

## REALCOMPACTIFICATION AND REPLETENESS OF WALLMAN SPACES

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(Received August 20, 1991 and in revised form January 23, 1992)

**ABSTRACT.** The extension of bounded lattice continuous functions on an arbitrary set  $X$  to the set of lattice regular zero-one measures on an algebra generated by a lattice (a Wallman-type space) is investigated.

Next the subset of lattice regular zero-one measures on an algebra generated by a lattice which integrates all lattice continuous functions on  $X$  is introduced and various properties of it are presented.

Finally conditions are established using repleteness criteria whereby the space of lattice regular zero-one measures on an algebra generated by a lattice which are countably additive (a Wallman-type space) is realcompact.

**KEY WORDS AND PHRASES.** Realcompact, repleteness, Wallman spaces, normal lattice, lattice continuous functions.

**1991 AMS SUBJECT CLASSIFICATION CODES.** 28C15, 28A12, 54D60.

### 1. INTRODUCTION.

Let  $X$  be an arbitrary set and  $\underline{L}$  a lattice of subsets of  $X$ .  $A(\underline{L})$  denotes the algebra generated by  $\underline{L}$ , and  $M(\underline{L})$  those bounded finitely additive measures on  $A(\underline{L})$ , and  $M_R(\underline{L})$  those elements of  $M(\underline{L})$  which are  $\underline{L}$ -regular while  $M_R^\sigma(\underline{L})$  denotes those elements of  $M_R(\underline{L})$  which are countably additive. The zero-one valued members of the above are designated by  $I(\underline{L})$ ,  $I_R(\underline{L})$ , and  $I_R^\sigma(\underline{L})$  respectively. For  $A \in A(\underline{L})$ ,  $w(A) = \{u \in I_R(\underline{L}) \mid u(A) = 1\}$ ,  $w(\underline{L}) = \{w(L) \mid L \in \underline{L}\}$ , then  $I_R(\underline{L})$  with the topology of closed sets  $\tau w(\underline{L})$  of arbitrary intersections of sets of  $w(\underline{L})$  is a compact,  $T_1$  topological space. It is one of the Wallman type spaces. Assuming  $\underline{L}$  is disjunctive then it is  $T_2$  if and only if  $\underline{L}$  is normal.

We begin by considering briefly, because of their importance, certain fundamental properties of normal lattices. Then we proceed to a consideration of  $I_R(\underline{L})$ , and the extension of bounded lattices continuous functions on  $X$  to  $I_R(\underline{L})$ . These results are generally known (see [8]) but we give somewhat shorter more direct proofs here.

We next consider the space  $Q(\underline{L})$  of measures in  $I_R(\underline{L})$  which integrate all lattice continuous functions on  $X$ , and show its relationship to  $I_R^\sigma(\underline{L})$ , and under suitable conditions, its relationship to the  $G_\delta$ -closure of  $X$  in  $I_R(\underline{L})$ .

Finally, we consider the Wallman type space  $I_R^\sigma(\underline{L})$ , and the lattice  $w_\sigma(\underline{L})$ , where for  $A \in A(\underline{L})$ ,  $w_\sigma(A) = \{u \in I_R^\sigma(\underline{L}) \mid u(A) = 1\}$ , and where  $w_\sigma(\underline{L}) = \{w_\sigma(L) \mid L \in \underline{L}\}$ . It is well-known that if  $\underline{L}$  is disjunctive then  $w_\sigma(\underline{L})$  is replete. We consider in this space the lattice of closed sets  $\tau w_\sigma(\underline{L})$  and its associated lattice of zero sets, and investigate their repleteness - thus obtaining sufficient conditions for the space  $I_R^\sigma(\underline{L})$  to be realcompact.

Our notations and terminology is consistent with [1, 3, 5, 6, 11]. However, the main definitions and notations used throughout the paper are presented for the reader's convenience in section 2(a). We note also that a number of results on normal lattices in section 2(b) are related to work of [4, 9].

## 2.(a) BACKGROUND AND NOTATION.

Let  $X$  be an abstract set, and  $\underline{L}$  the lattice of subsets of  $X$ . We assume that  $\phi, X \in \underline{L}$  for most of our results. First:

### Lattice Terminology:

$A(\underline{L})$  is the algebra generated by  $\underline{L}$ .

$\sigma(\underline{L})$  is the  $\sigma$ -algebra generated by  $\underline{L}$ .

$\delta(\underline{L})$  is the lattice of all countable intersections of sets from  $\underline{L}$ .  $\underline{L}$  is a delta lattice ( $\delta$ -lattice) if  $\delta(\underline{L}) = \underline{L}$ .

$\tau(\underline{L})$  is the lattice of arbitrary intersections of sets of  $\underline{L}$ .

$\underline{L}$  is complemented if  $L \in \underline{L} \Rightarrow L' \in \underline{L}$  (prime denotes complement), that is,  $\underline{L}$  is an algebra.

$\underline{L}$  is separating, if for any two elements  $x \neq y$  of  $X$ , there exists an element  $L \in \underline{L}$  such that  $x \in L$  and  $y \notin L$ .

$\underline{L}$  is  $T_2$  if, for any two elements  $x \neq y$  of  $X$ , there exists  $A, B \in \underline{L}$  such that  $x \in A'$  and  $y \in B'$  and  $A' \cap B' = \phi$ .

$\underline{L}$  is disjunctive if for any  $x \in X$  and  $A \in \underline{L}$  such that  $x \notin A$ , there exists a  $B \in \underline{L}$  such that  $x \in B$  and  $A \cap B = \phi$ .

$\underline{L}$  is regular if for any  $x \in X$ , and  $A \in \underline{L}$  such that  $x \notin A$  there exist  $B, C \in \underline{L}$  such that  $x \in B'$ ,  $A \subset C'$  and  $B' \cap C' = \phi$ .

$\underline{L}$  is normal if for all  $L_1, L_2 \in \underline{L}$  such that  $L_1 \cap L_2 = \phi$  there exists  $\tilde{L}_1', \tilde{L}_2' \in \underline{L}$  such that  $L_1 \subset \tilde{L}_1'$ ,  $L_2 \subset \tilde{L}_2'$ , and  $\tilde{L}_1' \cap \tilde{L}_2' = \phi$ .

$\underline{L}$  is compact if every covering of  $X$  by elements of  $\underline{L}'$  has a finite subcovering.

$\underline{L}$  is countably compact if every countable covering of  $X$  by elements of  $\underline{L}'$  has a finite subcovering.

$\underline{L}$  is Lindelöf if every covering of  $X$  by elements of  $\underline{L}'$  has a countable subcovering.

$\underline{L}$  is countably paracompact if whenever  $A_n \downarrow \phi, A_n \in \underline{L}$  there exists  $B_n \in \underline{L}$  such that  $A_n \subset B_n'$  and  $B_n' \downarrow \phi$ .

$\underline{L}$  is complement generated if, for  $L \in \underline{L}$  there exists  $L_n \in \underline{L}$  such that  $L = \bigcap_{n=1}^{\infty} L_n'$ .

It is well known that if  $\underline{L}$  is complement generated then  $\underline{L}$  is countably paracompact.

### Measure Terminology

We denote by  $M(\underline{L})$  the finitely additive bounded measures on  $A(\underline{L})$  (we may and do assume all elements of  $M(\underline{L})$  are  $\geq 0$ ).

$u \in M(\underline{L})$  is  $\underline{L}$ -regular if for any  $A \in A(\underline{L})$ ,  $u(A) = \sup\{u(L) \mid L \subset A, L \in \underline{L}\}$ ; (equivalently)  $= \inf\{u(L') \mid A \subset L', L' \in \underline{L}\}$ .

$u \in M(\underline{L})$  is  $\sigma$ -smooth on  $\underline{L}$  if  $L_n \in \underline{L}$ ,  $n = 1, 2, \dots$  and  $L_n \downarrow \phi = > u(L_n) \rightarrow 0$

$u \in M(\underline{L})$  is  $\sigma$ -smooth on  $A(\underline{L})$  if  $A_n \in A(\underline{L})$ ,  $n = 1, 2, \dots$  and  $A_n \downarrow \phi = > u(A_n) \rightarrow 0$ . Note  $u$  is  $\sigma$ -smooth on  $A(\underline{L})$  iff  $u$  is countably additive.

We will use the following notations.

$M_R(\underline{L}) =$  the set of  $\underline{L}$ -regular measures of  $M(\underline{L})$ .

$M_\sigma(\underline{L}) =$  the set of  $\sigma$ -smooth measures on  $\underline{L}$  of  $M(\underline{L})$ .

$M^\sigma(\underline{L}) =$  the set of  $\sigma$ -smooth measures on  $A(\underline{L})$  of  $M(\underline{L})$ .

$M_R^\sigma(\underline{L}) =$  the set of  $\underline{L}$ -regular measures of  $M^\sigma(\underline{L})$ .

Note that if  $u \in M_R(\underline{L})$  and  $u \in M_\sigma(\underline{L})$  then  $u \in M_R^\sigma(\underline{L})$ .

Also we denote by  $I(\underline{L})$ ,  $I_R(\underline{L})$ ,  $I_\sigma(\underline{L})$ ,  $I^\sigma(\underline{L})$ , and  $I_R^\sigma(\underline{L})$  the subsets of  $M(\underline{L})$ ,  $M_R(\underline{L})$ ,  $M_\sigma(\underline{L})$ ,  $M^\sigma(\underline{L})$ , and  $M_R^\sigma(\underline{L})$  consisting of zero-one valued measures.

Now for  $u_1, u_2 \in I(\underline{L})$ ,  $u_1 \leq u_2(\underline{L})$  means  $u_1(L) \leq u_2(L)$  for  $L \in \underline{L}$ .

Let  $J(\underline{L})$  denote those  $u \in I(\underline{L})$  such that whenever  $L_n \in \underline{L}$ ,  $n = 1, 2, \dots$  and  $\bigcap_{n=1}^\infty L_n \in \underline{L}$  then  $u(\bigcap_{n=1}^\infty L_n) = \inf_n u(L_n)$ .

Clearly,  $I^\sigma(\underline{L}) \subset J(\underline{L}) \subset I_\sigma(\underline{L})$ .

For  $u \in M(\underline{L})$  the support of  $u$ ,  $S(u) = \cap \{L \in \underline{L} \mid u(L) = u(X)\}$ .  $\underline{L}$  is replete if for any  $u \in I_R^\sigma(\underline{L})$ ,  $u \neq 0$ ,  $S(u) \neq \phi$ .

Let  $C(\underline{L})$  be the set of all real-valued  $\underline{L}$ -continuous functions defined on  $X$ , where  $f: X \rightarrow R$  is called  $\underline{L}$ -continuous if  $f^{-1}(E) \in \underline{L}$  for any closed set  $E \subset R$ . If  $X$  is a topological space,  $C(X)$  denotes the continuous functions on  $X$  or equivalently we can write  $C(X) = C(F)$  where  $F$  is the lattice of closed sets of  $X$ .  $z(\underline{L})$  is the lattice of zero sets of functions in  $C(\underline{L})$ .

$C_b(\underline{L}) =$  set of all real valued bounded  $\underline{L}$ -continuous functions defined on  $X$ .

Next we define  $w(A) = \{u \in I_R(\underline{L}) \mid u(A) = 1\}$  for  $A \in A(\underline{L})$ , and  $w(\underline{L}) = \{w(L) \mid L \in \underline{L}\}$ .

We have for  $A, B \in A(\underline{L})$ :

- (1)  $w(A \cup B) = w(A) \cup w(B)$
- (2)  $w(A \cap B) = w(A) \cap w(B)$
- (3)  $w(A)' = w(A')$
- (4)  $w(A(\underline{L})) = A(w(\underline{L}))$
- (5)  $A \subset B = > w(A) \subset w(B)$

Note  $w(\underline{L})$  is a lattice and if  $\underline{L}$  is disjunctive then  $w(A) = w(B)$  if and only if  $A = B$ .

The Wallman topology is obtained by taking  $w(\underline{L})$  as a base for the closed sets of a topology on  $I_R(\underline{L})$ .  $\langle I_R(\underline{L}), \tau w(\underline{L}) \rangle$  is the general Wallman space associated with  $X$  and  $\underline{L}$ . Note we have  $w(L) = \bar{L}$  for  $L \in \underline{L}$  if  $\underline{L}$  is separating and disjunctive. We also define:  $w_\sigma(A) = \{u \in I_R^\sigma(\underline{L}) \mid u(A) = 1\}$  where  $A \in A(\underline{L})$ , and note  $w(\underline{L}) \cap I_R^\sigma(\underline{L}) = w_\sigma(\underline{L})$ .

We now consider two lattices. Let  $\underline{L}_1$  and  $\underline{L}_2$  denote lattices of subsets, of  $X$  where  $\underline{L}_1 \subset \underline{L}_2$ .  $\underline{L}_1$  semi-separates  $\underline{L}_2$  if  $A \in \underline{L}_1$ ,  $B \in \underline{L}_2$  and  $A \cap B = \phi$  implies there exists  $C \in \underline{L}_1$ ,  $B \subset C$  and  $A \cap C = \phi$ .  $\underline{L}_1$  separates  $\underline{L}_2$  if  $A, B \in \underline{L}_2$  and  $A \cap B = \phi$  implies there exists  $C, D \in \underline{L}_1$  such that  $A \subset C, B \subset D$ , and  $C \cap D = \phi$ .  $\underline{L}_2$  is  $\underline{L}_1$ -countable paracompact if for every sequences  $\{B_n\}$  of sets of  $\underline{L}_2$ , such that  $B_n \downarrow \phi$  there exists  $\{A_n \in \underline{L}_1\}$  such that  $A_n' \downarrow \phi$  and  $B_n \subset A_n'$ .

$\underline{L}_2$  is  $\underline{L}_1$ -cb if given  $B_n \perp \phi$ ,  $B_n \in \underline{L}_2$  there exists  $\{A_n\}$ ,  $A_n \in \underline{L}_1$  such that  $A_n \perp \phi$  and  $B_n \subset A_n$ . Clearly if  $\underline{L}_1$  separates  $\underline{L}_2$  then  $\underline{L}_1$  semiseparates  $\underline{L}_2$ .

If  $\nu \in M(\underline{L}_2)$  then by  $\nu|A(\underline{L}_1)$  we mean  $\nu$  restricted to  $A(\underline{L}_1)$ . We state the following well known results:

Let  $\underline{L}_1 \subset \underline{L}_2$  be two lattices of subsets of  $X$ . If  $\underline{L}_1$  semiseparates  $\underline{L}_2$  then for  $\nu \in M_R(\underline{L}_2)$ ,  $u = \nu|A(\underline{L}_1) \in M_R(\underline{L}_1)$ .

Suppose  $\underline{L}_1 \subset \underline{L}_2$  are two lattices of subsets of  $X$ . Then if  $u \in M_R(\underline{L}_1)$ ,  $u$  extends to  $\nu \in M_R(\underline{L}_2)$ . Moreover, the extension is unique if  $\underline{L}_1$  separates  $\underline{L}_2$ .

We will frequently assume in the sequel that  $\underline{L}_1 \subset \underline{L}_2$  and  $\underline{L}_2$  is  $\underline{L}_1$  countably paracompact or countably bounded, but we note that this is unnecessary in certain situations as the following facts listed below show:

(1) If  $\underline{L}_2$  is  $\underline{L}_1$  countably bounded and if  $\underline{L}_1$  is countably paracompact (e.g., if  $\underline{L}_1$  is complement generated) then  $\underline{L}_2$  is  $\underline{L}_1$  countably paracompact.

(2) If  $\underline{L}_2$  is countably paracompact and if  $\underline{L}_1$  separates  $\underline{L}_2$  then  $\underline{L}_2$  is  $\underline{L}_1$  countably paracompact.

(3) Suppose  $\underline{L}_2$  is  $\underline{L}_1$  countably paracompact and  $\underline{L}_1$  semiseparates  $\underline{L}_2$  then  $\underline{L}_2$  is  $\underline{L}_1$  countably bounded.

(4) If  $\underline{L}_2$  is countably paracompact and if  $\underline{L}_1$  separates  $\underline{L}_2$  then  $\underline{L}_2$  is  $\underline{L}_1$  countably bounded.

## 2.(b)NORMAL LATTICES AND MEASURES.

In this section we will consider a number of measure implications of normal lattices and other special lattices as well as converse implications. We first note:

**THEOREM 2.1.** Let  $\underline{L}$  be a complemented generated lattice. The  $u \in I_\sigma(\underline{L}')$  implies  $u \in I_R^\sigma(\underline{L})$ .

**PROOF.** Since  $\underline{L}$  is complemented generated then  $\underline{L}$  is countably paracompact and therefore  $I_\sigma(\underline{L}') \subset I_\sigma(\underline{L})$ . Therefore it suffices to show  $u \in I_R(\underline{L})$ , but this is easy for if  $L \in \underline{L}$  then  $L = \bigcap_{n=1}^\infty L_n', L_n \in \underline{L}$  all  $n$ , and we may assume that the  $L_n' \perp \phi$ . Now if  $u(L) = 0$ , and if all  $u(L_n') = 1$  then  $\bigcap_{n=1}^\infty L_n' \cap L = \phi$  and  $u(L_n' \cap L) = 1$  all  $n$  which is a contradiction since  $u \in I_\sigma(\underline{L}')$ . It follows that  $u(L) = \inf\{u(\tilde{L}') \mid L \subset \tilde{L}', \tilde{L} \in \underline{L}\}$  and this implies  $u \in I_R(\underline{L})$ .

**REMARK.** It is equally easy to show if  $\underline{L}$  is complement generated and  $u \in M_\sigma(\underline{L}')$  then  $u \in M_R^\sigma(\underline{L})$ .

**THEOREM 2.2.** Let  $u \in J(\underline{L})$  and let  $\underline{L}$  be a  $\delta$ -lattice then  $u\left(\bigcup_{i=1}^\infty L_i'\right) \leq \sum_{i=1}^\infty u(L_i')$  where all  $L_i \in \underline{L}$ .

**PROOF.** Suppose  $u\left(\bigcup_{i=1}^\infty L_i'\right) = 1$  and  $\sum_{i=1}^\infty u(L_i') = 0$ . Now  $\sum_{i=1}^\infty u(L_i') = 0$  implies  $u(L_i') = 0$  all  $i$  and  $\bigcap_{i=1}^\infty L_i = \left(\bigcup_{i=1}^\infty L_i'\right)'$  therefore  $u\left(\bigcap_{i=1}^\infty L_i\right) = 0$  where obviously  $\bigcap_{i=1}^\infty L_i \in \underline{L}$ . Also  $u\left(\bigcap_{i=1}^\infty L_i\right) = \inf u(L_i)$  since  $u \in J(\underline{L})$ . So  $u\left(\bigcap_{i=1}^\infty L_i\right) = 0$  implies there exists an  $i_0$  such that  $\bigcap_{i=1}^\infty L_i \subset L_{i_0}$  and  $u(L_{i_0}) = 0$ . Therefore  $u(L_{i_0}') = 1$  which is a contradiction, therefore theorem is proved.

**THEOREM 2.3.** If  $\underline{L}$  is normal and complement generated then  $u \in J(\underline{L}) \Rightarrow u \in I_R^\sigma(\underline{L})$ .

**PROOF.** Let  $u \in J(\underline{L})$ ; we know that  $u \leq \nu$  on  $\underline{L}$  where  $\nu \in I_R(\underline{L})$ . This gives  $\nu \leq u$  on  $\underline{L}'$ . Suppose  $u \neq \nu$ . Then there exists  $L \in \underline{L}$  such that  $u(L) = 0$ ,  $\nu(L) = 1$ . However,  $L = \bigcap_{n=1}^\infty L_n'$  since  $\underline{L}$  is complement generated so  $L \subset L_n'$ . Therefore  $\nu(L) = 1 > \nu(L_n') = 1$  for all  $n$  which implies  $u(L_n') = 1$  for all  $n$  as  $\nu \leq u$  on  $\underline{L}'$ . Now  $L = \bigcap_{n=1}^\infty L_n' \Rightarrow L \cap L_n = \phi$  therefore since  $\underline{L}$  is normal there exists  $A_n', B_n' \in \underline{L}'$  such that  $L \subset A_n'$ ,  $L_n \subset B_n'$ , and  $A_n' \cap B_n' = \phi$ . Therefore  $L \subset A_n' \subset B_n \subset L_n$  from this

which gives  $\nu(A_n') = 1$  and  $\nu(B_n) = 1$  by monotonicity of  $\nu$ . Therefore  $u(B_n) = 1$  as  $u \leq \nu$  on  $\underline{L}$ . Also  $L \subset A_n' \subset B_n \subset L_n' = > L \subset \bigcap_{n=1}^{\infty} A_n' \subset \bigcap_{n=1}^{\infty} B_n \subset \bigcap_{n=1}^{\infty} L_n' = L$  which implies that  $L = \bigcap_{n=1}^{\infty} A_n' = \bigcap_{n=1}^{\infty} B_n = \bigcap_{n=1}^{\infty} L_n'$ , so  $u(L) = 0 = > u(\bigcap_{n=1}^{\infty} B_n) = 0$  which  $= > u(B_n) = 0$  by  $u \in J(\underline{L})$ . This is a contradiction as  $u(B_n) = 1$ . Therefore  $u = v \in I_R(\underline{L}) = > u \in I_R(\underline{L})$ . Now  $u \in J(\underline{L}) = > u \in I^\sigma(\underline{L})$ , therefore  $u \in I_R^\sigma(\underline{L})$ .

**THEOREM 2.4.** Let  $\underline{L}$  be a normal lattice,  $u \in I_R(\underline{L})$ ,  $u \leq \rho(\underline{L}')$  where  $\rho \in I_R(\underline{L}')$ . Then for  $L \in \underline{L}$ ,  $u(L') = \sup\{\rho(\tilde{L}) \mid \tilde{L} \subset L', \tilde{L} \in \underline{L}\}$ .

**PROOF.** Suppose  $u(L') = 1$ , where  $L \in \underline{L}$  then since  $u \in I_R(\underline{L})$  there exists  $\tilde{L} \subset L', \tilde{L} \in \underline{L}$ ,  $u(\tilde{L}) = 1$ . Since  $\tilde{L} \subset L' = > L \cap \tilde{L} = \phi$ , therefore by normality there exists  $A, B \in \underline{L}$  such that  $L \subset A', \tilde{L} \subset B', A' \cap B' = \phi$ . Therefore  $\tilde{L} \subset B' \subset A \subset L'$ , also  $u(\tilde{L}) = 1 = > u(B') = 1$  by monotonicity of  $u$ . Therefore  $\rho(B') = 1$  as  $u \leq \rho(\underline{L}')$ .  $\rho(A) = 1$  follows by monotonicity of  $\rho$ , proving the theorem.

**REMARK.** This theorem is equivalent to the following: Let  $\underline{L}$  be normal and let  $\nu \leq u(\underline{L})$  where  $\nu \in I(\underline{L})$  and  $u \in I_R(\underline{L})$ . Then  $u(L') = \sup\{\nu(\tilde{L}) \mid \tilde{L} \subset L', \tilde{L} \in \underline{L}\}$ . Next we show that actually the property in Theorem 2.4 or equivalently the one in the remark characterizes normal lattices, i.e.,

**THEOREM 2.5.** Suppose  $u \in I_R(\underline{L})$  and  $\rho \leq u(\underline{L})$  where  $\rho \in I_R(\underline{L}')$  and  $u(L') = 1$ ,  $L \in \underline{L}$  implies  $L' \supset A \in \underline{L}$  such that  $\rho(A) = 1$ . Then  $\underline{L}$  is normal.

**PROOF.** Let  $\rho \leq u(\underline{L})$ ,  $\rho \leq \nu(\underline{L})$  where  $u, \nu \in I_R(\underline{L})$  and  $\rho \in I_R(\underline{L}')$ . Assume  $u \neq \nu$ , this implies  $u(L_1) = 0$ ,  $\nu(L_1) = 1$ ,  $u(L_2) = 1$ ,  $\nu(L_2) = 0$  where  $L_1, L_2 \in \underline{L}$  and  $L_1 \cap L_2 = \phi$ . Now  $u(L_1) = 0$  implies  $u(L_1') = 1$  which implies there exists  $L_1' \supset A \in \underline{L}$  such that  $\rho(A) = 1$  and  $\nu(L_2) = 0$  implies  $\nu(L_2') = 1$  which implies there exists  $L_2' \supset B \in \underline{L}$  such that  $\rho(B) = 1$ . Since  $A \subset L_1'$ ,  $B \subset L_2'$  then  $L_1 \subset A'$  and  $L_2 \subset B'$ . So  $\rho(B) = 1$  implies  $\rho(B') = 0$  which implies  $u(B') = 0$  as  $u \leq \rho(\underline{L}')$ . However, by monotonicity  $u(L_2) \leq u(B')$  and  $u(L_2) = 1$  which implies  $u(B') = 1$  which contradicts  $u(B') = 0$ . Therefore  $u = \nu$  which means  $\underline{L}$  is normal.

**THEOREM 2.6.** Let  $\underline{L}$  be a normal lattice,  $u \in I_\sigma(\underline{L})$ ,  $u \leq \nu(\underline{L})$  where  $\nu \in I_R(\underline{L})$ . Then  $\nu \in I_\sigma(\underline{L}')$ .

**PROOF.** Suppose  $u \in I_\sigma(\underline{L})$  we know  $\rho \leq u \leq \nu(\underline{L})$  where  $\nu \leq u \leq \rho(\underline{L}')$  and  $\nu \in I_R(\underline{L})$ ,  $\rho \in I_R(\underline{L})$ . Suppose  $L_n' \not\downarrow \phi$ ,  $\nu(L_n') = 1$  all  $n$ ,  $L_n' \in \underline{L}'$ . Then there exists  $\tilde{L}_n \subset L_n'$  such that  $\rho(\tilde{L}_n) = 1$  all  $n$  by Theorem 2.5. Therefore  $u(\tilde{L}_n) = 1$  since  $\rho \leq u(\underline{L})$ . Now  $\tilde{L}_n \not\downarrow \phi$  since  $\bigcap_{n=1}^{\infty} \tilde{L}_n \subset \bigcap_{n=1}^{\infty} L_n'$ . This contradicts the fact that  $u \in I_\sigma(\underline{L})$ , therefore  $\nu \in I_\sigma(\underline{L}')$ .

**COROLLARY 2.7.** If  $\underline{L}$  is normal and countably paracompact then the  $\nu$  (from Theorem 2.6) belongs to  $I_R^\sigma(\underline{L})$ .

**PROOF.** Since  $\underline{L}$  is countably paracompact then  $I_\sigma(\underline{L}') \subset I_\sigma(\underline{L})$  by Theorem 2.2. Then  $\nu \in I_\sigma(\underline{L})$  and since  $\nu \in I_R(\underline{L})$  it follows that  $\nu \in I_R^\sigma(\underline{L})$ .

Next we consider a pair of lattices  $\underline{L}_1, \underline{L}_2$  of  $X$  such that  $\underline{L}_1 \subset \underline{L}_2$ , then we have:

**THEOREM 2.8.** If  $\underline{L}_1$  separates  $\underline{L}_2$  then  $\underline{L}_1$  is normal if and only if  $\underline{L}_2$  is normal.

**PROOF.** The proof is not difficult. We just show  $\underline{L}_2$  normal implies  $\underline{L}_1$  normal. Now let  $\underline{L}_2$  be normal and  $u \in I(\underline{L}_1)$ ,  $u \leq \nu_1(\underline{L}_1)$ ,  $u \leq \nu_2(\underline{L}_1)$  where  $\nu_1, \nu_2 \in I_R(\underline{L}_1)$ . Now we can extend  $u \in I(\underline{L}_1)$  to  $\lambda \in I(\underline{L}_2)$  and extend  $\nu_1$  to  $\tau_1 \in I_R(\underline{L}_2)$ ,  $\nu_2$  to  $\tau_2 \in I_R(\underline{L}_2)$ . Now we have  $\lambda \leq \tau_1(\underline{L}_2)$ ,  $\lambda \leq \tau_2(\underline{L}_2)$  which is not difficult to see since  $\underline{L}_1$  separates  $\underline{L}_2$ . Now  $\underline{L}_2$  is normal, therefore  $\tau_1 = \tau_2$  and  $\nu_1 = \tau_1 \mid A(\underline{L}_1) = \tau_2 \mid A(\underline{L}_1) = \nu_2$ . Therefore  $\underline{L}_1$  is normal.

### 3. THE WALLMAN SPACE $I_R(\underline{L})$ .

We give here a brief discussion of the general Wallman space (see also [11]). Consider the set  $I_R(\underline{L})$  and the lattice of subsets  $w(\underline{L})$ . It is well-known that  $w(\underline{L})$  is compact and it is not difficult to show:

**THEOREM 3.1.** The following are equivalent:

- (a)  $w(\underline{L})$  is normal;
- (b)  $w(\underline{L})$  is regular;
- (c)  $w(\underline{L})$  is  $T_2$ .

Now since  $w(\underline{L})$  is compact,  $\tau w(\underline{L})$  the topology of closed sets, is compact and  $w(\underline{L})$  separates  $\tau w(\underline{L})$ , and by Theorem 2.8  $w(\underline{L})$  is normal if and only if  $\tau w(\underline{L})$  is normal.  $\langle I_R(\underline{L}), \tau w(\underline{L}) \rangle$  is a compact topological space and it is always  $T_1$ . Assuming  $\underline{L}$  is disjunctive, it is  $T_2$  if and only if  $\underline{L}$  is normal. Next, let  $\underline{L}$  be a  $\delta$ -normal lattice of subsets of  $X$ , then the Alexandroff representation theorem (see [1]) yields for the conjugate space of  $C_b(\underline{L})$ , namely  $C_b(\underline{L})' = M_R(\underline{L})$  where to any  $\Phi \in C_b(\underline{L})'$  there corresponds a unique  $u \in M_R(\underline{L})$  such that  $\Phi(f) = \int_X f du$ , for all  $f \in C_b(\underline{L})$ .

A net  $\{u_\alpha\}$  in  $M_R(\underline{L})$  converges to  $u$  in  $M_R(\underline{L})$  in the weak  $*$  topology if and only if  $\int_X f du_\alpha \rightarrow \int_X f du$  for all  $f \in C_b(\underline{L})$ . We shall denote weak  $*$  convergence by  $\underline{w}^*$ .

**THEOREM 3.2.** Now let  $\underline{L}$  be  $\delta$ -normal and consider convergence in  $M_R^+(\underline{L})$ . The following are equivalent:

- (1)  $u_\alpha \xrightarrow{\underline{w}^*} u$
- (2)  $u_\alpha(X) \rightarrow u(X)$  and  $\overline{\lim}_\alpha u_\alpha(A) \leq u(A)$  for all  $A \in \underline{L}$
- (3)  $u_\alpha(X) \rightarrow u(X)$  and  $\underline{\lim}_\alpha u_\alpha(A') \geq u(A')$  for all  $A' \in \underline{L}'$ . For the proof in this particular setting see ([7]).

**THEOREM 3.3.** Let  $u_\alpha \in I_R(\underline{L}) \xrightarrow{\underline{w}^*} u \in M_R(\underline{L})$  then  $u \in I_R(\underline{L})$ . Thus  $I_R(\underline{L})$  is  $w^*$ -closed in  $M_R(\underline{L})$ .

**PROOF.** Suppose  $u_\alpha \in I_R(\underline{L}) \xrightarrow{\underline{w}^*} u \in M_R(\underline{L})$ . Therefore  $u_\alpha(X) \rightarrow u(X)$  by Theorem 3.2. Now  $u_\alpha(X) = 1$  since  $u_\alpha \in I_R(\underline{L})$ , therefore  $u(X) = 1$ , which means for  $A \in A(\underline{L})$ :  $0 \leq u(A) \leq 1$ . Suppose  $A \in A(\underline{L})$  and  $0 < u(A) < 1$ . Since  $u \in M_R(\underline{L})$  there exists  $L \in \underline{L} \subset A$  such that  $0 < u(L) \leq u(A)$  and there exists  $A' \subset \tilde{L}' \in \underline{L}'$  such that  $u(A') \leq u(\tilde{L}') < 1$ . Therefore  $0 < u(L) \leq u(\tilde{L}') < 1$ . Now  $L \subset \tilde{L}'$  therefore  $L \cap \tilde{L} = \emptyset$  which implies there exists  $A, B \in \underline{L}$  such that  $L \subset A, \tilde{L} \subset B, A' \cap B' = \emptyset$  by  $\underline{L}$  normal. Therefore  $L \subset A' \subset B \subset \tilde{L}'$  which implies  $0 < u(L) \leq u(A') \leq u(B) \leq u(\tilde{L}') < 1$  so that  $u(A') \leq u(B) < 1$ . Now since  $u_\alpha \xrightarrow{\underline{w}^*} u$  then  $\overline{\lim}_\alpha u_\alpha(B) \leq u(B)$  for  $B \in \underline{L}$ . Now  $u(B) < 1$ , therefore  $\overline{\lim}_\alpha u_\alpha(B) < 1$  which means  $u_\alpha(B) = 0$  since  $u_\alpha \in I_R(\underline{L})$ . Also  $\underline{\lim}_\alpha u_\alpha(A') \geq u(A')$  for  $A' \in \underline{L}'$  but  $u(A') < 1$ , therefore  $\underline{\lim}_\alpha u_\alpha(A') > 0$  as  $0 < u(A') < 1$ . Therefore  $u_\alpha(A') = 1$  since  $u_\alpha \in I_R(\underline{L})$ . However for  $A' \subset B$  we have  $u_\alpha(A') = 1, u_\alpha(B) = 0$  which is impossible. Therefore  $u(A) = 0$  or  $1$ , which implies  $u \in I_R(\underline{L})$ .

**THEOREM 3.4.**  $[\{u_x\}] = M_R(\underline{L})$

**PROOF.** The proof of this is not difficult and can be modelled after the well-known special case of  $\underline{L}$  being the lattice of zero sets in a Tychonoff space.

**THEOREM 3.5.** The  $w^*$ -topology of  $M_R(\underline{L})$  when restricted to  $I_R(\underline{L})$  gives the Wallman topology  $\tau w(\underline{L})$  for closed sets.

**PROOF.** Let  $u_\alpha \xrightarrow{\underline{w}^*} u$  we will show  $u_\alpha \xrightarrow{w} u$  where  $w$  is convergence in Wallman. Consider  $u_0 \in w(\underline{L})$ , therefore  $u_0(L) = 1$ . Using Theorem 3.2 we have  $\underline{\lim}_\alpha u_\alpha(L) \geq u_0(L)$ , therefore  $\underline{\lim}_\alpha u_\alpha(L) = 1$ . But  $1 = \underline{\lim}_\alpha u_\alpha(L) \leq \overline{\lim}_\alpha u_\alpha(L) \leq 1$ , therefore  $\overline{\lim}_\alpha u_\alpha(L) = 1$ . So there exists  $\alpha_0$  such that for all  $\alpha \geq \alpha_0$   $u_\alpha(L) = 1$ , therefore  $u_\alpha \in w(\underline{L})$  for all  $\alpha \geq \alpha_0$ . This gives  $u_\alpha \xrightarrow{w} u$  which proves the theorem.

We assume now that  $\underline{L}$  is  $\delta$ -normal, separating and disjunctive. Let  $f \in C_b(\underline{L})$  we define  $\hat{f}$  on  $I_R(\underline{L})$  by  $\hat{f}(u) = \int_X f du$  where  $u \in I_R(\underline{L})$ .

**THEOREM 3.6.**  $\hat{f} \in C(I_R(\underline{L}))$ .

PROOF. Let  $u_\alpha \xrightarrow{w} u_0$ . We must show that  $\hat{f}(u_\alpha) \rightarrow \hat{f}(u_0)$  which means  $u_\alpha \xrightarrow{w^*} u_0$ . For  $u_0 \in w(L')$  we have  $u_\alpha \in w(L')$  for all  $\alpha \geq \alpha_0$  as  $u_\alpha \xrightarrow{w} u_0$ . Therefore,  $u_\alpha(L') = 1$ ,  $\alpha \geq \alpha_0$ , which implies  $\lim u_\alpha(L') = 1$ . Therefore  $\lim u_\alpha(L') = \overline{\lim} u_\alpha(L') = \lim u_\alpha(L') = 1$  and  $u_0(L') = 1$  as  $u_0 \in w(L')$ . So  $\lim u_\alpha(L') \geq u_0(L')$  and therefore by Theorem 3.2 we have  $u_\alpha \xrightarrow{w^*} u_0$  which proves  $\hat{f} \in C(I_R(\underline{L}))$ .

THEOREM 3.7. The correspondence  $f \rightarrow \hat{f}$  is a bijection between  $C_b(\underline{L})$  and  $C(I_R(\underline{L}))$ ; the continuous functions on the Wallman space  $I_R(\underline{L})$ .

PROOF. Let  $A = \{\hat{f} \mid f \in C_b(\underline{L})\}$ . Then  $A \subset C(\tau w(\underline{L})) = C(I_R(\underline{L}))$ . Since  $u_\alpha \xrightarrow{w^*} u \Rightarrow \hat{f}(u) \in C(I_R(\underline{L}))$ . Now it is easy to show the following:

- (1)  $\widehat{f+g} = \hat{f} + \hat{g}$
- (2)  $\widehat{af} = a\hat{f}$  for  $a \in R$
- (3)  $\widehat{fg} = \hat{f}\hat{g}$
- (4)  $\|\hat{f}\| = \|f\|$ , therefore  $A$  is a closed subalgebra of  $C(\tau w(\underline{L}))$
- (5)  $A$  separates points. We can prove this by showing for  $u, v \in I_R(\underline{L})$ ,  $u \neq v$  there exists  $\hat{f} \in A$  such that  $\hat{f}(u) = 1$  and  $\hat{f}(v) = 0$ . This is done by using the normality of  $\underline{L}$ .
- (6)  $1 \in A$ . Therefore given  $u \in I_R(\underline{L})$  there exists  $\hat{f} \in A$  such that  $\hat{f}(u) \neq 0$ .

So by the Stone-Weierstrass theorem  $A = \bar{A} = C(\tau w(\underline{L}))$  which proves the theorem.

#### 4. THE SPACE $Q(\underline{L})$ .

In this section, we consider the important measures of  $I_R(\underline{L})$  which integrate all  $f \in C(\underline{L})$  and consider their relationship to  $I_R^\sigma(\underline{L})$ . Let  $\underline{L}$  be  $\delta$ -normal lattice. We define  $Q(\underline{L}) = \{u \in I_R(\underline{L}) \mid \int_X |f| du < \infty \text{ for all } f \in C(\underline{L})\}$ .

THEOREM 4.1.  $I_R^\sigma(\underline{L}) \subset Q(\underline{L})$ .

PROOF. Let  $v \in I_R^\sigma(\underline{L})$  and  $L_n = \{|f| \geq n\}$ . One can see  $L_n \downarrow \phi$  which implies  $v(L_n) \rightarrow 0$  since  $v \in I_R^\sigma(\underline{L})$ . Therefore  $v(L_N) = 0$  for  $N$  big. Now

$$\begin{aligned} \int_X |f| dv &= \int_{L_N} |f| dv + \int_{L_N'} |f| dv \\ &\leq NV L_{N'} \\ &\leq N \end{aligned}$$

Therefore  $\int_X |f| dv \leq N$  which proves  $v \in Q(\underline{L})$ .

THEOREM 4.2.  $I_R(\underline{L}) \cap I_\sigma(\underline{L}) \subset Q(\underline{L})$ .

PROOF. Let  $(|f| > n) = A_n$ . Clearly  $A_n' \downarrow \phi$  and  $A_n \in \underline{L}$  for all  $n$ . Now let  $v \in I_R(\underline{L}) \cap I_\sigma(\underline{L})$ , therefore  $v(A_n) = 0$ ,  $n \geq N$ . Now  $\int_X |f| dv = \int_{A_N} |f| dv + \int_{A_N'} |f| dv$ . Therefore  $\int_X |f| dv \leq NV(A_N)$ , so  $\int_X |f| dv \leq NV(A_N) < \infty$ . Therefore,  $v \in Q(\underline{L})$  which provides the theorem.

THEOREM 4.3.  $I_R^\sigma(\underline{L}) \subset I_R(\underline{L}) \cap I_\sigma(\underline{L}) \subset Q(\underline{L})$ .

PROOF. By Theorems 4.1 and 4.2 and the trivial observation that  $I_R^\sigma(\underline{L}) \subset I_R(\underline{L}) \cap I_\sigma(\underline{L})$ , the result is proved.

Following Varadarajan who considered the lattice of zero sets in a Tychonoff space, we introduce

DEFINITION. The Sequence  $\{B_n\}$  in  $\underline{L}$  is called regular if  $B_n' \downarrow \phi$  and there exists  $A_n$  in  $\underline{L}$  such that  $B_n \subset A_n' \subset B_{n+1}$  for all  $n$ .

THEOREM 4.4. Let  $\{B_n\}$  be a regular sequence. Then there exists  $\{f_n\}$ ,  $f_n \in C_b(\underline{L})$ ,  $0 \leq f_n \leq 1$  such that  $f_n \downarrow \phi$ ,  $f_n(B_n) = 0$ ,  $f_n(B_{n+1}') = 1$  for  $n = 1, 2, \dots$

PROOF. Omitted.

**THEOREM 4.5.** Let  $X$  be an abstract set and  $\underline{L}$  a  $\delta$ -normal lattice of subsets which is also countably paracompact. Let  $\{A_n\}$  in  $\underline{L}$ ,  $A_n \perp \phi$ . Then there exists a regular sequence  $\{C_n\}$  such that  $C_n \subset A_n'$  for all  $n$ .

**PROOF.** Since  $A_n \perp \phi$  and since  $\underline{L}$  is countably paracompact then there exists  $\{B_n\}$  in  $\underline{L}$  with  $A_n \subset B_n' \perp \phi$ . Now we show by induction that for any  $n$  we have  $\{C_K\}, \{D_K\}$  in  $\underline{L}$  with  $A_K \subset C_K' \subset D_K \subset (B_K' \cap C_{K-1}')$  where  $K = 1, \dots, n$ : (1) For  $n = 1$ , take  $C_0 = \phi$ , and  $A_1 \subset C_1' \subset D_1 \subset B_1'$  follows by normality. (2) Assume expression is true for  $n$ . Now  $A_{n+1} \subset B_{n+1}'$  and  $A_{n+1} \subset A_n \subset C_n'$ , therefore  $A_{n+1} \subset B_{n+1}' \cap C_n'$ . Using normality, there exists  $C_{n+1}, D_{n+1} \in \underline{L}$  such that  $A_{n+1} \subset C_{n+1}' \subset D_{n+1} \subset (B_{n+1}' \cap C_n')$  which finishes the induction argument. Since  $C_n \subset A_n'$  we must show  $\{C_n\}$  is regular. Now  $C_n' \subset B_n'$  implies  $C_n' \perp \phi$  and  $C_n \subset D_{n+1}' \subset C_{n+1}$ . Therefore  $\{C_n\}$  is regular as  $D_{n+1} \in \underline{L}$ . Finally using the previous two results it is not difficult to show using an argument similar to Varadarajan that the following holds:

**THEOREM 4.6.** Let  $\underline{L}$  be  $\delta$ -normal and countably paracompact, then  $Q(\underline{L}) \subset I_R^\sigma(\underline{L})$ .

So using Theorems 4.1 and 4.6 we have:

**THEOREM 4.7.** Let  $\underline{L}$  be  $\delta$ -normal and countably paracompact, then  $Q(\underline{L}) = I_R^\sigma(\underline{L})$ .

We also have:

**THEOREM 4.8.** If  $Q(\underline{L}) = I_R(\underline{L}) \cap I_\sigma(\underline{L}')$  and if  $I_\sigma(\underline{L}') \subset I_\sigma(\underline{L})$  then  $Q(\underline{L}) = I_R^\sigma(\underline{L})$ .

**PROOF.**  $Q(\underline{L}) = I_R(\underline{L}) \cap I_\sigma(\underline{L}') \subset I_R(\underline{L}) \cap I_\sigma(\underline{L})$ , but we know if  $v \in M_R(\underline{L})$  and  $v \in M_\sigma(\underline{L})$  then  $v \in M_R^\sigma(\underline{L})$ . Therefore  $Q(\underline{L}) \subset I_R(\underline{L}) \cap I_\sigma(\underline{L}) = I_R^\sigma(\underline{L})$ , so  $Q(\underline{L}) \subset I_R^\sigma(\underline{L})$ . However, from Theorem 4.1 we have  $I_R^\sigma(\underline{L}) \subset Q(\underline{L})$ , therefore  $Q(\underline{L}) = I_R^\sigma(\underline{L})$ .

Note:  $I_\sigma(\underline{L}') \subset I_\sigma(\underline{L})$  if  $\underline{L}$  is countably paracompact, also if  $\underline{L}$  is regular and Lindelöf.

Now we consider two lattices  $\underline{L}_1$  and  $\underline{L}_2$  such that  $\underline{L}_1 \subset \underline{L}_2$ . Then  $C(\underline{L}_1) \subset C(\underline{L}_2)$ .

**THEOREM 4.9.** Let  $\underline{L}_1, \underline{L}_2$  be lattices of subsets such that  $\underline{L}_1$  semi-separates  $\underline{L}_2$ . If  $v \in Q(\underline{L}_2)$  and if  $u = \nu|A(\underline{L}_1)$ , then  $u \in Q(\underline{L}_1)$ .

**PROOF.** Since  $\underline{L}_1$  semi-separates  $\underline{L}_2, u \in I_R(\underline{L}_1)$ . Also, since  $C(\underline{L}_1) \subset C(\underline{L}_2)$  and since  $v$  integrates all  $f \in C(\underline{L}_2)$ ,  $u$  integrates all  $g \in C(\underline{L}_1)$ . Hence  $u \in Q(\underline{L}_1)$ .

**THEOREM 4.10.** Let  $\underline{L}_1, \underline{L}_2$  be lattice of subsets such that  $\underline{L}_1$  separates  $\underline{L}_2$ . Let  $v \in Q(\underline{L}_2)$  and  $u = \nu|A(\underline{L}_1)$ . If  $Q(\underline{L}_1) = I_R^\sigma(\underline{L}_1)$  then  $v \in I_\sigma(\underline{L}_2')$ .

**PROOF.** By the previous theorem  $u \in Q(\underline{L}_1) = I_R^\sigma(\underline{L}_1)$  by hypothesis, and since  $\underline{L}_1$  separates  $\underline{L}_2$  it is easy to see  $\nu$ , the extension of  $u$ , is in  $I_\sigma(\underline{L}_2')$ .

**THEOREM 4.11.** Let  $\underline{L}_1, \underline{L}_2$  be lattice of subsets such that  $\underline{L}_1$  separates  $\underline{L}_2$ . If  $Q(\underline{L}_1) = I_R^\sigma(\underline{L}_1)$  then  $Q(\underline{L}_2) = I_R(\underline{L}_2) \cap I_\sigma(\underline{L}_2')$ .

**PROOF.**  $v \in Q(\underline{L}_2)$  implies  $v \in I_R(\underline{L}_2)$ , but  $v \in I_\sigma(\underline{L}_2')$  from Theorem 4.10, therefore  $Q(\underline{L}_2) \subset I_R(\underline{L}_2) \cap I_\sigma(\underline{L}_2')$ . However we know if  $v \in I_R(\underline{L}_2) \cap I_\sigma(\underline{L}_2')$  then  $v \in Q(\underline{L}_2)$  from Theorem 4.2 which proves the result.

We have the following application: For  $\underline{L}$   $\delta$ -normal,  $z(\underline{L}) \subset \underline{L}$  where  $z(\underline{L})$  consists of all sets of  $\underline{L}$  of the form  $L = \bigcap_{n=1}^\infty L_n'$ ,  $L_n \in \underline{L}$  for all  $n$ , (see [1]). Now  $z(\underline{L})$  separates  $\underline{L}$  and  $z(\underline{L})$  is normal and countably paracompact. Therefore by Theorem 4.7 we have  $I_R^\sigma(z(\underline{L})) = Q(z(\underline{L}))$ . Now using Theorem 4.11 we have  $Q(\underline{L}) = I_R(\underline{L}) \cap I_\sigma(\underline{L}')$ . Also if  $I_\sigma(\underline{L}') \subset I_\sigma(\underline{L})$  then  $Q(\underline{L}) = I_R^\sigma(\underline{L})$  by Theorem 4.8.

**REMARK.** We recall that if  $X$  is Tychonoff space and if  $\underline{L} = z$ , the lattice of zero sets then  $(I_R^\sigma(z), \tau w_\sigma(z))$  is the realcompactification  $\nu(X)$  of  $X$ .

Now we consider other criterion for  $Q(\underline{L}) = I_R(\underline{L}) \cap I_\sigma(\underline{L}')$ . If  $X$  is a topological space and if  $A \subset X$  we denote by  $\bar{A}^\delta$  the  $G_\delta$ -closure of  $A$ . Now if  $X$  is an abstract set and  $\underline{L}$  as usual is a



separating disjunctive  $\delta$ -normal lattice of subsets then we can view  $X$  embedded in  $Q(\underline{L})$ ; we have  $X \subset Q(\underline{L}) \subset I_R(\underline{L})$ . In fact, using Theorem 4.3 we have  $X \subset I_R^\sigma(\underline{L}) \subset I_R(\underline{L}) \cap I_\sigma(\underline{L}') \subset Q(\underline{L}) \subset I_R(\underline{L})$ .

THEOREM 4.12.  $\bar{X}^\delta \subset Q(\underline{L})$  where  $\bar{X}^\delta$  is the  $G_\delta$ -closure of  $X$  in the Wallman space  $I_R(\underline{L})$ .

PROOF. Suppose  $u \in \bar{X}^\delta$ . If  $u \notin Q(\underline{L})$  then there exists  $f \in C(\underline{L})$ ,  $f \geq 0$  such that  $\int f du = \infty$ . Let  $A_n' = \{f > n\} \in \underline{L}'$ . Then  $A_n' \downarrow \phi$  and  $u(A_n') = 1$ . Therefore  $u \in \bigcap_{n=1}^\infty w(A_n') \subset I_R(\underline{L}) - X$  which contradicts the fact  $u \in \bar{X}^\delta$ . Therefore  $u \in Q(\underline{L})$ , so  $\bar{X}^\delta \subset Q(\underline{L})$ .

THEOREM 4.13. If  $Q(\underline{L}) \subset \bar{X}^\delta$ , then  $G_\delta$ -closure of  $X$  in  $I_R(\underline{L})$ , then  $u \in I_\sigma(\underline{L}')$  where  $u \in Q(\underline{L})$ .

PROOF. Suppose  $u \in Q(\underline{L})$  which implies  $u \in I_R(\underline{L})$ . If  $u \notin I_\sigma(\underline{L}')$  then there exists  $L_n' \downarrow \phi$ ,  $L_n \in \underline{L}$ ,  $u(L_n') = 1$ . Therefore  $u \in \bigcap_{n=1}^\infty w(L_n') \subset I_R(\underline{L}) - X$ . Therefore  $u \notin \bar{X}^\delta$ , so  $Q(\underline{L}) \subset \bar{X}^\delta$  implies  $u \in I_\sigma(\underline{L}')$ .

THEOREM 4.14.  $Q(\underline{L}) = \bar{X}^\delta$  if and only if  $u \in I_\sigma(\underline{L}')$  for all  $u \in Q(\underline{L})$ .

PROOF. If  $Q(\underline{L}) = \bar{X}^\delta$  and if  $u \in Q(\underline{L})$  then  $u \in I_\sigma(\underline{L}')$  by the previous theorem. While if  $Q(\underline{L}) \subset I_\sigma(\underline{L}')$  then we must have  $Q(\underline{L}) \subset \bar{X}^\delta$  for if not then there exists  $G \in G_\delta$  such that  $u \in G \subset I_R(\underline{L}) - X$  where  $u \in I_R(\underline{L})$ . Therefore  $u \in \bigcap_{n=1}^\infty O_n \subset I_R(\underline{L})$  where  $O_n$  is an open set, which implies  $u \in O_n$  for all  $n$ . Now  $w(L_n')$  is an open set for  $L_n \in \underline{L}$ , therefore  $u \in w(L_n') \subset O_n$  which yields  $u \in \bigcap_{n=1}^\infty w(L_n') \subset \bigcap_{n=1}^\infty O_n$ . Therefore there exists  $u \in Q(\underline{L})$  such that  $u \in \bigcap_{n=1}^\infty w(L_n')$  where the  $w(L_n') \downarrow \phi$  and where  $\bigcap_{n=1}^\infty L_n \in \underline{L}$  and  $\bigcap_{n=1}^\infty w(L_n') \subset I_R(\underline{L}) - X$ , but then  $u(L_n') = 1$  for all  $n$  and  $L_n' \downarrow \phi$  which is a contradiction. Thus  $Q(\underline{L}) \subset \bar{X}^\delta$  and then by Theorem 4.12,  $Q(\underline{L}) = \bar{X}^\delta$ .

Using the previous theorem and Theorem 4.2 we have:

COROLLARY 4.15. If  $\underline{L}$  is  $\delta$ -normal separating and disjunctive then  $Q(\underline{L}) = \bar{X}^\delta$  if and only if  $Q(\underline{L}) = I_R(\underline{L}) \cap I_\sigma(\underline{L}')$ .

REMARK. We note that  $Q(\underline{L}) = I_R(\underline{L})$  if and only if  $C_b(\underline{L}) = C(\underline{L})$ ; this situation arises in particular if  $C(\underline{L})$  consists only of constant functions. (see below)

## 5. THE WALLMAN SPACE $I_R^\sigma(\underline{L})$ .

First we note  $I_R^\sigma(\underline{L})$  may be empty. Let  $X = \{0, 1, 2, \dots\}$  where  $\underline{L}$  consists of  $\phi$  and all sets of the form  $\{n, n+1, \dots\}$  for all  $n$ , and  $v_1, v_2 \in I_R(\underline{L})$ . If  $v_1 \neq v_2$  then there exists  $L_1, L_2 \in \underline{L}$  such that  $v_1(L_1) = 1$ ,  $v_2(L_1) = 0$ ,  $v_1(L_2) = 0$ ,  $v_2(L_2) = 1$  and  $L_1 \cap L_2 = \phi$ . However, this is impossible here as  $L_1 \cap L_2 \neq \phi$  unless  $L_1$  or  $L_2 = \phi$ . Therefore  $I_R(\underline{L}) = \{u\}$ . Now clearly if  $L_n = \{n, n+1, \dots\}$ ,  $L_n \in \underline{L}$  and  $L_n \downarrow \phi$ . However,  $u(L_n) = 1$  for all  $n$ , therefore  $I_R^\sigma(\underline{L}) = \phi$ . We also have in this example:  $C(\underline{L}) = C_b(\underline{L}) = \text{constant functions}$ ;  $\underline{L}$  is not disjunctive,  $\underline{L}$  is not countably paracompact;  $\underline{L}$  is not regular;  $\underline{L}$  is a  $\delta$ -lattice.

Now we state a familiar result:

THEOREM 5.1. Let  $\underline{L}$  be disjunctive then  $\langle I_R^\sigma(\underline{L}), w_\sigma(\underline{L}) \rangle$  is replete.

Next we give facts about  $C(\underline{L})$ : we denote by  $M_R^I(\underline{L})$  the set  $\{u \in M_R^\sigma(\underline{L}) \mid \int |f| d|u| < \infty \text{ for all } f \in C(\underline{L})\}$ . Note  $I_R^\sigma(\underline{L}) \subset M_R^I(\underline{L})$  and we denote by, similar to Varadarajan,  $\tilde{W}_I$  the topology on  $M_R^I(\underline{L})$ . A net  $\{u_\alpha\}$  in  $M_R^I(\underline{L})$  converges to  $u$  in  $M_R^I(\underline{L})$  with respect to  $W_I$  if and only if  $\int_X f du_\alpha \rightarrow \int_X f du$  for all  $f \in C(\underline{L})$ . The topology  $W_I$  restricted to  $I_R^\sigma(\underline{L})$  is the Wallman topology. Now using this it is easy to show that  $\hat{f}(u) = \int f du$ ,  $u \in I_R^\sigma(\underline{L})$  is continuous with respect to the Wallman topology  $\tau w_\sigma(\underline{L})$  on  $I_R^\sigma(\underline{L})$ , i.e.,  $\hat{f}(u) \in C(I_R^\sigma(\underline{L})) = C(\tau w_\sigma(\underline{L}))$ . Let  $\underline{L}$  be separating, disjunctive and  $\delta$ -normal throughout and  $f \in C(\underline{L})$ .

THEOREM 5.2. Let  $f \in C(\underline{L})$  then  $\hat{f}^{-1}[a, \infty) = Z(\hat{g})$  where  $g = (f - a) \wedge 0 \in C(\underline{L})$  and similarly  $\hat{f}^{-1}(-\infty, a] = Z(\hat{h})$  where  $h \in C(\underline{L})$ .

PROOF. Omitted.

THEOREM 5.3. Let  $z(\underline{L})$  be the zero lattice of  $C(\underline{L})$  then  $w_\sigma(z(\underline{L})) = z(w_\sigma(\underline{L}))$ .

PROOF. Let  $Z \in z(\underline{L}) \subset \underline{L}$ . Therefore by a theorem of Alexandroff  $Z = \bigcap_{n=1}^{\infty} L_n', L_n \in \underline{L}$  all  $n$ . Thus  $w_{\sigma}(Z) = \bigcap_{n=1}^{\infty} w_{\sigma}(L_n)'$ . But  $w_{\sigma}(\underline{L})$  is  $\delta$ -normal therefore by Alexandroff theorem again we get  $w_{\sigma}(Z) \in z(w_{\sigma}(\underline{L}))$ . Converse if  $w_{\sigma}(L) \in z(w_{\sigma}(\underline{L}))$ , where  $L \in \underline{L}$  then  $w_{\sigma}(L) = \bigcap_{n=1}^{\infty} w_{\sigma}(L_n) = w_{\sigma}(\bigcap_{n=1}^{\infty} L_n)$  and since  $\underline{L}$  is disjunctive,  $L = \bigcap_{n=1}^{\infty} L_n' \in z(\underline{L})$  again by Alexandroff's result and the proof is completed.

We have seen that if  $f \in C(\underline{L})$  then  $\hat{f} \in C(\tau w_{\sigma}(\underline{L}))$ , i.e., it is continuous with respect to Wallman topology on  $I_R^{\sigma}(\underline{L})$ . However we can do better.

THEOREM 5.4. If  $f \in C(\underline{L})$  then  $\hat{f} \in C(w_{\sigma}(\underline{L}))$  (where  $\hat{f}(u) = \int f du$  for all  $u \in I_R^{\sigma}(\underline{L})$ ).

PROOF. We must show  $\hat{f}^{-1}(E) \in w_{\sigma}(\underline{L})$  for any closed set  $\overset{X}{E} \subset R$ . It will suffice to show this for  $E = [a, b] \subset R$ . Now  $[a, b] = (-\infty, b] \cap [a, \infty)$  so  $\hat{f}^{-1}[a, b] = \hat{f}^{-1}[(-\infty, b] \cap [a, \infty)] = \hat{f}^{-1}(-\infty, b] \cap \hat{f}^{-1}[a, \infty) = Z(\hat{h}) \cap Z(\hat{g})$  using Theorem 5.2. Next we note if  $g \in C(\underline{L})$  then  $Z(\hat{g}) = \overline{Z(g)}$  where the closure is taken in the Wallman space  $I_R^{\sigma}(\underline{L})$  with topology of closed sets  $\tau w_{\sigma}(\underline{L})$ . Therefore  $\hat{f}^{-1}[a, b] = \overline{Z(g)} \cap \overline{Z(h)}$ , and  $Z(g), Z(h) \in z(\underline{L}) \subset \underline{L}$  so  $\hat{f}^{-1}[a, b] = \overline{Z(g)} \cap \overline{Z(h)}$  (using  $\overline{A \cap B} = \overline{A} \cap \overline{B}$  for  $A, B \in \underline{L}$ ). In addition  $\hat{f}^{-1}[a, b] = \overline{Z(g^2 + h^2)}$  and  $Z(g^2 + h^2) = Z \in z(\underline{L})$  and  $\overline{Z(g^2 + h^2)} = \overline{Z} = w_{\sigma}(Z)$ . Therefore  $\hat{f}^{-1}[a, b] = \overline{Z} = w_{\sigma}(Z)$  which implies  $\hat{f}^{-1}[a, b] \in w_{\sigma}(z(\underline{L}))$ . However using Theorem 5.3 we get  $\hat{f}^{-1}[a, b] \in z(w_{\sigma}(\underline{L}))$ . However  $z(w_{\sigma}(\underline{L})) \subset w_{\sigma}(\underline{L})$  therefore  $\hat{f}^{-1}[a, b] \in w_{\sigma}(\underline{L})$  which implies  $\hat{f} \in C(w_{\sigma}(\underline{L}))$ .

Now we intend to prove the converse. Suppose that  $h \in (w_{\sigma}(\underline{L}))$  then clearly  $h|_X \in C(\underline{L})$  and let  $h|_X = f \in C(\underline{L})$  then  $h = \hat{f}$  since both are continuous with respect to the Wallman topology and they agree on  $X$  which is dense in  $I_R^{\sigma}(\underline{L})$ .

Using the above results we have the following:

THEOREM 5.5. The correspondence  $f \rightarrow \hat{f}$  is a bijection between  $C(\underline{L})$  and  $C(w_{\sigma}(\underline{L}))$ ; the  $w_{\sigma}(\underline{L})$ -continuous functions on the Wallman space  $I_R^{\sigma}(\underline{L})$ .

Next let  $u \in I_R(\underline{L})$ , then we define  $M^u = \{f \in C(\underline{L}) \mid u \in \overline{Z(f)}^{I_R(\underline{L})}\}$ . The following facts we list for completeness (proofs can be found for this setting in [8]):

- 1) If  $u_1, u_2 \in I_R(\underline{L})$  and if  $u_1 \neq u_2$  then  $M^{u_1} \neq M^{u_2}$ .
- 2)  $M^u$  is a maximal ideal in  $C(\underline{L})$ .
- 3) (Generalized Gelfand-Kolmogoroff) If  $M$  is a maximal ideal in  $C(\underline{L})$  then there exists  $u \in I_R(\underline{L})$  such that  $M = M^u$ .

Thus there exists a one to one correspondence between elements of  $I_R(\underline{L})$  and maximal ideals of  $C(\underline{L})$ .

Now we return to the Wallman space  $\langle I_R^{\sigma}(\underline{L}), \tau w_{\sigma}(\underline{L}) \rangle$  and give conditions when this topological space is realcompact. We know that for  $\underline{L}$  disjunctive  $w_{\sigma}(\underline{L})$  is replete; the question we are now concerned with is: when is the lattice  $z(\tau w_{\sigma}(\underline{L}))$  replete? or i.e., when is the Wallman space realcompact?

THEOREM 5.6. Let  $\underline{L}$  be  $\delta$ -normal, separating, disjunctive, and countably paracompact then  $Q(\underline{L}) = I_R^{\sigma}(\underline{L})$  and if  $I_R^{\sigma}(\underline{L})$  with the Wallman topology is a c.b. space then it is realcompact.

PROOF.  $Q(\underline{L}) = I_R^{\sigma}(\underline{L})$  by Theorem 4.7. Now  $\langle I_R^{\sigma}(\underline{L}), w_{\sigma}(\underline{L}) \rangle$  is replete from Theorem 5.1. Now  $w_{\sigma}(\underline{L}) \subset \tau w_{\sigma}(\underline{L})$  (of course) and consequently  $z(\tau w_{\sigma}(\underline{L})) \subset \tau w_{\sigma}(\underline{L})$ . Now  $\underline{L}$   $\delta$ -normal implies  $w_{\sigma}(\underline{L})$   $\delta$ -normal and  $\underline{L}$  countably paracompact implies  $w_{\sigma}(\underline{L})$  is countably paracompact. Then by Theorem 5.3 of [2] we have  $\tau w_{\sigma}(\underline{L})$  is replete. Now by hypothesis  $\tau w_{\sigma}(\underline{L})$  is  $z(\tau w_{\sigma}(\underline{L}))$  countably bounded (c.b.). Thus  $z(\tau w_{\sigma}(\underline{L}))$  is replete by Theorem 3.4 of [2]. Hence  $\langle I_R^{\sigma}(\underline{L}), \tau w_{\sigma}(\underline{L}) \rangle$  is realcompact.

Note. If  $\tau w_{\sigma}(\underline{L})$  is  $z(\tau w_{\sigma}(\underline{L}))$  countably paracompact the same conclusion can be drawn.

We continue to assume that  $\underline{L}$  is separating, disjunctive and  $\delta$ -normal. Let  $h \in C(I_R^\sigma(\underline{L}))$  or, i.e.,  $h \in C(\tau w_\sigma(\underline{L}))$  in lattice notation, then  $f = h|_X \in C(\tau \underline{L})$ , clearly. If  $f \in C(\underline{L})$  then by our earlier work in this section we would have  $h = \hat{f} \in C(w_\sigma(\underline{L}))$ . This situation arises if  $X$  is a Tychonoff topological space and  $\underline{L} = z$  lattice of zero sets of continuous functions on  $X$  for in this case if  $h \in C(I_R^\sigma(z))$  then  $h|_X \in C(\tau z) = C(F) = C(z)$  where  $F$  is the lattice of closed sets of  $X$ . Thus, in this case,  $w_\sigma(z) = z(\tau w_\sigma(z))$  and since  $w_\sigma(z)$  is replete, we have that  $I_R^\sigma(z)$  is realcompact with respect to the Wallman space.

**THEOREM 5.7.** Let  $\underline{L}$  be separating, disjunctive and  $\delta$ -normal. If  $C(\tau w_\sigma(\underline{L})) = C(w_\sigma(\underline{L}))$  then  $z(w_\sigma(\underline{L})) = z(\tau w_\sigma(\underline{L}))$  and if  $w_\sigma(\underline{L})$  is  $z(w_\sigma(\underline{L}))$  c.b. or countably paracompact then  $I_R^\sigma(\underline{L})$  with the Wallman topology is realcompact.

**PROOF.** Since  $w_\sigma(\underline{L}) \subset \tau w_\sigma(\underline{L})$  then  $z(w_\sigma(\underline{L})) \subset z(\tau w_\sigma(\underline{L}))$ . Now let  $Z(f) \in z(\tau w_\sigma(\underline{L}))$  where  $f \in C(\tau w_\sigma(\underline{L}))$ , but  $C(\tau w_\sigma(\underline{L})) = C(w_\sigma(\underline{L}))$ . This implies  $Z(f) \in z(w_\sigma(\underline{L}))$ . Therefore  $z(\tau w_\sigma(\underline{L})) \subset z(w_\sigma(\underline{L}))$ . Now if  $w_\sigma(\underline{L})$  is  $z(w_\sigma(\underline{L}))$  countably bounded or countably paracompact then since  $w_\sigma(\underline{L})$  is replete we have using the same argument as in the proof of Theorem 5.6 that  $z(w_\sigma(\underline{L}))$  is replete, therefore  $z(\tau w_\sigma(\underline{L}))$  is replete.

Finally we extend Theorem 5.7 but first note  $z(w_\sigma(\underline{L})) \subset w_\sigma(\underline{L}) \subset \tau w_\sigma(\underline{L})$  and  $z(w_\sigma(\underline{L})) \subset z(\tau w_\sigma(\underline{L})) \subset \tau w_\sigma(\underline{L})$ .

**THEOREM 5.8.** Let  $\underline{L}$  be separating, disjunctive and  $\delta$ -normal. If  $\underline{L}$  is  $z(\underline{L})$  countably bounded (c.b.) or  $\underline{L}$  is  $z(\underline{L})$ -countably paracompact and assume  $z(\tau w_\sigma(\underline{L})) \subset \tau z(w_\sigma(\underline{L}))$ , then  $z(\tau w_\sigma(\underline{L}))$  is replete, i.e.,  $I_R^\sigma(\underline{L})$  with the Wallman topology is realcompact.

**PROOF.**  $z(w_\sigma(\underline{L}))$  is complement generated since  $z(\underline{L})$  is complement generated. (Use Theorem 5.3) and  $z(w_\sigma(\underline{L})) \subset z(\tau w_\sigma(\underline{L})) \subset \tau z(w_\sigma(\underline{L}))$ , therefore by Theorem 3.1 part (1) of [2] we have  $z(\tau w_\sigma(\underline{L}))$  is replete, as  $z(w_\sigma(\underline{L}))$  is replete from the fact  $\underline{L}$  is  $z(\underline{L})$  countably bounded or  $\underline{L}$  is  $z(\underline{L})$  countably paracompact.

**ACKNOWLEDGEMENT.** The author wishes to express his thanks to Jersey City State College for support of this paper through a grant of partial release time from teaching.

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