

THICKNESS IN TOPOLOGICAL TRANSFORMATION SEMIGROUPS

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ABSTRACT. This article deals with thickness in topological transformation semigroups (τ -semigroups). Thickness is used to establish conditions guaranteeing an invariant mean on a function space defined on a τ -semigroup if there exists an invariant mean on its functions restricted to a sub- τ -semigroup of the original τ -semigroup. We sketch earlier results, then give many equivalent conditions for thickness on τ -semigroups, and finally present theorems giving conditions for an invariant mean to exist on a function space.

KEY WORDS AND PHRASES. Thickness, topological transformation semigroup, transformation semigroup, invariant mean

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1. Left-Thickness in Semigroups

Mitchell introduced the concept of left-thickness in a semigroup [Mitchell, 1965]: a subset T of semigroup S is *left-thick* in $S \Leftrightarrow \forall$ finite $U \subseteq S, \exists t \in U: Ut \subseteq T$.

Any left ideal of a semigroup is left-thick, but not conversely. The complete relationship between left ideals and left-thick subsets is this: Let $\beta(S)$ be the Stone-Čech compactification of semigroup S endowed with the discrete topology, and let $T \subseteq S$. Then T is left-thick in $S \Leftrightarrow$ the closure of T in $\beta(S)$ contains a left ideal of $\beta(S)$ [Wilde & Witz, 1967, lemma 5.1]. (See Theorem 4.3.g *infra* for a more general formulation of this result.)

It can be shown that in the definition t can be taken in T or U can be a singleton.

Let $B(S)$ = the set of all bounded complex- or real-valued functions on semigroup S . For any $s \in S$ and $f \in B(S)$, $T_s f$ denotes the function in $B(S)$ defined by $T_s f(t) = f(st)$ ($\forall t \in S$).

A *mean* on $B(S)$ is a member of the dual space $B(S)^*$ of $B(S)$ which satisfies $\mu(1) = 1 = \|\mu\|$. Mean μ is *invariant* $\Leftrightarrow \mu(T_s f) = \mu f$ ($\forall s \in S, f \in B(S)$).

The importance of left-thickness for our subject is because of this theorem [Mitchell, 1965, theorem 9].

Theorem. Let T be a left-thick subsemigroup of semigroup S . Then $B(S)$ has a left-invariant mean $\Leftrightarrow B(T)$ has a left-invariant mean.

H. D. Junghenn generalized Mitchell's concept of left-thickness [Junghenn, 1979, p. 38].

First it is necessary to define more terms.

Subspace F of $B(S)$ is *left-translation invariant* $\Leftrightarrow T_s f \in F$ ($\forall s \in S, f \in F$). Let $\mu \in F^*$, the dual space of F ; define $T_\mu f$ ($\forall f \in F$) by $T_\mu f(s) = \mu(T_s f)$ ($\forall s \in S$). Then $T_\mu: F \rightarrow B(S)$. F is *left-introverted* $\Leftrightarrow T_\mu(F) \subseteq F$ ($\forall \mu \in F^*$).

Definition. Let S be a semigroup; $F \subseteq B(S)$ be a left-translation invariant, left-introverted, norm-closed subalgebra containing the constant functions; $T \subseteq S$ be non-empty;

$F(T) = \{g \in F \mid \chi_T \leq g \leq 1\}$. Then

T is *F-left thick* in $S \Leftrightarrow \forall \epsilon > 0, \exists g \in F(T)$, and finite $U = \{s_1, s_2, \dots, s_n\} \subseteq S \exists s \in S: g(s, s) > 1 - \epsilon$ ($i = 1, \dots, n$)

If $\chi_T \in F$, then Junghenn's definition of *F-left thickness* reduces to Mitchell's definition of left-thickness: let $g = \chi_T$, then for $0 < \epsilon < 1$, $1 - \epsilon < g(s, s) = \chi_T(s, s) \rightarrow s, s \in T$ ($i = 1, \dots, n$).

Junghenn generalizes Mitchell's theorem thus:

Theorem. *If T is a left-thick subsemigroup of S , then F has a left-invariant mean μ \Rightarrow $F|_T$ has a left-invariant mean.*

2. Transformation Semigroups

Thickness can be defined in the more general setting of a transformation semigroup. This section defines such semigroups and other necessary terms.

Definition 2.1. A *transformation semigroup* is a system $\langle S, X, \pi \rangle$ consisting of a semigroup S , a set X , and a mapping $\pi: S \times X \rightarrow X$ which satisfies

1. $\pi(s, \pi(t, x)) = \pi(st, x)$ ($\forall s, t \in S, x \in X$);
2. $\pi(e, x) = x$ ($\forall x \in X$) whenever S has two-sided identity e .

If $\pi(s, x) = sx$ expresses the image of (s, x) under π , then condition (1) becomes $s(tx) = (st)x$ and condition (2) becomes $ex = x$.

The abbreviated notion $\langle S, X \rangle$ will denote a transformation semigroup whenever the meaning of π is clear or whenever π is generic.

$\langle T, Y \rangle$ is a *subtransformation semigroup* of $\langle S, X \rangle \Rightarrow T$ is a subsemigroup of S , $Y \subseteq X$, and $TY \subseteq Y$.

Definition 2.2. Let semigroup S and set X both be endowed with Hausdorff topologies. Transformation semigroup $\langle S, X, \pi \rangle$ is a *topological transformation semigroup*, or τ -semigroup $\Rightarrow \pi$ is separately continuous in the variables s and x .

Again, a τ -semigroup will be denoted briefly by $\langle S, X \rangle$.

Let $C(X)$ denote the set of continuous and bounded complex- or real-valued functions on X .

Definition 2.3. Let $\langle S, X \rangle$ be a τ -semigroup. $T_s f$ denotes, for any $s \in S$ and $f \in C(X)$, the function in $C(X)$ defined by $T_s f(x) = f(sx)$ ($\forall x \in X$). If F is a linear subspace of $C(X)$, then F is S -invariant $\Rightarrow T_s f \in F$ ($\forall s \in S, f \in F$). Notation: $T_S = \{T_s | s \in S\}$ and $T_S F = \{T_s f | f \in F\}$.

Observe that $T_t T_s = T_{st}$ ($\forall s, t \in S$).

Definition 2.4. Let $\langle S, X \rangle$ be a τ -semigroup; F be a linear space $\subseteq C(X)$ which is norm-closed, conjugate-closed, S -invariant, and contains the constant functions; $G \subseteq C(S)$ a linear space, and let $\mu \in F^*$. Define $T_\mu f$ ($\forall f \in F$) by $T_\mu f(s) = \mu(T_s f)$ ($\forall s \in S$). Then $T_\mu: F \rightarrow B(S)$. F is G -introverted $\Rightarrow T_\mu(F) \subseteq G$ ($\forall \mu \in F^*$).

In the preceding definition F^* may be replaced by $C(X)^*$ since every functional in F^* can be extended to a functional in $C(X)^*$. Also it can be shown that F^* can be replaced by $M(F)$, the set of all means on F .

Definition 2.5. Let F be G -introverted, $\mu \in F^*$, and $\lambda \in G^*$. The *evolution product* of λ and μ , denoted $\lambda \mu$, is defined by $\lambda \mu f = \lambda(T_\mu f)$ ($\forall f \in F$).

Note that $\lambda \mu \in F^*$ and that if G is norm-closed, conjugate-closed, and contains the constant functions, then $\lambda \in M(G)$ and $\mu \in M(F)$ imply $\lambda \mu \in M(F)$.

A *mean* on $F \subseteq C(X)$ is defined in the same way as a mean on $B(S)$ was defined in section 1. If F is an algebra under pointwise multiplication, then mean μ is *multiplicative* $\Rightarrow \mu(fg) = \mu(f)\mu(g)$ ($\forall f, g \in F$).

Let $M(F) =$ set of all means on F , and $MM(F) =$ set of all multiplicative means on F . $M(F)$ and $MM(F)$ are both w^* -compact, being closed subsets of the unit ball in F^* .

Mean $\mu \in M(F)$ is *invariant* $\Rightarrow \mu(T_s f) = \mu(f)$ ($\forall f \in F, s \in S$). Note that μ is invariant $\Rightarrow e(s)T_\mu = T_\mu$ ($\forall s \in S$).

An *evaluation* at $x \in X$ is defined by $e(x)f = f(x)$ ($\forall f \in F$); clearly an evaluation is a mean. A *finite mean* on F is a convex combination of evaluations.

A mean is multiplicative if and only if it is the w^* -limit of evaluations.

A special case of transformation semigroup is furnished by letting $X = S$ and $\pi = \lambda(\cdot)$ where $\lambda_s: S \times S \rightarrow S$ is defined for any fixed $s \in S$ by $\lambda_s(t) = st$ ($\forall t \in S$). If $G \subseteq C(S)$ is a linear space, then $L_s g(t) = g(st)$ ($\forall s, t \in S, g \in G$); also, $\lambda_* \mu \in M(G) \rightarrow \lambda_* \mu \in M(G)$. If $F \subseteq C(X)$ is a linear space then $L_s T_\mu = T_\mu L_s$ ($\forall s \in S, \mu \in M(F)$). Mean $\mu \in M(G)$ is *left-invariant* $\Leftrightarrow \mu(L_s g) = \mu(g)$ ($\forall g \in G$).

3. Thickness in Transformation Semigroups

Junghenn's generalization of F -left thickness carries over in a straightforward way to transformation semigroups. The corresponding concept is defined in Definition 3.1, and a plethora of alternative characterizations is given by Theorem 3.3.

Assumptions:

$\langle S, X \rangle$ is a transformation semigroup;

$G \subseteq C(S)$ is a subalgebra;

$F \subseteq C(X)$ is an algebra which is norm-closed, S -invariant, G -introverted, and contains the constant functions;

$Y \subseteq X$.

Notation:

$F(Y) = \{g \in F \mid \chi_Y \leq g \leq 1\} = \{g \in F \mid 0 \leq g \leq 1, g \equiv 1 \text{ on } Y\}$

$Z(Y) = \{g \in F \mid g \equiv 0 \text{ on } Y\}$.

Definition 3.1. Y is *F, S -thick* in $X \Leftrightarrow \forall \epsilon > 0, \exists g \in F(Y)$, and finite $U = \{s_1, s_2, \dots, s_n\} \subseteq S, \exists x \in X: g(s_k x) > 1 - \epsilon$ ($k = 1, \dots, n$).

Remark 3.2. If $X = S$ and the action is left multiplication, then the definition is identical to Junghenn's.

Relative to Theorem 3.3 b,h,i,j *infra* it is necessary to recall that a norm-closed subalgebra F of $C(X)$ is also a closed lattice, so that, in particular, $f \in F \rightarrow |f| \in F$ [Simmons, p. 159, lemma].

Theorem 3.3. The following statements are equivalent:

- Y is *F, S -thick* in X ;
- $\forall \epsilon > 0$, finite $D = \{g_1, g_2, \dots, g_m\} \subseteq F(Y)$, finite $U = \{s_1, s_2, \dots, s_n\} \subseteq S$ $\exists x \in X: \inf \{g_i(s_k x) \mid g_i \in D, s_k \in U\} > 1 - \epsilon$;
- $\forall \epsilon > 0$, finite $D = \{g_1, g_2, \dots, g_m\} \subseteq F(Y)$, finite $U = \{s_1, s_2, \dots, s_n\} \subseteq S$ $\exists x \in X: \frac{1}{n} \sum_{k=1}^n g_i(s_k x) > 1 - \epsilon$ ($i = 1, \dots, m$) and $\frac{1}{m} \sum_{i=1}^m g_i(s_k x) > 1 - \epsilon$ ($k = 1, \dots, n$);
- $\exists \lambda \in MM(F), \forall s \in S, g \in F(Y): \lambda(T_s g) = 1$ and $\lambda(g) = 1$;
- $\exists \mu \in M(F), \forall s \in S, g \in F(Y): \mu(T_s g) = 1$ and $\mu(g) = 1$;
- $\exists \mu \in M(F), \forall v \in M(G), g \in F(Y): v\mu(g) = 1$;
- $Cle(Y)$ contains a compact $MM(G)$ -invariant set;
- $\forall \epsilon > 0, g \in Z(Y)$, finite $U = \{s_1, s_2, \dots, s_n\} \subseteq S \exists x \in X: |g(s_k x)| < \epsilon$ ($k = 1, \dots, n$);
- $\forall \epsilon > 0$, finite $D = \{g_1, g_2, \dots, g_m\} \subseteq Z(Y)$, finite $U = \{s_1, s_2, \dots, s_n\} \subseteq S$ $\exists x \in X: \sup \{|g_i(s_k x)| \mid g_i \in D, s_k \in U\} < \epsilon$;
- $\forall \epsilon > 0$, finite $D = \{g_1, g_2, \dots, g_m\} \subseteq Z(Y)$, finite $U = \{s_1, s_2, \dots, s_n\} \subseteq S$ $\exists x \in X: \frac{1}{n} \sum_{k=1}^n |g_i(s_k x)| < \epsilon$ ($i = 1, \dots, m$) and $\frac{1}{m} \sum_{i=1}^m |g_i(s_k x)| < \epsilon$ ($k = 1, \dots, n$);
- $\exists \lambda \in MM(F), \forall s \in S, g \in Z(Y): \lambda(T_s g) = 0$ and $\lambda(g) = 0$;
- $\exists \mu \in M(F), \forall s \in S, g \in Z(Y): \mu(T_s g) = 0$ and $\mu(g) = 0$;
- $\exists \mu \in M(F), \forall v \in M(G), g \in Z(Y): v\mu(g) = 0$.

PROOF: $a \rightarrow b$: $f(x) = \inf \{g_i(x) | g_i \in D\}$ is in $F(Y)$ because $0 \leq g_i \leq 1$, $g_i \neq 1$ on Y ($i=1, \dots, m$).

By (a) $\exists x \in X: f(s_k x) > 1 - \epsilon$ ($k=1, \dots, n$). Because U is finite, $\inf \{f(s_k x) | s_k \in U\} > 1 - \epsilon$.

$$b \rightarrow c: \inf \{g_i(s_k x) | g_i \in D, s_k \in U\} > 1 - \epsilon \rightarrow \sum_{k=1}^n g_i(s_k x) \geq n [\inf \{g_i(s_k x)\}] > n(1 - \epsilon)$$

and $\sum_{i=1}^m g_i(s_k x) \geq m [\inf \{g_i(s_k x)\}] > m(1 - \epsilon)$.

$$c \rightarrow d: \text{For each } (\epsilon, U, D) \text{ in (c) choose } x = x(\epsilon, U, D) \text{ so that } \frac{1}{n} \sum_{k=1}^n g(s_k x) > 1 - \frac{1}{n} \epsilon \quad (\forall g \in D)$$

$$> 1 - \frac{1}{n} \epsilon \quad (\forall g \in D). \text{ Let } r \in U, g \in D. \text{ Then } g(s_k x) \leq 1 \quad (k=1, \dots, n) \rightarrow \sum_{s_k \neq r} g(s_k x) \leq n - 1 = - \sum_{s_k = r} g(s_k x)$$

$$\geq -n + 1 \rightarrow g(rx) = \sum_{k=1}^n g(s_k x) - \sum_{s_k \neq r} g(s_k x) > 1 - \epsilon. \text{ Define } (\epsilon, U, D) \leq (\epsilon', U', D') \rightarrow$$

$\epsilon' \geq \epsilon, U \subseteq U', D \subseteq D'$. The net $\langle e(x(\epsilon, U, D)) \rangle \in MM(F)$ has a subnet $\langle e(x_m) \rangle$ which w^* -converges to some $\lambda' \in MM(F)$, since $MM(F)$ is compact. For $\delta > 0$ and $(\epsilon, U, D) \geq (\delta, \{s\}, \{g\})$ it follows that $1 - \delta \leq 1 - \epsilon < g(sx(\epsilon, U, D)) = e(x(\epsilon, U, D)) T_s g$ by the earlier inequality. Therefore, $1 - \delta \leq \lim_m [e(x_m)(T_s g)] = [\lim_m e(x_m)] (T_s g) = \lambda' (T_s g)$. Since δ was arbitrary, $1 \leq \lambda' T_s g$.

Because $0 \leq g \leq 1$, $T_s g \leq 1$, and so $\lambda' (T_s g) \leq 1$. Thus, the first part of (d) is proven. Let $v \in MM(G)$; then $\lambda = v \lambda' \in MM(F)$ and $(T_\lambda, T_s g)(t) = \lambda' [T_t T_s g] = \lambda' (T_{st} g) = 1 \rightarrow \lambda (T_s g) = v \lambda' (T_s g) = v [T_\lambda, T_s g] = v 1 = 1$; also $v \lambda' (g) = v [T_\lambda, g] = v 1 = 1$.

$$d \rightarrow e: MM(F) \subseteq M(F).$$

e \rightarrow f: Let $v \in M(G)$ and μ be as in (e), so that $(T_\mu g)(s) = (\mu T_s g) = 1$; then $v \mu(g) = v (T_\mu g) = v(1) = 1$.

f \rightarrow a: We prove (not (a)) \rightarrow (not (f)). Suppose $\exists \epsilon > 0, h \in F(Y), U =$

$$\{s_1, s_2, \dots, s_n\} \subseteq S \text{ such that } \forall x \in X, \exists s_x \in U: h(s_x x) \leq 1 - \epsilon. \text{ Define } v = \frac{1}{n} \sum_{k=1}^n e(s_k). \text{ Then } (\forall x \in X)$$

$$[v e(x)] h = \frac{1}{n} \sum_{k=1}^n h(s_k x) \leq 1 - \epsilon/n \text{ because } 0 \leq h \leq 1 \text{ and, for some } s_k = s_x, h(s_k x) \leq 1 - \epsilon. \text{ This}$$

inequality, valid for all evaluations $e(x)$, also holds for all finite means, and so for all limits

$$\mu \in M(F) \text{ of finite means: } v \mu(h) \leq 1 - \frac{\epsilon}{n}. \text{ Therefore (f) is impossible.}$$

$$d \rightarrow g: \text{Choose } \lambda \in MM(F) \text{ as in (d). } MM(G)\lambda \text{ is then an } MM(G)\text{-invariant set.}$$

Since $Cl[e(Y)]$ is closed, it suffices to show that $e(s)\lambda \in Cl[e(Y)]$ for $\forall s \in S$. Suppose that $\exists s_0: e(s_0)\lambda \notin Cl[e(Y)]$. Then, since $MM(F)$ is compact Hausdorff and so completely regular, $\exists h \in C(MM(F)): 0 \leq h \leq 1$, $h(e(s_0)\lambda) = 0$, and $h(Cl[e(Y)]) = 1$. $g = h \circ e \in F(Y)$ because for $y \in Y$ $g(y) = h(e(y)) = 1$. Then $\lambda (T_{s_0} g) = [e(s_0)\lambda] g = h(e(s_0)\lambda) = 0$, contradicting (d).

g \rightarrow d: Let I be an $MM(G)$ -invariant set $\subseteq Cl(e(Y))$. If $\lambda \in I$, then $e(s)\lambda \in I \subseteq Cl(e(Y)) \quad (\forall s \in S)$. Therefore, $\lambda (T_s g) = [e(s)\lambda] g = 1 \quad (\forall g \in F(Y))$. Clearly $\lambda(g) = 1 \quad (\forall g \in F(Y))$.

a \rightarrow h: Assume Y is F,S -thick in X . Let $\epsilon > 0$, $g \in Z(Y)$, finite $U \subseteq S$. If $g \neq 0$,

result is trivial; hence, assume that $g \neq 0$. Then $1 - \frac{1}{\|g\|} \cdot |g| \in F(Y)$. Consequently, $\exists x \in X$:

$$1 - \frac{1}{\|g\|} \cdot |g(s_k x)| \geq 1 - \frac{\epsilon}{\|g\|}, \text{ whence } |g(s_k x)| < \epsilon \ (k=1, \dots, n).$$

h \rightarrow a: Assume (h). Let $\epsilon > 0$, $g \in F(Y)$, finite $U \subseteq S$. Then $1 - g \in Z(Y)$.

Therefore, $\exists x \in X: |1 - g(s_k x)| < \epsilon \Rightarrow -\epsilon < 1 - g(s_k x) < \epsilon \Rightarrow -g(s_k x) < -1 + \epsilon \Rightarrow g(s_k x) > 1 - \epsilon \ (k=1, \dots, n)$.

h \rightarrow i: $\sup \{ |g_j| \mid g_j \in D \} \in Z(Y)$, because $g_j \neq 0$ on $Y \ (j=1, \dots, m)$.

i \rightarrow k: For each (ϵ, U, D) in (i) choose $x = x(\epsilon, U, D)$. Define

$(\epsilon, U, D) \leq (\epsilon', U', D') \Leftrightarrow \epsilon \geq \epsilon', U \subseteq U', D \subseteq D'$. The net $\langle e(x(\epsilon, U, D)) \rangle \in MM(F)$ has a subnet $\langle e(x_m) \rangle$ which converges to some $\lambda \in MM(F)$ since $MM(F)$ is compact. Let $\delta > 0$. If $(\epsilon, U, D) \geq (\delta, \{s\}, \{g\})$, then $\delta \geq \epsilon > \sup \{ |g_j(s_k x(\epsilon, U, D))| \mid g_j \in D, s_k \in U \} \geq |g(s_k x(\epsilon, U, D))|$. Ergo $\delta \geq \lim_m |c(x_m)| T_s g | = [\lim_m c(x_m)] |T_s g| = \lambda |T_s g|$. Since δ was arbitrary, the first part of (k) is proven. The second part is shown in the same manner as the second part of (c) \rightarrow (d).

i \rightarrow j: Trivial.

j \rightarrow i: In the first part of (j), replace ϵ by $\frac{\epsilon}{n}$: $\frac{\epsilon}{n} > \frac{1}{n} \sum_{k=1}^n |g_j(s_k x)| \ (j=1, \dots, n) \Rightarrow$

$$\epsilon > \sum_{k=1}^n |g_j(s_k x)| > \sup \{ |g_j(s_k x)| \mid g_j \in D, s_k \in U \}.$$

k \rightarrow l, l \rightarrow m: Trivial.

m \rightarrow h: We show (not (h)) \Rightarrow (not (m)). Suppose $\exists \epsilon > 0$, $h \in Z(Y)$, finite $U \subseteq S$

such that $\forall x \in X, \exists s_x \in U: |h(s_x x)| \geq \epsilon$. Define $v = \frac{1}{n} \sum_{k=1}^n c(s_k)$. Then $\forall x \in X: |v(e(x))| |h| =$

$\frac{1}{n} \sum_{k=1}^n |h(s_k x)| \geq \epsilon/n$, because $|h| \geq 0$ and for some $s_k = s_x$, $|h(s_k x)| \geq \epsilon$. Hence, replacing $e(x)$ by

any finite mean, then for any $\mu \in M(F)$, $\mu |h| \geq \epsilon/n$. Therefore (m) is impossible. QED

Remark 3.4. Parts d., e., k., and l., of Theorem 3.3 suggest that S behaves with regard to thickness as though it contained an identity. In fact, if S^1 denotes the semigroup S with a discrete identity 1 adjoined, then Y is F,S -thick in $X \Leftrightarrow Y$ is F,S^1 -thick in X where S^1 acts on X in the natural way.

Corollary 3.5. If the characteristic function $\chi_Y \in F$, then the following statements are equivalent:

- a. Y is F,S -thick in X ;
- b. \forall finite $U = \{s_1, s_2, \dots, s_n\} \subseteq S, \exists x \in X: s_k x \in Y \ (k=1, \dots, n)$;
- c. \forall finite $U = \{s_1, s_2, \dots, s_n\} \subseteq S, \exists y \in Y: s_k y \in Y \ (k=1, \dots, n)$;
- d. The family $\{s^{-1}Y \mid s \in S\}$ has the finite intersection property;
- e. $\bigcap_{s \in S} \text{Cl } e(s^{-1}Y) \neq \emptyset$ where $e(s^{-1}Y) = \{e(x) \mid s x \in Y\}$.

PROOF: $e \rightarrow a$: Let $\mu \in \bigcap_{s \in S} \text{Cl } e(s^{-1}Y)$; also let $s \in S$, $g \in F(Y)$. Then $\mu \in \text{Cl } e(s^{-1}Y)$, so \exists

net $\langle x_n \rangle$ such that $\mu = w^* - \lim e(x_n)$ and $s x_n \in Y \ (\forall n)$; whence $\mu T_s g = [w^* - \lim_n e(x_n)] T_s g = \lim_n [g(s x_n)] = \lim 1_n = 1$. Now let $\lambda \in M(G)$. Then $\lambda \mu \in M(F)$ and $\lambda \mu T_s g = \lambda [T_\mu (T_s g)] =$

$\lambda[L_s T_\mu g] = \lambda[L_s] = 1$; also $\lambda\mu(g) = \lambda[T_\mu g] = \lambda[\mu T_{(s)} g] = \lambda[1] = 1$. Therefore by 3.3.e Y is F.S-thick. QED

Results for transformation semigroups comparable to the theorems of section 1 can be generalized in the same way as in [Junghenn 1979, p. 40, theorem 2].

Theorem 3.6. Let $\langle S, X \rangle$ be a transformation semigroup;
 $\langle T, Y \rangle$ be a subtransformation semigroup of $\langle S, X \rangle$; and
 $F \subseteq B(X)$ be a translation invariant, conjugate-closed, norm-closed subalgebra which contains the constant functions.

If F has invariant mean μ with respect to $\langle T, X \rangle$ such that $\inf \{\mu(g) | g \in F(Y)\} > 0$, then $F|_Y$ has invariant mean with respect to $\langle T, Y \rangle$.

PROOF: X is embedded in the compact set $MM(F)$ by $e(\cdot)$, and $F \subseteq C(MM(F))$ by the Gelfand representation theorem. Also $Cl\ e(Y) \subseteq MM(F)$. By the Riesz representation theorem, the invariant mean μ defines a regular Borel probability measure $\hat{\mu}$ on $MM(F)$ such that $\mu(f) = \int_{MM(F)} \hat{f} d\hat{\mu} (\forall f \in F)$. Invariance of μ is reflected in $\hat{\mu}$ as follows:

$$\int_{MM(F)} T_{e(t)} \hat{f} d\hat{\mu} = \int_{MM(F)} \hat{T}_t f d\hat{\mu} = \mu(T_t f) = \mu(f) = \int_{MM(F)} \hat{f} d\hat{\mu} (\forall t \in T).$$

Since μ is regular, $\hat{\mu}(Cl\ e(Y)) = \inf \{\hat{\mu}(U) | U \text{ open, } Cl\ e(Y) \subseteq U\}$. Now let U be any open set such that $Cl\ e(Y) \subseteq U$. Because $MM(F)$ is normal, by Urysohn's lemma, $\exists g \in C(MM(F)) \cap F$ such that $\hat{g}(Cl\ e(Y)) = 1$, $\hat{g}(U^c) = 0$, and $0 \leq \hat{g} \leq 1$; thus $\hat{g} \leq \chi_U$ and g , the correlative of \hat{g} , is in $F(Y)$. $\mu(g) = \int_{MM(F)} \hat{g} d\hat{\mu} \leq \int_{MM(F)} \chi_U d\hat{\mu} = \hat{\mu}(U)$. Therefore by hypothesis $0 < \inf \{\mu(g) | g \in F(Y)\} \leq \inf \{\hat{\mu}(U) | U \text{ open, } Cl\ e(Y) \subseteq U\} = \hat{\mu}(Cl\ e(Y))$. Ergo,

$$v(f) = \frac{1}{\hat{\mu}(Cl\ e(Y))} \int_{Cl\ e(Y)} \hat{f} d\hat{\mu} \text{ is a mean on } F.$$

Define v_0 on $F|_Y$ by $v_0(f|_Y) = v(f)$. v_0 is well-defined because $f|_Y = g|_Y \rightarrow f - g \in Z(Y) \rightarrow \hat{f} - \hat{g} \in Cl\ e(Y) \rightarrow 0 = v(f - g) = v(f) - v(g)$. Also $v_0 \in M(F|_Y)$.

To show that v_0 is invariant it suffices to prove that $\int_{Cl\ e(Y)} T_{e(t)} \hat{f} d\hat{\mu} = \int_{Cl\ e(Y)} \hat{f} d\hat{\mu} (\forall t \in T)$.

Fix $t \in T$. Define $E_1 = e(t)^{-1}(Cl\ e(Y)) \setminus Cl\ e(Y)$, $E_n = e(t)^{-1}(E_{n-1})$ ($n \geq 2$). The E_n are pairwise disjoint: $\mu \in E_2 \rightarrow e(t)\mu \in E_1 \rightarrow e(t)\mu \in Cl\ e(Y) \rightarrow \mu \in E_1$, so $E_1 \cap E_2 = \emptyset$. Assume that E_m and E_n are pairwise disjoint ($1 \leq m < n$). Then $\mu \in E_{n+1} \rightarrow e(t)\mu \in E_n \rightarrow e(t)\mu \in E_m$ ($1 \leq m < n$) $\rightarrow \mu \in e(t)^{-1}E_m = E_{m+1} = E_p$ ($2 \leq p = m+1 < n+1$), so $E_{n+1} \cap E_p = \emptyset$. Also $\mu \in E_{n+1} \rightarrow e(t)^n \mu \in E_1$ (by induction) $\rightarrow e(t)^n \mu \in Cl\ e(Y)$, but $\mu \in E_1 \rightarrow e(t)\mu \in Cl\ e(Y) \rightarrow e(t)^n \mu \in Cl\ e(Y)$ (by invariance of Y), so $E_{n+1} \cap E_1 = \emptyset$. The E_n are Borel sets since $\mu - e(t)\mu$ is w^* -continuous for $\forall \mu \in MM(F)$.

Because $(\forall n \geq 2) T_{e(t)} \chi_{E_{n-1}}(\mu) = \chi_{E_{n-1}}(e(t)\mu) = \chi_{e(t)^{-1}E_{n-1}}(\mu)$, it follows that

$$\hat{\mu}(E_n) = \hat{\mu}(e(t)^{-1}E_{n-1}) = \int_{MM(F)} \chi_{e(t)^{-1}E_{n-1}} d\hat{\mu} = \int_{MM(F)} T_{e(t)} \chi_{E_{n-1}} d\hat{\mu} = \int_{MM(F)} \chi_{E_{n-1}} d\hat{\mu} = \hat{\mu}(E_{n-1}).$$

Therefore, $\hat{\mu}(E_1 \cup E_2 \cup \dots \cup E_n) = \sum_{j=1}^n \hat{\mu}(E_j) = n \hat{\mu}(E_1)$. Since this holds for arbitrary n,

$$\hat{\mu}(E_1) = 0.$$

Because Y is invariant, $e(T)Cl\ e(Y) \subseteq Cl\ e(Y)$, whence $Cl\ e(Y) \setminus e(t)^{-1}Cl\ e(Y) = \emptyset$. Since $Cl\ e(Y) \Delta e(t)^{-1}Cl\ e(Y) = [Cl\ e(Y) \setminus e(t)^{-1}Cl\ e(Y)] \cup E_1 = E_1$, $\hat{\mu}[Cl\ e(Y) \Delta e(t)^{-1}Cl\ e(Y)] = 0$, so $\int_{Cl\ e(Y)} T_{e(t)} \hat{f} d\hat{\mu} = \int_{e(t)^{-1}Cl\ e(Y)} T_{e(t)} \hat{f} d\hat{\mu} = \int_{MM(F)} T_{e(t)} [\hat{f} \chi_{Cl\ e(Y)}] d\hat{\mu} = \int_{Cl\ e(Y)} \hat{f} d\hat{\mu}$. QED

Theorem 3.7. Let $\langle S, X \rangle$ be a τ -semigroup;
 $\langle T, Y \rangle$ be a sub τ -semigroup of $\langle S, X \rangle$;
 $F \subseteq B(X)$ be a translation invariant, norm-closed, G -introverted subalgebra which contains the constant functions.

1. If $F|_Y$ has an invariant mean with respect to $\langle T, Y \rangle$ and T is G -thick in S , then F has an invariant mean with respect to $\langle S, X \rangle$.
2. If G has a left-invariant mean and Y is F, S -thick in X , then $F|_Y$ has an invariant mean with respect to $\langle T, Y \rangle$.

PROOF: 1. Functional $\bar{\mu}$ in $F|_Y$ defines a functional μ in F^* by $\mu f = \bar{\mu} f|_Y$ ($\forall f \in F$), thus $\mu T_t f = \bar{\mu} T_t f|_Y$ ($\forall f \in F, t \in T$). Therefore, because F is G -introverted, $F|_Y$ is $G|_T$ -introverted.

Relative to the algebra $F|_Y$ defined on $\langle T, Y \rangle$: Let $\bar{\mu}$ be an invariant mean of $F|_Y$; then $e(t)\bar{\mu} = \bar{\mu}(T_s e) = \bar{\mu}$ ($\forall t \in T$) where $e(t) \in \text{MM}(G|_T)$. Let $\bar{\lambda} \in \text{Cl } e(T) = \text{MM}(G|_T)$, and let $\langle e(t_\alpha) \rangle \subseteq e(T) \subseteq \text{MM}(G|_T)$ be a net such that $\bar{\lambda} = w^* - \lim e(t_\alpha)$. Ergo,

$$\bar{\lambda} \bar{\mu} = [w^* - \lim_\alpha e(t_\alpha)] \bar{\mu} = \lim_\alpha [e(t_\alpha) \bar{\mu}] = \lim_\alpha \bar{\mu} = \bar{\mu}. \text{ That is, } \bar{\lambda} \bar{\mu} = \bar{\mu} \text{ } (\forall \bar{\lambda} \in \text{Cl } e(T)).$$

Relative to the algebra F defined on $\langle S, X \rangle$: \exists left-ideal K of $\text{Cl } e(S)$ in $\text{Cl } e(T) \subseteq \text{MM}(G)$ [Wilde & Witz, 1967, lemma 5.1]. Choose $\lambda_0 \in K$. Then $e(s)\lambda_0 \in K \subseteq \text{Cl } e(T) \subseteq \text{MM}(G)$ ($\forall s \in S$).

Any $\lambda \in \text{Cl } e(T) \subseteq \text{MM}(G)$ gives rise to a $\bar{\lambda} \in \text{Cl } e(T) \subseteq \text{MM}(G|_T)$ in the following way: $\bar{\lambda} = w^* - \lim_\alpha e(t_\alpha) \in \text{MM}(G)$. Now $\langle e(t_\alpha) \rangle$ is a net in $e(T) \subseteq \text{MM}(G|_T)$ so has a convergent subnet $\langle e(t_\beta) \rangle$ with $\bar{\lambda} = w^* - \lim e(t_\beta) \in \text{MM}(G|_T)$. $\bar{\lambda}$ may not be unique. For $\bar{\mu} \in F|_Y$ define $\mu \in F^*$ by $\mu f = \bar{\mu} f|_Y$ ($\forall f \in F$) as we have done earlier. Then for all $f \in F$ $\bar{\lambda} \bar{\mu} f|_Y = \bar{\lambda} (T_{\bar{\mu}} f|_Y) = \lim_\beta [e(t_\beta) T_{\bar{\mu}} f|_Y] = \lim_\beta [\bar{\mu} T_{t_\beta} f|_Y]$; also, $\lambda \mu f = \lambda (T_\mu f) = \lim_\alpha [\bar{\mu} T_{t_\alpha} f|_Y]$; ergo $\lambda \mu(f) = \bar{\lambda} \bar{\mu}(f|_Y)$, regardless of the choice of $\bar{\lambda}$ which is associated with λ .

Finally, choose $\bar{\mu}$ to be an invariant mean of $F|_Y$, and define $\mu \in M(F)$ as before. Then $\lambda \mu(f) = \bar{\lambda} \bar{\mu}(f|_Y) = \bar{\mu}(f|_Y) = \mu(f)$, that is, $\lambda \mu = \mu$ ($\forall \lambda \in \text{Cl } e(T) \subseteq \text{MM}(G)$). In particular, $e(s)\lambda_0 \mu = \mu$ ($\forall s \in S$), so that $\lambda_0 \mu$ is invariant.

2. Because Y is F, S -thick in X , then by Theorem 3.3.f $\exists \mu \in M(F)$ such that $\nu \mu(f) = 1$ ($\forall \nu \in M(G), f \in F(Y)$). Let ν be an invariant mean of G . Then $\nu \mu$ is an invariant mean of F such that $\nu \mu(f) = 1$ ($\forall f \in F(Y)$). By Theorem 3.6 $F|_Y$ has an invariant mean with respect to $\langle T, Y \rangle$. QED

In the preceding theorem the thickness condition on T in (1) implies the thickness condition on Y in (2) according to the following lemma:

Lemma 3.8. Let $\langle S, X \rangle$ be a τ -semigroup;
 $\langle T, Y \rangle$ be a sub τ -semigroup of $\langle S, X \rangle$;
 $F \subseteq B(X)$ be a translation-invariant, norm-closed, G -introverted subalgebra which contains the constant functions.

If T is G -thick in S , then Y is F, S -thick in X .

PROOF: Let $f \in F(Y)$: $0 \leq f \leq 1$, $f = 1$ on Y . Then $T_{e(y)} f \in F(T)$ ($\forall y \in Y$). By Theorem 3.3.e applied to $L(S, G) \ni \mu \in M(G)$ such that $1 = \mu(L_s T_{e(y)} f) = \mu(T_{e(y)} T_s f) = \mu e(y) T_s f$ and $1 = \mu T_{e(y)} f = \mu e(y) f$. Then $\mu e(y) \in M(F)$ has the properties required by Theorem 3.3.e for Y to be F, S -thick. QED

Junghenn's theorem of section 1 is obtained from Theorem 3.7 and Lemma 3.8 by letting $X = S$, $Y = T$, and the action be left multiplication.

4. Multiplicative Means and Thickness

Several results connect multiplicative means with thickness. F is assumed to be an S -invariant, norm-closed algebra $\subseteq C(X)$ which contains the constant functions.

Theorem 4.1 If F has an invariant multiplicative mean, then for any finite partition $\{A_i\}_1^n$ of X $\exists k$ such that A_k is F,S-thick.

PROOF: Let $v \in MM(F)$ be invariant. v induces a regular Borel probability measure \hat{v}

defined on $MM(F)$, and $\sum_1^n \hat{v}(\text{Cl } e(A_i)) \geq 1$. Because v is multiplicative, for each i $\hat{v}(\text{Cl } e(A_i)) = 0$ or $\hat{v}(\text{Cl } e(A_i)) = 1$. Hence, $\exists k$ such that $\hat{v}(\text{Cl } e(A_k)) = 1$. Therefore, $v(f) = 1$ ($\forall f \in F(A_k)$) because $\chi_{A_k} \leq f \leq 1 \Rightarrow \chi_{\text{Cl } e(A_k)} \leq \hat{f} \leq 1$ and $1 = \hat{v}(\text{Cl } e(A_k)) = \int \chi_{\text{Cl } e(A_k)} d\hat{v} \leq \int \hat{f} d\hat{v} = v(f) \leq 1$.

Then, by Theorem 3.3.d A_k is F,S-thick.

QED

Definition 4.2 $K(f,s) = \{\mu \in MM(F) | \mu(T_s f - f) = 0\}$

Theorem 4.3. The following are equivalent:

a. F has an invariant multiplicative mean;

b. It is not the case that $MM(F) \subseteq \bigcup_{\substack{f \in F \\ s \in S}} K^c(f,s)$;

c. It is not the case that $\exists f_1, \dots, f_n \in F; \exists s_1, \dots, s_n \in S: MM(F) \subseteq \bigcup_{i=1}^n K^c(f_i, s_i)$;

d. $\forall f_1, \dots, f_n \in F; \forall s_1, \dots, s_n \in S; \forall \delta > 0; \exists x_\delta: e(x_\delta) \sum_{i=1}^n |T_{s_i} f_i - f_i| < \delta$;

e. $\forall f_1, \dots, f_n \in F; \forall s_1, \dots, s_n \in S; \forall \delta > 0, \exists x_\delta: |T_{s_i} f_i(x_\delta) - f_i(x_\delta)| < \delta$ ($i = 1, \dots, n$);

f. $\forall f_1, \dots, f_n \in F; \forall s_1, \dots, s_n \in S; \exists \lambda \in MM(F): \lambda |T_{s_i} f_i - f_i| = 0$ ($i = 1, \dots, n$);

g. $\forall f_1, \dots, f_n \in F; \forall s_1, \dots, s_n \in S; \exists \lambda \in MM(F): \lambda (T_{s_i} f_i - f_i) = 0$ ($i = 1, \dots, n$);

h. $\forall \epsilon > 0; \forall f_1, \dots, f_n \in F; \forall s_1, \dots, s_n \in S: \exists c_1, \dots, c_n \in \mathbb{C} \exists Y \subseteq X: |f_k - c_k| < \epsilon$ and $|T_{s_k} f_k - c_k| < \epsilon$ on Y ($k = 1, \dots, n$) and Y is F,S-thick in X .

PROOF: a \Rightarrow b: F has an invariant multiplicative mean $\Rightarrow \exists \lambda \in MM(F): \lambda \in K(f,s)$ ($\forall f \in F, s \in S$) \Rightarrow the $K^c(f,s)$ do not cover all of $MM(F)$.

$\neg b \Rightarrow \neg c$: $MM(F)$ is compact and the $K^c(f,s)$ are open.

$\neg c \Rightarrow \neg d$: Let $f_1, \dots, f_n \in F$ and $s_1, \dots, s_n \in S$ be as in the negation of (c). If for any $\delta > 0 \exists x_\delta \in X$ such that $e(x_\delta) \sum |T_{s_k} f_k - f_k| = \sum |T_{s_k} f_k(x_\delta) - f_k(x_\delta)| < \delta$, then the net

$\langle e(x_\delta) \rangle_{\delta > 0} \subseteq MM(F)$ contains a convergent subnet $\langle e(x_{\delta_\alpha}) \rangle_{\alpha \in A}$ of $\langle e(x_\delta) \rangle$ and

$\lim_{\alpha} e(x_{\delta_\alpha}) = \lambda \in MM(F)$; thus, for any $\epsilon > 0 \exists \alpha_0 \in A: \alpha \geq \alpha_0 \Rightarrow |\lambda \sum |T_{s_k} f_k - f_k| -$

$e(x_{\delta_\alpha}) \sum |T_{s_k} f_k - f_k| | < \frac{\epsilon}{2}$. Let $\alpha_1 \in A$ be $\geq \alpha_0$ and such that $\delta_{\alpha_1} < \frac{\epsilon}{2}$, so that

$e(x_{\delta_{\alpha_1}}) \sum |T_{s_k} f_k - f_k| < \frac{\epsilon}{2}$. Then $0 \leq \lambda \sum |T_{s_k} f_k - f_k| < e(x_{\delta_{\alpha_1}}) \cdot \sum |T_{s_k} f_k - f_k| +$

$\frac{\epsilon}{2} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$. Since ϵ was arbitrary, $\lambda \sum |T_{s_k} f_k - f_k| = 0 \Rightarrow |T_{s_k} f_k - f_k| =$

0 ($\forall k$) $\Rightarrow \lambda (T_{s_k} f_k - f_k) = 0$ ($\forall k$). The last equation contradicts that $\lambda \in \bigcup_{i=1}^n K^c(f_i, s_i)$.

$\neg d \rightarrow \neg c$: Suppose that $\exists f_1, \dots, f_n \in F$ and $s_1, \dots, s_n \in S$ and $\delta > 0$ such that

($\forall x$) $c(x) \sum |T_{s_k} f_k - f_k| \geq \delta$. Let $\lambda \in \text{MM}(F)$, so that $\lambda = w^* - \lim c(x_v)$ with $x_v \in X$ ($\forall v$).

Then $\lambda \sum |T_{s_k} f_k - f_k| = w^* - \lim c(x_v) \sum |T_{s_k} f_k - f_k| \geq \delta \rightarrow \exists k^0$ such that

$\frac{\delta}{n} \leq \lambda |T_{s_{k^0}} f_{k^0} - f_{k^0}| = |\lambda (T_{s_{k^0}} f_{k^0} - f_{k^0})| (|\lambda| g) = |\lambda g|$ because λ is multiplicative)

$$\Rightarrow \lambda (T_{s_{k^0}} f_{k^0} - f_{k^0}) \neq 0 \Rightarrow \lambda \notin K(f_{k^0}, s_{k^0}) \Rightarrow \lambda \in K^c(f_{k^0}, s_{k^0}) \Rightarrow \lambda \in \bigcup_{k=1}^n K^c(f_k, s_k).$$

$c \rightarrow f$: $\langle e(x_\delta) \rangle_{\delta > 0}$ is a net in $\text{MM}(F)$ so has a convergent subnet $\langle e(x_{\delta_\alpha}) \rangle_{\alpha \in A}$. Let λ denote the w^* -limit of $\langle e(x_{\delta_\alpha}) \rangle$. Then by the same reasoning as in $\neg c \rightarrow \neg d$, $\exists \alpha_1 \in A$ such

that $0 \leq \lambda |T_{s_k} f_k - f_k| < e(x_{\delta_{\alpha_1}}) |T_{s_k} f_k - f_k| + \frac{\epsilon}{2} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$. Since ϵ is arbitrary,

$$\lambda |T_{s_k} f_k - f_k| = 0.$$

$f \rightarrow e$: Since $\lambda \in \text{MM}(F)$, $\lambda = w^* - \lim c(x_v)$ for some net $\langle e(x_v) \rangle$ with $x_v \in X$ ($\forall v$). By the definition of w^* -convergence, for any $\delta > 0 \exists e(x_\delta) \in \langle e(x_v) \rangle$ such that

$$e(x_\delta) |T_{s_i} f_i - f_i| < \delta \quad (i=1, \dots, n).$$

$a \rightarrow h$: Assume (a) and let $f_1, \dots, f_n \in F$; $s_1, \dots, s_n \in S$; and $\epsilon > 0$.

Notation: $L(r_1, \dots, r_n) = f^{-1}(S_\epsilon(r_1)) \cap f_2^{-1}(S_\epsilon(r_2)) \cap \dots \cap f_n^{-1}(S_\epsilon(r_n)) \cap (T_{s_1} f_1)^{-1}(S_\epsilon(r_1)) \cap \dots \cap (T_{s_n} f_n)^{-1}(S_\epsilon(r_n))$ for $r_1, \dots, r_n \in \mathbb{C}$, where $S_\epsilon(r_k) = \{x \in \mathbb{C} \mid |x - r_k| < \epsilon\}$ ($k=1, \dots, n$). If some $L(r_1, \dots, r_n)$ is F, S -thick, then it suffices for the Y of (h) with $r_1 = c_1, \dots, r_n = c_n$. Assume that no $L(r_1, \dots, r_n)$ is F, S -thick. A contradiction shall be deduced. For each non-empty $L(r_1, \dots, r_n)$ and for each $\lambda \in \text{MM}(F)$, $\exists s \in S$, $\exists g \in Z(L(r_1, \dots, r_n))$ such that $\lambda(T_s(g)) \neq 0$ by (k) of Theorem 4.3. In particular, if λ is invariant, then $\lambda(g) = \lambda(T_s(g)) \neq 0$. Let $\langle e(x_v) \rangle$ be a net in $\text{MM}(F)$ such that

$\lambda = w^* - \lim_v e(x_v)$. Then for $i=1, \dots, n$, $\exists N_i$ such that $v \geq N_i \Rightarrow |f_i(x_v) - \lambda f_i| < \epsilon$ and

$|T_{s_i} f_i(x_v) - \lambda f_i| < \epsilon$; this entails that $v \geq N_1, N_2, \dots, N_n \Rightarrow |f_i(x_v) - \lambda f_i| < \epsilon$ and

$|T_{s_i} f_i(x_v) - \lambda f_i| < \epsilon \quad (i=1, \dots, n) \Rightarrow x_v \in L(\lambda f_1, \dots, \lambda f_n)$. For $L(\lambda f_1, \dots, \lambda f_n)$, $\exists g \in Z(L(\lambda f_1, \dots, \lambda f_n))$ with

$\lambda(g) \neq 0$, as previously noted, so $g(x_v) = 0$ for all $v \geq N_1, N_2, \dots, N_n$. Therefore,

$\lambda(g) = \lim_v e(x_v)g = 0$, a contradiction.

$d \rightarrow e$, $e \rightarrow d$, $f \rightarrow g$, $h \rightarrow e$: Easy

QED

REFERENCES

1. Berglund, John F., Junghenn, Hugo D., and Milnes, Paul. *Compact Right Topological Semigroups and Generalizations of Almost Periodicity*. Lecture Notes in Mathematics No. 663. Berlin: Springer-Verlag, 1978.
2. Junghenn, Hugo D. "Amenability of Function Spaces on Thick Sub-semigroups." *Proceedings of the American Mathematical Society*, Vol. 75, No. 1 (1979), 37–41.
3. Lau, Anthony To-Ming. "Extremely Amenable Algebras." *Pacific Journal of Mathematics*, Vol. 33, No. 2 (1970), 329–336.
4. ———. "Action of topological semigroups, invariant means, and fixed points." *Studia Mathematica*, Vol. 43 (1972), 139–156.
5. Mitchell, Theodore. "Constant Functions and Left Invariant Means on Semigroups." *Transactions of the American Mathematical Society*, Vol. 119 (1965 August), 244–261.

6. Sakai, Kokichi. "On amenable transformation semigroups."
 - I. *Journal of Mathematics, Kyoto Univ.* 16-3 (1976), 555-595.
 - II. *Ibid.*, 597-626.
 - III. *Scientific Reports, Kagoshima Univ.*, No. 25 (1976 September), 31-51.
7. Simmons, George F. *Introduction to Topology and Modern Analysis*. New York: McGraw-Hill Book Co., 1963.
8. Wilden, Carroll, and Witz, Klaus. "Invariant Means and the Stone-Čech Compactification." *Pacific Journal of Mathematics*, Vol. 21 (1967), 577-586.

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