\tilde{\mu}(G) < \tilde{\mu}(I_R(\mathcal{L}))$. Also since $W_\sigma(\mathcal{L})$ is normal and T_2 then $F \in \tau W_\sigma(\mathcal{L})$. Now $\mu \in M_R^{\tau}(\mathcal{L})$ projects onto $I_R^{\sigma}(\mathcal{L})$ and μ' is the projection on $W_\sigma(\mathcal{L})$ and μ'' is the projection on $\tau W_\sigma(\mathcal{L})$. For $E \in \mathcal{A}(W_\sigma(\mathcal{L}))$ let $\lambda(E) = \mu''(E \cap F)$ then $0 \le \lambda(E) \le \mu''(E) = \mu'(E)$ so $0 \le \lambda \le \mu'$ on $\mathcal{A}[W_\sigma(\mathcal{L})]$. Now if

$$W_\sigma(L_\alpha) \downarrow \varnothing, L_\alpha \in \mathcal{L} \text{ then } W_\sigma(L_\alpha) \cap F \downarrow \varnothing \text{ and } \lambda[W_\sigma(L_\alpha)] = \mu''[W_\sigma(L_\alpha) \cap F] \to 0$$

then

$$\lambda \in M_R^{\tau}(W_\sigma(\mathcal{L})).$$

Since λ is τ-smooth and $W_\sigma(\mathcal{L})$ is regular. Also $\lambda \in M_R^{\tau}[W_\sigma(\mathcal{L})]$ since $\forall \in > 0$, $\lambda(I_R^{\sigma}(\mathcal{L})) = \mu''(F)$ then

$$\lambda^*(F) = \lambda^*\left(\bigcap_\alpha W_\sigma(\mathcal{L}_\alpha)\right) = \inf \lambda[W_\sigma(L_\alpha)]$$

$$= \inf \mu''[W_\sigma(L_\alpha) \cap F] = \mu''(W_\sigma(L_\alpha) \cap F) = \mu''(F).$$

Therefore

$$\lambda^*(F) = \mu''(F) = \lambda(I_R^{\sigma}(\mathcal{L})) > \lambda(I_R^{\sigma}(\mathcal{L})) - \varepsilon.$$

Thus

$$\lambda \in M_R^\tau[W_\sigma(\mathcal{L})]$$

Therefore

$$\lambda = \gamma' \in M_R^\tau[W_\sigma(\mathcal{L})]$$

so

$$0 \le \gamma' \le \mu' \text{ on } \mathcal{A}[W_\sigma(\mathcal{L})] \text{ and } 0 \le \gamma \le \mu \text{ on } \mathcal{A}(\mathcal{L})$$

and $\gamma' \in M_R^\tau[W_\sigma(\mathcal{L})]$ and $\lambda = \gamma' \neq 0$ contradiction. Hence

$$\bar{\mu}^*(I_R(\mathcal{L}) - I_R^\sigma(\mathcal{L})) = \bar{\mu}(I_R(\mathcal{L}))$$

2.b) Let $\gamma' \in M_R^\sigma(W_\sigma(\mathcal{L}))$ then $\gamma \in M_R^\sigma(\mathcal{L})$ also $\gamma' \in M_R^\tau[W_\sigma(\mathcal{L})]$ because γ' is $W_\sigma(\mathcal{L})$-tight. Now

$$0 \le \gamma' \le \mu' \text{ on } \mathcal{A}[W_\sigma(\mathcal{L})] \Rightarrow 0 \le \gamma'' \le \mu'' \text{ on } \mathcal{A}[\tau W_\sigma(\mathcal{L})]$$

also $I_R^\sigma(\mathcal{L})$ is $\bar{\gamma}^*$-measurable since $\gamma' \in M_R^\tau[W_\sigma(\mathcal{L})]$ then $\bar{\gamma}^*(I_R^\sigma(\mathcal{L})) = \bar{\gamma}(I_R(\mathcal{L}))$ from previous work. Therefore $\exists F$, $W_\sigma(\mathcal{L})$-compact, $F \subset I_R^\sigma(\mathcal{L})$ such that

$$\gamma''(F) > \frac{1}{2}\gamma''(I_R^\sigma(\mathcal{L})) = \frac{1}{2}\gamma'[I_R^\sigma(\mathcal{L})]$$

so

$$\gamma''(F) \le \mu''(F) = 0 \text{ since } F \subset I_R^\sigma(\mathcal{L})$$

and since by hypothesis

$$\bar{\mu}^*(I_R(\mathcal{L}) - I_R^\sigma(\mathcal{L})) = \bar{\mu}[I_R(\mathcal{L})]$$

then

$$\bar{\mu}^*(I_R^\sigma(\mathcal{L})) = 0$$

and

$$\bar{\mu}(F) = 0$$

but then

$$\gamma'(I_r^\sigma(\mathcal{L})) = 0 \Rightarrow \gamma' = 0 \Rightarrow \gamma = 0$$

therefore μ is s. τ. a.

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