

**ON STRONG LAWS OF LARGE NUMBERS FOR ARRAYS  
OF ROWWISE INDEPENDENT RANDOM ELEMENTS**

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**ABSTRACT.** Let  $\{X_{nk}\}$  be an array of rowwise independent random elements in a separable Banach space of type  $r$ ,  $1 \leq r \leq 2$ . Complete convergence of  $n^{1/p} \sum_{k=1}^n X_{nk}$  to 0,  $0 < p < r \leq 2$  is obtained when  $\sup_{1 \leq k \leq n} E\|X_{nk}\|^r = O(n^\alpha)$ ,  $\alpha \geq 0$  with  $\nu\left(\frac{1}{p} - \frac{1}{r}\right) > \alpha + 1$ . An application to density estimation is also given.

**KEY WORDS AND PHRASES.** *Random elements, strong laws of large numbers, complete convergence, Rademacher type  $r$  spaces.*

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**1. INTRODUCTION AND PRELIMINARIES.**

Let  $(\mathcal{E}, \|\cdot\|)$  be a real separable Banach space. Let  $(\Omega, \mathcal{A}, P)$  denote a probability space. A random element  $X$  in  $\mathcal{E}$  is a function from  $\Omega$  into  $\mathcal{E}$  which is  $\mathcal{A}$ -measurable with respect of the Borel subsets  $B(\mathcal{E})$ . The  $p^{th}$  absolute moment of a random element  $X$  is  $E\|X\|^p$  where  $E$  is the expected value of the random variable  $\|X\|^p$ . The expected value of a random element  $X$  is defined to be the Bochner integral (when  $E\|X\| < \infty$ ) and is denoted by  $EX$ . The concepts of independence and identical distributions for real-valued random variables extend directly to  $\mathcal{E}$ . A separable Banach space is said to be of (Rademacher) type  $r$ ,  $1 \leq r \leq 2$ , if there exist a constant  $C$  such that

$$E \left\| \sum_{k=1}^n X_k \right\|^r \leq C \sum_{k=1}^n E \|X_k\|^r$$

for all independent random elements  $X_1, \dots, X_n$  with zero means and finite  $r^{\text{th}}$  moments. Every separable Hilbert space and finite dimensional Banach space is of type 2. Every separable Banach space is at least type 1 while  $l^r$  and  $L^r$  spaces are of type  $\min(2, r)$  for  $r \geq 1$ .

Throughout this paper  $\{X_{nk} : 1 \leq k \leq n, n \geq 1\}$  will denote rowwise independent random elements in  $\mathcal{E}$  such that

$$EX_{nk} = 0 \quad \text{for all } n \text{ and } k. \quad (1.1)$$

The major results of this paper show that

$$\frac{1}{n^{1/p}} \sum_{k=1}^n X_{nk} \rightarrow 0 \quad \text{completely} \quad (1.2)$$

where complete convergence is defined (as in Hsu and Robbins [1]) by

$$\sum_{n=1}^{\infty} P \left[ \left\| \frac{1}{n^{1/p}} \sum_{k=1}^n X_{nk} \right\| > \epsilon \right] < \infty \quad (1.3)$$

for each  $\epsilon > 0$ .

Erdős [2] showed that for an array of i.i.d. random variables  $\{X_{nk}\}$ , (1.3) holds if and only if  $E|X_{11}|^{2p} < \infty$ . Jain [3] obtained a uniform strong law of large numbers for sequences of i.i.d. random elements in separable Banach spaces of type 2 which would yield (1.2) with  $p = 1$  for an array of i.i.d. random elements  $\{X_{nk}\}$  in a type 2 space. Woyczynski [4] showed that

$$\frac{1}{n^{1/p}} \sum_{k=1}^n X_k \rightarrow 0 \quad \text{completely} \quad (1.4)$$

for any sequence  $\{X_n\}$  of independent random elements in a Banach space of type  $r$ ,  $1 \leq p < r \leq 2$  with  $EX_n = 0$  for all  $n$  which is uniformly bounded by a random variable  $X$  satisfying  $E|X|^p < \infty$ . Recall that an array  $\{X_{nk}\}$  of random elements is said to be uniformly bounded by a random variable  $X$  if for all  $n$  and  $k$  and for every real number  $t > 0$

$$P[\|X_{nk}\| > t] \leq P[|X| > t]. \quad (1.5)$$

Note that i.i.d. random elements are uniformly bounded by  $\|X_{11}\|$ . Moricz, Hu, and Taylor [5] showed that Erdős' result could be obtained by replacing the i.i.d. condition by the uniformly bounded condition (1.5). Taylor and Hu [6] obtained complete convergence in type  $r$  spaces,  $1 < r \leq 2$  for uniformly bounded; rowwise independent random elements. The results of this paper relaxes the assumption of uniformly bounded random elements in Taylor and Hu [6]. Moreover, a major application of the main result of this paper is

indicated for kernel density estimators where uniformly bounded random variables can not be assumed.

## 2. MAJOR RESULTS.

The following lemma from Woyczynski [4] will be used in obtaining the major result, Theorem 2.

LEMMA 1. Let  $1 \leq r \leq 2$  and  $q \geq 1$ . The following properties are equivalent:

- (i)  $\mathcal{E}$  is of type  $r$
- (ii) There exists a  $C$  such that for all independent random elements  $X_1, \dots, X_n$  in  $\mathcal{E}$  with  $EX_k = 0$ , and  $E\|X_k\|^q < \infty$ ,  $k = 1, 2, \dots, n$

$$E\left\|\sum_{k=1}^n X_k\right\|^q \leq CE\left(\sum_{k=1}^n \|X_k\|^r\right)^{q/r}$$

THEOREM 2. Let  $\{X_{nk}\}$  be an array of rowwise independent random elements in a separable Banach space of type  $r$ . If  $EX_{nk} = 0$  and

$$\sup_{1 \leq k \leq n} E\|X_{nk}\|^\nu = O(n^\alpha), \quad \alpha \geq 0 \quad (2.1)$$

where  $\nu\left(\frac{1}{p} - \frac{1}{r}\right) > \alpha + 1$ ,  $0 < p < r \leq 2$ . Then

$$\frac{1}{n^{1/p}} \sum_{k=1}^n X_{nk} \rightarrow 0 \quad \text{completely.}$$

PROOF: Let  $\epsilon > 0$  be given. By Markov's inequality

$$\begin{aligned} \sum_{n=1}^{\infty} P\left(\left\|\frac{1}{n^{1/p}} \sum_{k=1}^n X_{nk}\right\| > \epsilon\right) &\leq \sum_{n=1}^{\infty} \frac{1}{\epsilon^\nu} E\left\|\frac{1}{n^{1/p}} \sum_{k=1}^n X_{nk}\right\|^\nu \\ &\leq C_1 \sum_{n=1}^{\infty} \frac{1}{n^{\nu/p}} E\left\|\sum_{k=1}^n X_{nk}\right\|^\nu. \end{aligned} \quad (2.2)$$

By Lemma 1 and Hölder's inequality,

$$\begin{aligned} C_1 \sum_{n=1}^{\infty} \frac{1}{n^{\nu/p}} E\left\|\sum_{k=1}^n X_{nk}\right\|^\nu &\leq C_1 \sum_{n=1}^{\infty} \frac{1}{n^{\nu/p}} E\left(\sum_{k=1}^n \|X_{nk}\|^r\right)^{\nu/r} \left(\sum_{k=1}^n 1\right)^{1-\frac{\nu}{r}} \\ &\leq C_1 \sum_{n=1}^{\infty} \frac{n^{\nu/r-1}}{n^{\nu/p}} \sum_{k=1}^n E\|X_{nk}\|^\nu \\ &\leq C_1 \sum_{n=1}^{\infty} \frac{n^{\nu/r-1}}{n^{\nu/p}} \cdot n \sup_{1 \leq k \leq n} E\|X_{nk}\|^\nu \\ &\leq C_2 \sum_{n=1}^{\infty} \frac{n^{\nu/r-1} \cdot n \cdot n^\alpha}{n^{\nu/p}} \\ &= C_2 \sum_{n=1}^{\infty} \frac{1}{n^{\nu(\frac{1}{p}-\frac{1}{r})-\alpha}} < \infty \end{aligned}$$

since  $\nu \left( \frac{1}{p} - \frac{1}{r} \right) > 1 + \alpha$ . Therefore,

$$\frac{1}{n^{1/p}} \sum_{k=1}^n X_{nk} \rightarrow 0 \quad \text{completely.}$$

### REMARK 1.

For values of  $p$  and  $r$ ,  $1 \leq p \leq r \leq 2$ , it follows that  $\nu > 2$ . Moreover, as  $p$  and  $r$  move close to each other  $\nu$  increases without bound. However, for certain values of  $p$  strictly less than one, a value of  $\nu = 1$  is possible to obtain complete convergence. To see this let  $p = \frac{1}{3}$ ,  $r = 1$  and  $\alpha = 0$ . It follows that  $\nu \left( \frac{1}{p} - \frac{1}{r} \right) = \nu(3 - 1) = 2\nu > 1$  implies that  $\nu > \frac{1}{2}$ . However, the proof of Theorem 2 requires that  $\nu \geq 1$ . Thus,  $\nu = 1$  is the smallest moment necessary (given suitable conditions on  $p$ ,  $r$  and  $\alpha$ ) to obtain complete convergence, via Theorem 2

### REMARK 2.

The condition  $\sup_{1 \leq k \leq n} E\|X_{nk}\|^\nu = O(n^\alpha)$  is somewhat stronger than (1.5) used by Taylor and Hu [6]. However, the bound in each row increases as  $n \rightarrow \infty$  which is a substantial improvement in Theorem 4 of Taylor, Moricz and Hu [5]. This substantial improvement will be illustrated in Example 1.

An immediate corollary to Theorem 2 is obtained for i.i.d. random elements.

**COROLLARY 3.** *Let  $\{X_{nk}\}$  be an array of i.i.d. random elements in a Banach space  $\mathcal{E}$  of type  $r$  such that  $EX_{11} = 0$ . Let  $E\|X_{11}\|^\nu < \infty$  where  $\nu \left( \frac{1}{p} - \frac{1}{r} \right) > 1$ ,  $0 < p < r \leq 2$ . Then,*

$$\frac{1}{n^{1/p}} \sum_{k=1}^n X_{nk} \rightarrow 0 \quad \text{completely.}$$

### REMARK 3.

The moment condition in Corollary 3 can be considerably smaller than the moment condition in Theorem 6 of Taylor and Hu [6], (see Remark 1) but in general will be much larger.

### 3. EXAMPLE 1.

Let  $X_1, \dots, X_n$  be i.i.d. random variables with common density function  $f$ . The kernel estimator for  $f$  with constant bandwidths  $h_n$  is given by

$$f_n(t) = \frac{1}{nh_n} \sum_{k=1}^n K\left(\frac{t - X_k}{h_n}\right) \quad (3.1)$$

where  $K$  is a bounded (integrable) kernel with compact support  $[a, b]$  and the sequence  $\{h_n\}$  is bounded and monotonically decreasing to 0 as  $n \rightarrow \infty$ . Let  $X_{nk}$  be defined as follows:

$$X_{nk} = \frac{1}{h_n} \left[ K\left(\frac{t - X_k}{h_n}\right) - E\left(K\left(\frac{t - X_1}{h_n}\right)\right) \right]. \quad (3.2)$$

Since the sequence  $\{X_n\}$  is i.i.d., it follows that  $\{X_{nk} : k = 1, 2, \dots\}$  is i.i.d. for each  $n$ . Verification of Condition (2.1) depends on the choice of  $K$ , the bandwidth sequence  $\{h_n\}$  and the particular Banach space. Typically,  $h_n = O(n^{-d})$  where  $0 < d < \frac{1}{2}$ . To illustrate the applicability of Condition (2.1), consider the Banach space  $L^r$ ,  $1 < r \leq 2$ . Then for each  $k$  and  $n$

$$\begin{aligned} E\|X_{nk}\|^\nu &\leq 2^\nu \left( \int_{-\infty}^{\infty} \left| \frac{1}{h_n} K\left(\frac{t-X_1}{h_n}\right) \right|^r dt \right)^{\nu/r} \\ &\leq C_1 h_n^{\nu(1-r)/r} \\ &\leq C_2 n^{d\nu(r-1)/r}. \end{aligned}$$

Since  $d < \frac{1}{2}$  and  $r > 1$ ,  $\nu$  can be chosen so that

$$\sup_{1 \leq k \leq n} E\|X_{nk}\|^\nu = O(n^\alpha)$$

and

$$\nu \left( \frac{1}{p} - \frac{1}{r} \right) > \alpha + 1 \quad \text{by letting } p = 1 \text{ and } \alpha = d\nu \left( \frac{r-1}{r} \right).$$

Verification of (2.1) follows easily for  $L^q$ ,  $q \geq 2$ , since they are of type 2. Thus,  $n^{-1} \sum_{k=1}^n X_{nk} \rightarrow 0$  completely or  $(nh_n)^{-1} \sum_{k=1}^n \left( K\left(\frac{t-X_k}{h_n}\right) - E\left(K\left(\frac{t-X_1}{h_n}\right)\right) \right) \rightarrow 0$  completely. Hence, consistency for (3.1) follows since  $(h_n)^{-1} E\left(K\left(\frac{t-X_1}{h_n}\right)\right) \rightarrow f(t)$  by traditional techniques.

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