

# **SOLUTION TO A PARABOLIC EQUATION WITH INTEGRAL TYPE BOUNDARY CONDITION**

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(Received September 28, 1992 and in revised form January 21, 1993)

**ABSTRACT** In this paper we study the existence, and continuous dependence of the solution  $\vartheta = \vartheta(x, t)$  on a Hölder space  $H^{2+\gamma, 1+\gamma/2}(\overline{Q}_\tau)$  ( $\overline{Q}_\tau = [0, 1] \times [0, \tau]$ ,  $0 < \gamma < 1$ ) of a linear parabolic equation, prescribing  $\vartheta(x, 0) = f(x)$ ,  $\vartheta_x(1, \tau) = g(\tau)$  the integral type condition  $\int_0^b \vartheta(x, \tau) dx = E(\tau)$ .

**KEY WORD AND PHRASES.** Parabolic equation, integral boundary condition.

**1991 AMS SUBJECT CLASSIFICATION CODE:** 35K20.

## 1. INTRODUCTION.

Consider the problem of finding  $\vartheta = \vartheta(x, \tau)$  such that

$$\vartheta_\tau = (r(x, \tau)\vartheta_x)_x, \quad 0 < x < 1, \quad 0 < \tau \leq T, \quad (1.1)$$

$$\vartheta_x(1, \tau) = g(\tau), \quad 0 \leq \tau \leq T, \quad (1.2)$$

$$\vartheta(x, 0) = f(x), \quad 0 \leq x \leq 1, \quad (1.3)$$

$$\int_0^b \vartheta(x, \tau) dx = E(\tau), \quad 0 \leq \tau \leq T, \quad (1.4)$$

with  $E(0) = \int_0^b f(x) dx$ , for  $b$  fixed with  $0 < b < 1$  and  $r(x, \tau) \geq r_0 > 0$  on  $[0, 1] \times [0, T]$ .

In Cannon, Yanpin Lin [1] it is proved a result on existence, uniqueness and continuous dependence for this problem. In this paper we give conditions for which the solution of (1.1)-(1.4) belongs to a Hölder space and we prove that this solution depends continuously upon the data with respect to the corresponding Hölder norms. Similar problems are considered in [2,3,5,6,8,9,10].

Notice that function  $\vartheta$  satisfies (1.1)-(1.4) if and only if  $u(x, t) = \vartheta(x, \tau)$ , with  $t = \int_0^\tau \frac{ds}{r(b, s)}$ ,

satisfies

$$\begin{aligned} u_t &= u_{xx} + \left[ \left( \frac{a(x,t) - a(b,t)}{a(b,t)} \right) u_x \right]_x \\ &= \left( \frac{a(x,t)}{a(b,t)} u_x \right)_x, \quad 0 < x < 1, \quad 0 < t \leq T, \end{aligned} \quad (1.5)$$

$$u_x(1,t) = \tilde{g}(t), \quad 0 < t \leq T, \quad (1.6)$$

$$u(x,0) = f(x), \quad 0 \leq x \leq 1, \quad (1.7)$$

$$\int_0^b u(x,t) dx = \tilde{E}(t), \quad 0 \leq t \leq T, \quad (1.8)$$

where  $\tilde{E}(t) = E(t)$ ,  $\tilde{g}(t) = g(t)$ ,  $a(x,t) = r(x,t)$ ,  $T = \int_0^T \frac{dx}{r(b,x)}$ , and  $\tilde{E}(0) = E(0) = \int_a^b f(x) dx$ .

(A) and (B) will denote problems (1.1)-(1.4) and (1.5)-(1.8), respectively. The results on existence, uniqueness and continuous dependence will be based on a standard fixed point argument for a contraction defined on a subset of an appropriate functional space. We shall follow Ladyzenskaja et al. [11] to define the spaces of Hölder continuous functions:

Let  $Q_T = (0,1) \times (0,T)$ ,  $\bar{Q}_T = [0,1] \times [0,T]$ . For  $M > 0$ ,  $k = 0, 1, 2$  and  $0 < \gamma < 1$ ,  $H^{k+\gamma}[0,M]$  shall denote the spaces of functions  $h = h(t)$  in  $[0,M]$ , with  $\|h\|_M^{(k+\gamma)} < \infty$ ; where

$$\|h\|_M^{(k+\gamma)} = \sum_{n=0}^k \|h^{(n)}\|_M + \|h^{(k)}\|_M^{(\gamma)},$$

$$\|h\|_M = \sup_{t \in [0,M]} |h(t)|,$$

$$\|h\|_M^{(\gamma)} = |h(0)| + \sup_{t,t' \in [0,M]} \frac{|h(t) - h(t')|}{|t - t'|^\gamma},$$

where  $h^{(n)}$  denotes the derivative of  $h$  of order  $n$ .

For  $u : \bar{Q}_T \rightarrow \mathbf{R}$ , let

$$H_{x,\gamma}^T(u) = \sup_{\substack{x,x' \in [0,1] \\ t \in [0,T]}} \frac{|u(x,t) - u(x',t)|}{|x - x'|^\gamma}$$

$$H_{t,\gamma}^T(u) = \sup_{\substack{x \in [0,1] \\ t,t' \in [0,T]}} \frac{|u(x,t) - u(x,t')|}{|t - t'|^\gamma}$$

$$\|u\|_{Q_T} = \sup_{(x,t) \in Q_T} |u(x,t)|$$

Then  $H^{\gamma,\gamma/2}(\bar{Q}_T)$  and  $H^{2+\gamma,1+\gamma/2}(\bar{Q}_T)$  will denote the space of all functions  $u : \bar{Q}_T \rightarrow \mathbf{R}$  such that

$$\|u\|_T^{\gamma,\gamma/2} = \|u\|_{Q_T} + H_{x,\gamma}^T(u) + H_{t,\gamma/2}^T(u) < \infty$$

and

$$\begin{aligned} \|u\|_T^{2+\gamma, 1+\gamma/2} &= \|u\|_{Q_T} + \|u_x\|_{Q_T} + \|u_{xx}\|_{Q_T} + \|u_t\|_{Q_T} \\ &+ H_{t, \frac{1+\gamma}{2}}^T(u_x) + H_{x, \gamma}^T(u_t) + H_{x, \gamma}^T(u_{xx}) < \infty, \end{aligned}$$

respectively.

$K = K(x, t)$  will denote the fundamental solution to the heat equation

$$K(x, t) = \frac{1}{(4\pi t)^{1/2}} e^{-\frac{x^2}{4t}}, \quad x \in \mathbf{R}, \quad t > 0,$$

and  $\theta = \theta(x, t)$  shall be the Theta function

$$\theta(x, t) = \sum_{m=-\infty}^{\infty} K(x + 2m, t), \quad (\text{see [4]}).$$

## 2. EXISTENCE, UNIQUENESS AND CONTINUOUS DÉPENDENCE.

DEFINITION. A function  $u(x, t)$  on  $\overline{Q}_T$  is called a solution of problem (B), if

- 1)  $u$  and  $u_x$  are continuous in  $\overline{Q}_T$ ,
- 2)  $u_{xx}$  is bounded in  $\overline{Q}_T$ ,
- 3)  $u$  satisfies (1.5)-(1.8).

We notice that if  $u$  is such that  $u_x$  is continuous in  $\overline{Q}_T$  and satisfies (1.5)-(1.7), then  $u$  is a solution of problem (B) if and only if

$$a(b, t)\tilde{E}'(t) = a(b, t)u_x(b, t) - a(0, t)u_x(0, t) \quad (2.1)$$

or

$$E'(\tau) = r(b, \tau)\vartheta_x(b, \tau) - r(0, \tau)\vartheta_x(0, \tau), \quad (2.2)$$

for  $0 \leq \tau \leq T$ ,  $0 \leq t \leq T$ , provided  $E$  is differentiable.

We shall assume the following compatibility hypothesis:

$$\text{H1)} \quad \tilde{g}(0) = f'(1),$$

$$\text{H2)} \quad a(b, 0)\tilde{E}'(0) = a(b, 0)f'(b) - a(0, 0)f'(0), \text{ and the regularity conditions:}$$

$$\text{R1)} \quad \tilde{E} \in H^{1+(\frac{1+\gamma}{2})}[0, T], \tilde{g} \in H^{\frac{1+\gamma}{2}}[0, T], f \in H^{2+\gamma},$$

$$\text{R2)} \quad a, a_x, a_{xx} \in H^{\gamma, \gamma/2}(\overline{Q}_T) \text{ and } H_{x, \delta}^T(a_t) < \infty \text{ for some } \delta > 0.$$

Let  $V_T = \{\varphi \in H^{1+(\frac{1+\gamma}{2})}[0, T] : \varphi(0) = f'(0)\}$ . We define a nonlinear operator  $\mathcal{F} : V_T \rightarrow V_T$  as follows: For  $\varphi \in V_T$ , let  $u^\varphi$  be the unique solution in  $H^{2+\gamma, 1+\gamma/2}(\overline{Q}_T)$  of (1.5)-(1.7), with  $u_x(0, t) = \varphi(t)$ , (cf [11], Theorem 5.3 p. 320). Then we define

$$\mathcal{F}\varphi(t) = \frac{a(b, t)}{a(0, t)}(u_x^\varphi(b, t) - \tilde{E}'(t)).$$

Since  $u^\varphi \in H^{2+\gamma, 1+\gamma/2}(\overline{Q}_T)$  and (H2) holds, we have  $\mathcal{F}\varphi \in V_T$ , furthermore, if  $\varphi$  is a fixed point of  $\mathcal{F}$  then  $u^\varphi$  is a solution of problem (B) and conversely.

LEMMA 2.1. There exists  $\epsilon > 0$  not depending on  $f, \tilde{g}, \tilde{E}$ , such that if  $0 < T^* < \epsilon$  then

- a)  $\|\mathcal{F}\varphi - \mathcal{F}\psi\|_{T^*} \leq \frac{1}{2}\|\varphi - \psi\|_{T^*}, \quad \varphi, \psi \in V_T,$   
 b)  $\|\mathcal{F}\varphi - \mathcal{F}\psi\|_{T^*}^{(\frac{1+\gamma}{2})} \leq \frac{1}{2}\|\varphi - \psi\|_{T^*}^{(\frac{1+\gamma}{2})}, \quad \varphi, \psi \in V_T.$

PROOF. Let  $T^* \leq T$ ,  $\varphi$  and  $\psi$  in  $V_{T^*}$ ,  $h = \varphi - \psi$  and  $w = u^\varphi - u^\psi$ . Then

$$\begin{aligned} w(x, t) &= -2 \int_0^t \theta(x, t - \tau) h(\tau) d\tau + \\ &\quad \int_0^t \int_0^1 \{ \theta(x - \xi, t - \tau) + \theta(x + \xi, t - \tau) \} F(\xi, \tau) d\xi d\tau, \end{aligned} \quad (2.3)$$

with  $F(x, t) = (\frac{a(x, t) - a(b, t)}{a(b, t)} w_x)_x$  (cf [4] p. 339).

It follows that for  $t \in [0, T^*]$ ,

$$\begin{aligned} w_x(b, t) &= -2 \int_0^t \theta_x(b, t - \tau) h(\tau) d\tau \\ &\quad + \int_0^t \int_0^1 \theta_x(b + \xi, t - \tau) F(\xi, \tau) d\xi d\tau + \int_0^t \int_0^1 \theta_x(b - \xi, t - \tau) F(\xi, \tau) d\xi d\tau \\ &= I_1 + I_2 + I_3. \end{aligned}$$

We clearly have

$$|I_1| \leq 2\|\varphi - \psi\|_{T^*} \int_0^{T^*} |\theta_x(b, \tau)| d\tau \leq C_1 T^* \|h\|_{T^*}.$$

Since term by term differentiation of the series in  $I_2$  is possible, then we have

$$\begin{aligned} I_2 &= \int_0^t \int_0^1 \theta_x(b + \xi, t - \tau) \left( \frac{a(\xi, \tau) - a(b, \tau)}{a(b, \tau)} w_\xi(\xi, \tau) \right) d\xi d\tau \\ &= - \int_0^t \theta_x(b, t - \tau) \left( \frac{a(0, \tau) - a(b, \tau)}{a(b, \tau)} \right) w_\xi(0, \tau) d\tau \\ &\quad - \int_0^t \int_0^1 \theta_{xx}(b + \xi, t - \tau) \left( \frac{a(\xi, \tau) - a(b, \tau)}{a(b, \tau)} \right) w_\xi(\xi, \tau) d\xi d\tau. \end{aligned}$$

Condition (R2) implies that equation (1.5) (satisfied by  $w$ ) can be differentiated (see [7, Sec. 3.5]) and then  $w_x$  satisfies a linear parabolic equation. Thus, by the weak maximum principle it follows that

$$\|w_x\|_{\overline{Q}_{T^*}} \leq e^{MT^*} \|\varphi - \psi\|_{T^*} = e^{MT^*} \|h\|_{T^*}, \quad \text{where } M = \sup_{\overline{Q}_T} \left| \left( \frac{a(x, t)}{a(b, t)} \right)_{xx} \right|,$$

(cf. [7, Th. 2.3.8]).

Then  $|I_2| \leq C_2 T^* \|h\|_{T^*}$ . Finally, if we write  $\theta(x, t) = K(x, t) + H(x, t)$ , with  $H(x, t) = \sum_{\substack{m=-\infty \\ m \neq 0}}^{\infty} K(x + 2m, t)$ , then

$$\begin{aligned} I_3 &= \int_0^t \int_0^1 H_x(b - \xi, t - \tau) F(\xi, \tau) d\xi d\tau \\ &\quad + \int_0^t \int_0^1 K_x(b - \xi, t - \tau) F(\xi, \tau) d\xi d\tau \\ &= J_1 + J_2. \end{aligned}$$

$J_1$  can be estimated just as  $I_2$ , to obtain

$$|J_1| \leq C_3 T^* \|h\|_{T^*} \text{ for } t \leq T.$$

To estimate  $J_2$  we have to take case of the singularity of  $K(x, t)$  at  $(0, 0)$ .

Since  $\left| \frac{a(\xi, \tau) - a(b, \tau)}{a(b, \tau)} \right| \leq C_4 |\xi - b|$ , then integrating by parts as before, we have

$$\begin{aligned} |J_2| &\leq \int_0^t |K_x(\cdot, t - \tau) \left( \frac{a(0, \tau) - a(b, \tau)}{a(b, \tau)} \right) w_\xi(0, \tau)| d\tau \\ &\quad + C_5 \|h\|_{T^*} \int_0^t \int_0^1 |K_{xx}(b - \xi, t - \tau)(\xi - b)| d\xi d\tau \\ &\leq C_6 (T^* + T^{*1/2}) \|h\|_{T^*}. \end{aligned}$$

Hence  $|w_x(b, t)| \leq |I_1| + |I_2| + |I_3| \leq CT^{*1/2} \|h\|_{T^*}$ ,  $t \leq T^*$ , where  $C$  depends on  $T$ ,  $b$  and function  $a(x, t)$ . From this (a) follows immediately.

Now we estimate  $\|w_x(b, \cdot)\|_{T^*}^{(1+\frac{1}{2})}$ :

For  $t < s$  we have

$$\begin{aligned} w_x(b, s) - w_x(b, t) &= -2 \int_0^t \theta_x(b, \tau) (h(s - \tau) - h(t - \tau)) d\tau \\ &\quad - 2 \int_t^s \theta_x(b, \tau) h(s - \tau) d\tau \\ &\quad + \int_0^t \int_0^1 \theta_x(b + \xi, \tau) (F(\xi, s - \tau) - F(\xi, t - \tau)) d\xi d\tau \\ &\quad + \int_t^s \int_0^1 \theta_x(b + \xi, \tau) F(\xi, s - \tau) d\xi d\tau \\ &\quad + \int_0^t \int_0^1 H_x(b - \xi, \tau) (F(\xi, s - \tau) - F(\xi, t - \tau)) d\xi d\tau \\ &\quad + \int_t^s \int_0^1 H_x(b - \xi, \tau) F(\xi, s - \tau) d\xi d\tau \\ &\quad + \int_0^t \int_0^1 K_x(b - \xi, \tau) (F(\xi, s - \tau) - F(\xi, t - \tau)) d\xi d\tau \\ &\quad + \int_t^s \int_0^1 K_x(b - \xi, \tau) F(\xi, s - \tau) d\xi d\tau \\ &= L_1 + L_2 + L_3 + L_4 + L_5 + L_6 + L_7 + L_8. \end{aligned}$$

We claim that

$$|L_i| \leq M_i T^* \|h\|_{T^*}^{(1+\frac{1}{2})} |s - t|^{\frac{1+\gamma}{2}}, \quad i = 1, \dots, 6, \quad (2.4)$$

$$|L_7| \leq M_7 T^{*5/2} \|h\|_{T^*}^{(1+\frac{1}{2})} |s - t|^{\frac{1+\gamma}{2}}, \quad (2.5)$$

$$|L_8| \leq M_8 T^{*1/2} \|h\|_{T^*}^{(1+\frac{1}{2})} |s - t|^{\frac{1+\gamma}{2}}, \quad (2.6)$$

where  $M_i$  depends on  $T$ ,  $b$  and function  $a(x, t)$ ,  $i = 1, \dots, 8$ .

The proof of (2.4) follows as the proof of part (a). For (2.5) we let  $c(x, t) = \frac{a(x, t) - a(b, t)}{a(b, t)}$ , then

$$L_7 = - \int_0^t K_x(b, \tau) (c(0, s - \tau) w_x(0, s - \tau) - c(0, t - \tau) w_x(0, t - \tau)) d\tau$$

$$\begin{aligned}
& + \int_0^t \int_0^1 K_{xx}(b - \xi, \tau) c(\xi, s - \tau) (w_x(\xi, s - \tau) - w_x(\xi, t - \tau)) d\xi d\tau \\
& + \int_0^t \int_0^1 K_{xx}(b - \xi, \tau) (w_x(\xi, t - \tau) (c(\xi, s - \tau) - c(\xi, t - \tau))) d\xi d\tau \\
& = J_1 + J_2 + J_3.
\end{aligned}$$

Since  $c(\xi, t) = \mathcal{O}(|\xi - b|)$ , we obtain

$$|J_1| \leq K_1 T^* \|h\|_{T^*}^{(\frac{1+\gamma}{2})} |t - s|^{\frac{1+\gamma}{2}} \quad (2.7)$$

$$|J_2| \leq K_2 T^{*1/2} \|w\|_{T^*}^{2+\gamma, 1+\gamma/2} |t - s|^{1+\gamma}, \quad (2.8)$$

and by (R2),

$$J_3 = \int_0^t \int_0^1 |\xi - b|^6 K_{xx}(b - \xi, \tau) w_x(\xi, t - \tau) \int_t^s \frac{\partial}{\partial r} \frac{c(\xi, r - \tau)}{|\xi - b|^6} d\xi d\tau.$$

Hence

$$|J_3| \leq K_3 T^{*6/2} \|w\|_{T^*}^{2+\gamma, 1+\gamma/2} |t - s|. \quad (2.9)$$

We obtain (2.5) from (2.7), (2.8), (2.9) and the fact that  $\|w\|_{T^*}^{2+\gamma, 1+\gamma/2} \leq M \|h\|_{T^*}^{(\frac{1+\gamma}{2})}$ , where  $M$  does not depend on  $T^*$  (see [11] Theorem 5.4, p. 322). With a similar argument we obtain (2.6), and the proof of the Lemma follows from (2.4) (2.5) and (2.6).

REMARK. Notice that Lemma 2.1(a) holds for any two functions  $\varphi, \psi$  for which  $u^\varphi, u^\psi$  are well defined,  $u_x^\varphi, u_x^\psi$  are continuous in  $\overline{Q}_{T^*}$  and  $u_{xx}^\varphi, u_{xx}^\psi$  are bounded in  $\overline{Q}_{T^*}$ .

THEOREM 2.2. Assume that  $H_1, H_2, R_1, R_2$  hold. Then there exists a unique solution  $u = u(x, t)$  of Problem (B). This solution belongs to  $H^{2+\gamma, 1+\gamma/2}(\overline{Q}_T)$  and satisfies

$$\|u\|_T^{2+\gamma, 1+\gamma/2} \leq C(T) \left\{ \|\tilde{E}\|^{1+(\frac{1+\gamma}{2})} + \|\tilde{g}\|_T^{(\frac{1+\gamma}{2})} + \|f\|_1^{2+\gamma} \right\}.$$

PROOF. Let  $\epsilon > 0$  as in Lemma 2.1 and  $T^* < \epsilon$ , then if we define the sequence  $\varphi_i(x) = f''(0), \varphi_{i+1} = \mathcal{F}\varphi_i, i = 1, 2, \dots$ , then Lemma 2.1 implies that the sequence of restrictions  $\{\varphi_i|_{[0, T^*]}\}_{i \in \mathbb{N}}$  converges in  $C[0, T^*]$  and in  $H_{T^*}^{(\frac{1+\gamma}{2})}$  to a function  $\varphi_0$ .

Furthermore

$$\begin{aligned}
\|\varphi_n\|_{T^*}^{(\frac{1+\gamma}{2})} & \leq \sum_{i=1}^{\infty} \|\varphi_{i+1} - \varphi_i\|_{T^*}^{(\frac{1+\gamma}{2})} + \|\varphi_1\|_{T^*}^{(\frac{1+\gamma}{2})} \\
& \leq 2\|\varphi_2 - \varphi_1\|_{T^*}^{(\frac{1+\gamma}{2})} + \|\varphi_1\|_{T^*}^{(\frac{1+\gamma}{2})} \\
& \leq C_1 \left\{ \|\tilde{E}\|_{T^*}^{1+(\frac{1+\gamma}{2})} + \|\tilde{g}\|_{T^*}^{(\frac{1+\gamma}{2})} + \|f\|_1^{2+\gamma} \right\}.
\end{aligned}$$

Then for  $u : \overline{Q}_{T^*} \rightarrow \mathbb{R}$  defined by  $u = u^{\varphi_0}$ , we have

$$\|u\|_{T^*}^{2+\gamma, 1+\gamma/2} \leq C_2 \left\{ \|\tilde{E}\|_{T^*}^{1+(\frac{1+\gamma}{2})} + \|\tilde{g}\|_{T^*}^{(\frac{1+\gamma}{2})} + \|f\|_1^{2+\gamma} \right\}.$$

Hence  $u$  is solution to the local problem. Since  $C_1$  and  $C_2$  depend on  $T^*$  only, a global solution  $u$  can be obtained by a standard step by step construction, and  $u$  satisfies

$$\|u\|_T^{2+\gamma, 1+\gamma/2} \leq C \left\{ \|\tilde{E}\|_T^{1+(\frac{1+\gamma}{2})} + \|\tilde{g}\|_T^{(\frac{1+\gamma}{2})} + \|f\|_1^{2+\gamma} \right\}.$$

Finally, the remark after Lemma 2.1 implies that any solution of (B) in  $\overline{Q}_T$  has to be  $u$ .

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