

THE CAUCHY PROBLEM OF THE ONE DIMENSIONAL SCHRÖDINGER EQUATION  
WITH NON-LOCAL POTENTIALS

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ABSTRACT. For a large class of operators  $A$ , not necessarily local, it is proved that the Cauchy problem of the Schrödinger equation:

$$-\frac{d^2f(z)}{dz^2} + Af(z) = s^2f(z), \quad f(0) = 0, \quad f'(0) = 1$$

possesses a unique solution in the Hilbert ( $H_2(\Delta)$ ) and Banach ( $H_1(\Delta)$ ) spaces of analytic functions in the unit disc  $\Delta = \{z: |z| < 1\}$ .

KEY WORDS AND PHRASES. Cauchy problem, Schrödinger equation, Hardy-Lebesgue spaces.

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1. INTRODUCTION.

Let  $C[0, \pi]$  be the Banach space of continuous functions on the interval  $[0, \pi]$ . The norm of an element  $f(x)$  of that space is defined by

$$\|f\| = \sup_{x \in [0, \pi]} |f(x)|.$$

Assume that  $A$  is a linear bounded operator on  $[0, \pi]$  not necessarily local, i.e.,  $A$  need not be the multiplication operator by a continuous function  $a(x)$ . It may, for instance, be an integral operator on  $C[0, \pi]$ . It is known that the Schrödinger equation:

$$-\frac{d^2f}{dx^2} + Af(x) = s^2f(x) \quad (1.1)$$

possesses a unique solution in  $C[0, \pi]$  satisfying the initial conditions:

$$f(0) = 0, \quad f'(0) = 1 \quad (1.2)$$

provided that  $|s| > c_0 \|A\|$ ,  $c_0 = \max_{0 < \alpha < 1} \frac{1 - \cos \alpha \pi}{\alpha}$  [1].

Also it is known [1] that the solution is bounded for every  $s$  in the region:

$$G = \{s: |s| \geq \alpha c_0 \|A\|, \quad \alpha > 1\}.$$

The purpose of this paper is to prove similar results for the initial valued problem [(1.1),(1.2)] in the Hardy spaces  $H_2(\Delta)$  and  $H_1(\Delta)$ . These are the spaces of analytic functions:  $f(z) = \sum_{n=1}^{\infty} \alpha_n z^{n-1}$  in the unit disk  $\Delta = \{z: |z| < 1\}$ , which satisfy respectively the conditions:

$\sum_{n=1}^{\infty} |\alpha_n|^2 < \infty$  and  $\sum_{n=1}^{\infty} |\alpha_n| < \infty$  or equivalently the conditions:

$$\sup_{0 < r < 1} \int_0^{2\pi} |f(re^{i\vartheta})|^2 d\vartheta < \infty \text{ and } \sup_{0 < r < 1} \int_0^{2\pi} |f(re^{i\vartheta})| d\vartheta < \infty,$$

for  $re^{i\vartheta} = z$ .

## 2. REDUCTION OF THE SCHRÖDINGER EQUATION.

$$-\frac{d^2f(z)}{dz^2} + Af(z) = s^2 f(z) \quad (2.1)$$

in  $H_2(\Delta)(H_1(\Delta))$  to an abstract operator form. (We follow the method prescribed in [2] and [3]).

Let  $H$  denote an abstract separable Hilbert space with an orthogonal basis  $\{e_n\}_1^{\infty}$  and let  $V$  be the unilateral shift operator on  $H$ , i.e.,

$$V: V e_n = e_{n+1}, \quad n = 1, 2, \dots, \quad V^*: V^* e_n = e_{n-1}, \quad n \neq 1, \quad V^* e_1 = 0$$

is the adjoint operator of  $V$ .

Every function  $f(z) = \sum_{n=1}^{\infty} \alpha_n z^{n-1}$  in  $H_2(\Delta)$  can be represented as follows:  $f(z) = (f_z, f)$ , where  $(\cdot, \cdot)$  means the scalar product in  $H$  and  $f_z = \sum_{n=1}^{\infty} z^{n-1} e_n$ ,  $|z| < 1$  are the eigenelements of  $V^*$ .

The space  $H_1$  is the Banach space which consists of those elements  $f = \sum_{n=1}^{\infty} \bar{\alpha}_n e_n$ , in  $H$ , (overbar means complex conjugate), that satisfy the condition  $\sum_{n=1}^{\infty} |(f, e_n)| < \infty$ . This space under the isomorphism  $f(z) = (f_z, f)$  is isomorphic to  $H_1(\Delta)$ .

The norm in  $H_1$  is denoted by:  $\|f\|_1 = \sum_{n=1}^{\infty} |(f, e_n)|$ . To any open set or dense linear manifold  $E$  in  $H(H_1)$  corresponds an open set or dense linear manifold  $\bar{E}$  in  $H_2(\Delta)(H_1(\Delta))$ . Suppose that  $A$  is a mapping in  $H_2(\Delta)(H_1(\Delta))$  and  $\bar{A}$  is a mapping in  $H(H_1)$ . Then if the relation  $Af(z) = (f_z, \bar{A}f)$  holds  $\forall f \in E$ , we call  $\bar{A}$  the abstract form of  $A$ . For example if  $A$  is the differential operator  $\frac{d^2}{dz^2}$  in  $H_2(\Delta)$ , i.e.,  $Af(z) = \frac{d^2f(z)}{dz^2}$ , then  $\bar{A} = (C_o V^*)^2 = C_o (C_o + I) V^* V^2$ , where  $C_o$  is the diagonal operator  $C_o e_n = n e_n, n = 1, 2, \dots$  (see for details in [2] and [3]).

Every bounded operator on  $H_2(\Delta)(H)$  is defined on  $H_1(\Delta)(H_1)$  and maps, in general, elements of  $H_1(\Delta)(H_1)$  into  $H_2(\Delta)(H)$ .

The following properties follow easily:

(i)  $H_1$  is invariant under the operators  $V, V^*$  and  $\|V\|_1 = \|V^*\|_1 = 1$ , where  $\|A\|_1$  means the norm of an operator on  $H_1$ .

(ii)  $H_1$  is invariant under every bounded diagonal operator  $D e_n = d_n e_n, n = 1, 2, \dots$  on  $H$  and  $\|D\|_1 = \|D\| = \sup_n |d_n|$ .

(iii) For every element  $f(z) = \sum_{n=1}^{\infty} \alpha_n z^{n-1}$  in  $H_1(\Delta)$  the uniform limit of the sequence  $\sum_{i=1}^n \bar{\alpha}_i V^{i-1}$ , i.e.,  $\lim_{n \rightarrow \infty} \sum_{i=1}^n \bar{\alpha}_i V^{i-1}$  exists and defines a bounded operator  $f^*(V) = \bar{\alpha}_1 + \bar{\alpha}_2 V + \bar{\alpha}_3 V^2 + \dots$  on  $H_1$ . Moreover  $\|f^*(V)\|_1 = \|f\|_1$ .

(iv) The null space of  $V^{*k}$  in  $H$  belongs to  $H_1$ .

Now we write equation (2.1) in the form:

$$\frac{d^2f}{dz^2} + s^2 f(z) - Af(z) = 0. \quad (2.2)$$

The abstract form of equation (2.2) is the following:

$$((C_o V^*)^2 - \bar{A}_1) f = 0 \quad (2.3)$$

or

$$(V^{*2} - B_1 \bar{A}_1) f = 0, \quad (2.4)$$

where  $\bar{A}_1 = \bar{A} - s^2 I$  and  $B_1$  is the diagonal operator on  $H: B_1 e_n = \frac{1}{n(n+1)} e_n$ ,  $n = 1, 2, \dots$ .

This means that equation (2.2) has a solution in  $H_2(\Delta)(H_1(\Delta))$  satisfying the conditions  $f(0) = 0, f'(0) = 1$  iff equation (2.4) has a solution in  $H(H_1)$  satisfying the conditions:

$$(f, e_1) = 0, \quad (f, e_2) = 1. \quad (2.5)$$

Note that  $H_1$  is imbedded in  $H$  in the sense that  $f$  in  $H_1$  implies  $f$  in  $H$  and  $\|f\| \leq \|f\|_1$ .

### 3. SOLUTION OF THE CAUCHY PROBLEM [(2.4), (2.5)] IN $H$ AND $H_1$ .

**THEOREM 1.** The equation  $(V^* - B_1 \bar{A}_1)f = 0$  has at least one solution in  $H$  which satisfies the condition  $(f, e_1) = 0$ .

**PROOF.** Set  $f = Vg$ , then  $(Vg, e_1) = (g, V^* e_1) = (g, 0) = 0$ . Also  $(V^{*2} - B_1 \bar{A}_1)(Vg) = 0$  implies  $V^*(I - VB_1 \bar{A}_1 V)g = 0$ .

Thus  $(I - VB_1 \bar{A}_1 V)g = ce_1$ .

Now since  $B_1$  is compact,  $V$  and  $\bar{A}$  bounded the operator  $VB_1 \bar{A}_1 V$  is compact and the Fredholm alternative implies that either:  $(I - VB_1 \bar{A}_1 V)g = 0$  for  $g \neq 0$  or  $(I - VB_1 \bar{A}_1 V)^{-1}$  exists and it is bounded.

In the first case  $g \neq 0$  is a solution of equation (2.4). In the second case we have  $g = c(I - VB_1 \bar{A}_1 V)^{-1}e_1 \neq 0$  for  $c \neq 0$ .  $\square$

Theorem 1 implies that the Schrödinger equation (2.1) has at least one solution in  $H_2(\Delta)$  which satisfies the condition  $f(0) = 0$ , for every real or complex  $s$ , and every bounded linear operator  $A$  on  $H_2(\Delta)$ .

**THEOREM 2.** If  $\|\bar{A}_1\| < 2$ , then equation (2.4) has a unique solution in  $H$  which satisfies the conditions (2.5).

**PROOF.** Set  $f = e_2 + V^2 g$ , then obviously  $(f, e_1) = 0$  and  $(f, e_2) = 1$ . Also from equation (2.4) we get:  $-B_1 \bar{A}_1 e_2 + Ig - B_1 \bar{A}_1 V^2 g = 0$  which implies that

$$(I - B_1 \bar{A}_1 V^2)g = B_1 \bar{A}_1 e_2. \quad (3.1)$$

(i) If  $\bar{A}_1 e_2 = 0$ , then  $e_2$  is the unique solution in  $H$  which satisfies the initial conditions, since  $(I - B_1 \bar{A}_1 V^2)g = 0$  implies that  $g = 0$ .

(ii) If  $\bar{A}_1 e_2 \neq 0$ , then from equation (3.1) since  $\|B_1\| = \frac{1}{2}$  and  $\|\bar{A}\| < 2$ , we easily get that  $\|B_1 \bar{A}_1 V^2\| < 1$ . Hence the inverse of  $(I - B_1 \bar{A}_1 V^2)$  exists and it is bounded on  $H$ . Therefore  $g = (I - B_1 \bar{A}_1 V^2)^{-1} B_1 \bar{A}_1 e_2$ ,  $g \neq 0$  and  $g$  is uniquely defined.  $\square$

There has been defined, in [3], a class of bounded operators on  $H(H_1)$  which have the so-called " $k$ -invariant property." Abstract forms of local potentials of the form:  $Af(z) = a(z)f(z)$  are included in this class.

The importance of such operators is due to the fact that if  $\bar{A}_1$  is  $k$ -invariant on the space  $H_2$ , then the operator  $A_2 = I - V^2 B_1 \bar{A}_1$  leaves invariant the space  $H_1$  and when restricted on it, has a bounded inverse (see [3], Theorem 3.2).

**DEFINITION.** A bounded operator  $\bar{A}$  on  $H$  is called  $k$ -invariant iff its adjoint  $\bar{A}^*$  has the property:  $\bar{A}^* e_i \in M_{i+k-1}$ , where  $M_{i+k-1}$  is the subspace spanned by  $\{e_1, e_2, \dots, e_{i+k-1}\}$ ,  $i = 1, 2, \dots$ .

Such operators are the diagonal operators in the basis  $\{e_n\}_1^\infty$ , analytic functions of the shift  $V$ , algebraic combinations of the above and polynomial functions of  $V^*$  of degree less than  $k$ .

In accordance with the above definition a bounded operator  $A$  on  $H_2(\Delta)(H_1(\Delta))$  is called  $2$ -invariant iff its adjoint  $A^*$  has the property:  $A^* z^i \in \{1, z, z^2, \dots, z^i\}$ , where  $\{1, z, z^2, \dots, z^i\}$ , is the subspace of  $H_2(\Delta)(H_1(\Delta))$  spanned by the elements  $1, z, z^2, \dots, z^i$ .

For example the operator:

$$A = Af(z) = af(z) + zf(z) + \frac{1}{z}(f(z) - f(0))$$

is a 2-invariant self adjoint operator on  $H_2(\Delta)$ .

**THEOREM 3.** The Cauchy problem:

$$-\frac{d^2f(z)}{dz^2} + Af(z) = s^2f(z) \quad (3.2)$$

$$f(0) = 0, \quad f'(0) = 1, \quad (3.3)$$

where  $A$  is any 2-invariant operator on  $H_1(\Delta)$ , has a unique solution in  $H_1(\Delta)$  for every  $s \in \mathbb{C}$ .

This solution is bounded for every  $z$  in the unit disc.

**PROOF.** The abstract form of (3.2) is:

$$(V^*V - B_1\bar{A} + s^2B_1)f = 0 \quad (3.4)$$

and the conditions (3.3) are equivalent to

$$(f, e_1) = 0, \quad (f, e_2) = 1. \quad (3.5)$$

Setting  $f = e_2 + V^2g$  which obviously satisfies the initial conditions (3.5) we get:

$$(I - B_1(\bar{A} - s^2)V^2)g = B_1(\bar{A} - s^2)e_2. \quad (3.6)$$

The operators  $V, V^*$  and  $B_1$  leave the space  $H_1$  invariant. The same holds for the operator  $(I - B_1(\bar{A} - s^2)V^2)$ , which restricted on  $H_1$  has a bounded inverse (see [3], Theorem 3.2). Also  $B_1(\bar{A} - s^2)e_2 = h \in H_1$  and the unique solution of (3.6) is given by:  $g = (I - B_1(\bar{A} - s^2)V^2)^{-1}h$ .

For every  $f(z) = \sum_{n=1}^{\infty} \alpha_n z^{n-1} \in H_1(\Delta)$  we have:  $|f(z)| \leq \sum_{n=1}^{\infty} |\alpha_n| = \|f(z)\|_{H_1(\Delta)} < \infty$ ,  $|z| \leq 1$ . This shows that the solution predicted by the theorem is bounded for  $|z| \leq 1$ .  $\square$

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