INTEGRAL MEANS OF CERTAIN CLASS OF ANALYTIC FUNCTIONS

GAO CHUNYI

Department of Mathematics
Changsha Communications Institute
Changsha, Hunan 410076
People's Republic of China

(Received June 20, 1991 and in revised form September 3, 1992)

ABSTRACT. In this paper we discuss the following class of functions

$$S_\lambda(\alpha,\beta)=\{f(z):\left|\frac{f(z)}{g(z)}-1\right|<\beta\left|\lambda\frac{f(z)}{g(z)}+1\right|, z\in D\} \text{ where } 0\le\lambda\le 1,\ 0<\beta\le 1,\ 0\le\alpha<1,$$

and $f(z)=z+\sum_{n=2}^{\infty}a_n z^n$ is analytic in $D=\{z:|z|<1\}$, $g(z)$ is a starlike function of order α. A subordination about this class is obtained, the integral means of functions in $S_\lambda(\alpha,\beta)$ and some extremal properties are studied.

KEY WORDS AND PHRASES. Analytic function, subordination, integral mean, distortion, coefficient inequality.
1991 AMS SUBJECT CLASSIFICATION CODE. 30C45.

1. INTRODUCTION.

Let A be the class consisting of all functions $f(z)=z+\sum_{n=2}^{\infty}a_nz^n$ which are analytic in $D=\{z:|z|<1\}$. Owa [1] has introduced the class $\tilde{S}_\lambda(\alpha,\beta)$. If $f(z)\in A$ and there exists $g(z)=z-\sum_{n=2}^{\infty}|b_n|z^n\in S^*(\alpha)$ $(0\le\alpha<1)$ such that

$$\left|\frac{f(z)}{g(z)}-1\right|<\beta\left|\lambda\frac{f(z)}{g(z)}+1\right| \qquad (0\le\lambda\le 1,\ 0<\beta\le 1,\ z\in D), \tag{1.1}$$

we say $f(z)\in\tilde{S}_\lambda(\alpha,\beta)$. Owa [1] discussed the coefficient estimates of functions in $\tilde{S}_\lambda(\alpha,\beta)$. In this paper, we discuss the general case, i.e., the class $S_\lambda(\alpha,\beta)$ which is generated by a function

$$g(z)=z+\sum_{n=2}^{\infty}b_n z^n\in S^*(\alpha).$$

We first gave a subordinate about this class, then we discuss the integral means of functions in $S_\lambda(\alpha,\beta)$, from this we can get some extremal properties about $S_\lambda(\alpha,\beta)$. We also discuss a subclass of $S_\lambda(\alpha,\beta)$.

2. A SUBORDINATION ABOUT $S_\lambda(\alpha,\beta)$.

We say that $g(z)$ is subordinate to $f(z)$ if there exists a function $\omega(z)$ analytic in D satisfying $\omega(0)=0$ and $|\omega(z)|<1$ such that $g(z)=f(\omega(z))$ $(|z|<1)$. This subordination is denoted by $g(z)\prec f(z)$. About the class $S_\lambda(\alpha,\beta)$, we have the following:

THEOREM 2.1. If $f(z)\in S_\lambda(\alpha,\beta)$, i.e., there exists a function $g(z)\in S^*(\alpha)$ such that the inequality (1.1) holds, then we have

$$\frac{f(z)}{g(z)}\prec\frac{1+\beta z}{1-\beta\lambda z}=p_{\beta,\lambda}(z). \tag{2.1}$$

PROOF. Let $p(z)=\frac{f(z)}{g(z)}$, then $p(0)=1$. Now we divide the proof into three cases.

CASE (a). Let $\lambda \neq 0$, β and λ are not equal to 1 at the same time. Now the inequality (1.1) can be written as $|p(z)-1| < |\beta\lambda p(z)+\beta|$, that is,
$|p(z)|^2-2Re p(z)+1<\beta^2\lambda^2|p(z)|^2+2\beta^2\lambda Re p(z)+\beta^2$. From this we can get

$$\left|p(z)-\frac{1-\beta}{1+\beta\lambda}-\frac{\beta(1+\lambda)}{1-\beta^2\lambda^2}\right| < \frac{\beta(1+\lambda)}{1-\beta^2\lambda^2}.$$

Because univalent function $p_{\beta,\lambda}(z)=\frac{1+\beta z}{1-\beta\lambda z}$ maps D onto the disk

$$\left\{w: \left|w-\frac{1-\beta}{1+\beta\lambda}-\frac{\beta(1+\lambda)}{1-\beta^2\lambda^2}\right| < \frac{\beta(1+\lambda)}{1-\beta^2\lambda^2}\right\},$$

so $p(D)\subset p_{\beta,\lambda}(D)$ and $p(0)=p_{\beta,\lambda}(0)=1$. From the principle of subordination of univalent functions, we have $p(z)\prec p_{\beta,\lambda}(z)$, that is (2.1).

CASE (b). Let $\lambda=0$. Now the inequality (1.1) becomes $|p(z)-1| < \beta$.
Because univalent function $p_{\beta,o}(z)=1+\beta z$ maps D onto the disk $\{w: |w-1|<\beta\}$, so $p(D)\subset p_{\beta,o}(D)$ and $p(0)=p_{\beta,o}(0)=1$. Thus $p(z)\prec p_{\beta,o}(z)$.

CASE (c). Let $\lambda=\beta=1$. The inequality (1.1) becomes $|p(z)-1| < |p(z)+1|$, that is $Re\, p(z)>0$. Because $p(0)=1$, so $p(z)\prec\frac{1+z}{1-z}=p_{1,1}(z)$.

Thus for any $0\le\lambda\le 1, 0<\beta\le 1$, we have proved (2.1).

3. THE INTEGRAL MEANS OF FUNCTIONS IN $S_\lambda(\alpha,\beta)$.

We first state some lemmas.

LEMMA 3.1 [2]. For any $g,h\in L^1[-\pi,\pi]$, the following statements are equivalent:

(a) For every convex non-decreasing function Φ on $(-\infty,\infty)$,

$$\int_{-\pi}^{\pi}\Phi(g(x))\,dx \le \int_{-\pi}^{\pi}\Phi(h(x))\,dx.$$

(b) For every $t\in(-\infty,\infty)$,

$$\int_{-\pi}^{\pi}(g(x)-t)^+\,dx \le \int_{-\pi}^{\pi}(h(x)-t)^+\,dx.$$

(c) $g^*(\theta)\le h^*(\theta)$, $(0\le\theta\le\pi)$.

LEMMA 3.2 [2]. If g,h are real integrable functions on $[-\pi,\pi]$, then $(g+h)^*(\theta)\le g^*(\theta)+h^*(\theta)$ $(0\le\theta\le\pi)$, with equality holding if and only if g,h are symmetric decreasing arrangement functions.

The definitions of $u^*(x)$ and the symmetric decreasing arrangement function can be found in [2].

LEMMA 3.3 [3]. Let $\Phi(t)$ be a convex increasing function, if $g(z)\prec f(z)$ in D, then

$$\int_{-\pi}^{\pi}\Phi(|g(re^{i\theta})|)\,d\theta \le \int_{-\pi}^{\pi}\Phi(|f(re^{i\theta})|)\,d\theta \qquad (0<r<1) \tag{3.1}$$

and if $u(z)$ is a harmonic function in D, $v(z)=u(\omega(z))$, where $\omega(z)$ is analytic in D, $\omega(0)=0$, $|\omega(z)|<1$, then

$$\int_{-\pi}^{\pi}\Phi(\pm v(re^{i\theta}))\,d\theta \le \int_{-\pi}^{\pi}\Phi(\pm u(re^{i\theta}))\,d\theta \qquad (0<r<1). \tag{3.2}$$

When $f(z)$ is not a constant, the equality in (3.1) holds if and only if $\omega(z)=e^{i\theta}z$ or $\Phi(u)=a\log u+b$ $(a<0)$.

Let

$$k_\alpha(z)=\frac{z}{(1-z)^{2(1-\alpha)}},$$

it is well known that $k_\alpha(z) \in S^*(\alpha)$. For any $g(z) \in S^*(\alpha)$, we have

$$g(z) = z\, exp\left\{2(1-\alpha)\int_{|x|=1} log\frac{1}{1-xz}\, d\mu(x)\right\},$$

so we can easily obtain

$$\frac{g(z)}{z} \prec \frac{1}{(1-z)^{2(1-\alpha)}} = \frac{k_\alpha(z)}{z}. \tag{3.3}$$

THEOREM 3.1. If $f(z) \in S_\lambda(\alpha,\beta)$, $F_x(z) = e^{-ix}\, k_\alpha(e^{ix}z)\cdot p_{\beta,\lambda}(e^{ix}z)$, $\Phi(t)$ is a convex non-decreasing function on $(-\infty,\infty)$, then

$$\int_{-\pi}^{\pi} \Phi(\pm log\frac{|f(re^{i\theta})|}{r})\, d\theta \le \int_{-\pi}^{\pi} \Phi(\pm log\frac{|F_o(re^{i\theta})|}{r})\, d\theta \qquad (0<r<1). \tag{3.4}$$

For a strictly convex function Φ, the equality holds only for $f(z) = F_x(z)$.

PROOF. From the definition of $S_\lambda(\alpha,\beta)$, we know there exists a function $g(z) \in S^*(\alpha)$ such that the inequality (1.1) holds. So we have, from Theorem 2.1

$$p(z) = \frac{f(z)}{g(z)} \prec \frac{1+\beta z}{1-\beta\lambda z} = P_{\beta,\lambda}(z)$$

Thus

$$\int_{-\pi}^{\pi} \Phi(\pm log|p(re^{i\theta})|)\, d\theta \le \int_{-\pi}^{\pi} \Phi(\pm log|p_{\beta,\lambda}(re^{i\theta})|)\, d\theta, \qquad \text{by Lemma 3.3.}$$

Then from Lemma 3.1 we have

$$(log|p(re^{i\theta})|)^* \le (log|p_{\beta,\lambda}(re^{i\theta})|)^*.$$

On the other hand, because $\frac{f(z)}{z} = p(z)\cdot\frac{g(z)}{z}$, we have, by Lemma 3.2,

$$\left(log\frac{|f(re^{i\theta})|}{r}\right)^* \le (log|p(re^{i\theta})|)^* + \left(log\frac{|g(re^{i\theta})|}{r}\right)^*$$

Using (3.3) and Lemmas 3.3 and 3.1, we can easily get

$$\left(log\frac{|g(re^{i\theta})|}{r}\right)^* \le \left(log\frac{|k_\alpha(re^{i\theta})|}{r}\right)^*.$$

So we obtain

$$\left(log\frac{|f(re^{i\theta})|}{r}\right)^* \le (log|p_{\beta,\lambda}(re^{i\theta})|)^* + \left(log\frac{|k_\alpha(re^{i\theta})|}{r}\right)^*.$$

By evaluation we know $log|p_{\beta,\lambda}(re^{i\theta})|$ and $log\frac{|k_\alpha(re^{i\theta})|}{r}$ are symmetric decreasing arrangement functions, so again from Lemma 3.2 we have

$$\left(log\frac{|f(re^{i\theta})|}{r}\right)^* \le \left(log\left|p_{\beta,\lambda}(re^{i\theta})\cdot\frac{k_\alpha(re^{i\theta})}{r}\right|\right)^* = \left(log\frac{|F_o(re^{i\theta})|}{r}\right)^*$$

Finally we obtain, by Lemma 3.1,

$$\int_{-\pi}^{\pi} \Phi\left(log\frac{|f(re^{i\theta})|}{r}\right) d\theta \le \int_{-\pi}^{\pi} \Phi\left(log\frac{|F_o(re^{i\theta})|}{r}\right) d\theta$$

We can similarly prove the case of negative sign. The condition of the equality can easily be obtained.

THEOREM 3.2. Let $f(z) \in S_\lambda(\alpha,\beta)$, then for $p>0$ we have

$$\int_{-\pi}^{\pi} \left|f(re^{i\theta})\right|^p d\theta \le \int_{-\pi}^{\pi} \left|k_\alpha(re^{i\theta})\, p_{\beta,\lambda}(re^{i\theta})\right|^p d\theta \qquad (0<r<1) \tag{3.5}$$

and

$$\int_{-\pi}^{\pi} \left|f\left(re^{i\theta}\right)\right|^{-p} d\theta \le \int_{-\pi}^{\pi} \left|k_\alpha\left(re^{i\theta}\right) p_{\beta,\lambda}\left(re^{i\theta}\right)\right|^{-p} d\theta \qquad (0 < r < 1) \tag{3.6}$$

where the equality holds only for $f(z) = \frac{1}{x} k_\alpha(xz)\, p_{\beta,\lambda}(xz)$, $|x| = 1$.

PROOF. We only need let $\Phi(t) = e^{pt}$ in Theorem 3.1.

COROLLARY 3.1. If $f(z) \in S_\lambda(\alpha, \beta)$, then we have the following sharp inequality:

$$\frac{r(1-\beta r)}{(1+r)^{2(1-\alpha)}(1+\beta\lambda r)} \le |f(z)| \le \frac{r(1+\beta r)}{(1-r)^{2(1-\alpha)}(1-\beta\lambda r)} \qquad (|z| = r). \tag{3.7}$$

PROOF. Take p-th root in both sides of (3.5) and (3.6), and let $p \to \infty$, we can get inequality (3.7).

COROLLARY 3.2. If $f(z) \in S_\lambda(\alpha, \beta)$, then we have $f(D) \supset \{w: |w| < d(\alpha, \beta, \lambda)\}$, where

$$d(\alpha, \beta, \lambda) = \frac{1-\beta}{2^{2(1-\alpha)}(1+\beta\lambda)}$$

cannot be replaced by any larger number.

PROOF. We can easily know $f(z)$ is univalent in D from the definition of $S_\lambda(\alpha, \beta)$, so

$$dist\,(0, \partial f(D)) = \lim_{|z| \to 1} inf\, |f(z)| \ge \lim_{|z| \to 1} \frac{|z|(1-\beta|z|)}{(1+|z|)^{2(1-\alpha)}(1+\beta\lambda|z|)} = \frac{1-\beta}{2^{2(1-\alpha)}(1+\beta\lambda)}.$$

The sharpness can bee seen from the function $\frac{z(1+\beta z)}{(1-z)^{2(1-\alpha)}(1-\beta\lambda z)} \in S_\lambda(\alpha, \beta)$.

4. A SUBCLASS $\in S_\lambda(\alpha, \beta)$.

Let $g(z) = z$, we obtain a subclass $\in S_\lambda(\alpha, \beta)$, we denote it by $S_\lambda(\beta)$. Corresponding to (2.1), for the class $S_\lambda(\beta)$, we have the following subordination:

$$\frac{f(z)}{z} \prec \frac{1+\beta z}{1-\beta\lambda z} = p_{\beta,\lambda}(z). \tag{4.1}$$

Thus for $S_\lambda(\beta)$ we have

THEOREM 4.1. Let $f(z) \in S_\lambda(\beta), \Phi(t)$ is a convex non-decreasing function on $(-\infty, \infty)$, then

$$\int_{-\pi}^{\pi} \Phi\left(\pm \log \frac{|f(re^{i\theta})|}{r}\right) d\theta \le \int_{-\pi}^{\pi} \Phi\left(\pm \log \left|\frac{1+\beta re^{i\theta}}{1-\beta\lambda re^{i\theta}}\right|\right) d\theta \qquad (0 < r < 1). \tag{4.2}$$

For a strictly convex function Φ, the equality holds only for function $f(z) = zp_{\beta,\lambda}(xz)$, $|x| = 1$

If we use subordination (4.1) and Lemma 3.3, we can obtain the following:

THEOREM 4.2. Let $f(z) \in S_\lambda(\beta), \Phi(t)$ is a convex non-decreasing function on $(-\infty, \infty)$, then

(a) $$\int_{-\pi}^{\pi} \Phi\left(\frac{|f(re^{i\theta})|}{r}\right) d\theta \le \int_{-\pi}^{\pi} \Phi\left(\left|\frac{1+\beta re^{i\theta}}{1-\beta\lambda re^{i\theta}}\right|\right) d\theta, \tag{4.3}$$

(b) $$\int_{-\pi}^{\pi} \Phi\left(\left|\log \frac{f(re^{i\theta})}{re^{i\theta}}\right|\right) d\theta \le \int_{-\pi}^{\pi} \Phi\left(\left|\log \frac{1+\beta re^{i\theta}}{1-\beta\lambda re^{i\theta}}\right|\right) d\theta, \tag{4.4}$$

(c) $$\int_{-\pi}^{\pi} \Phi\left(\pm \arg \frac{f(re^{i\theta})}{re^{i\theta}}\right) d\theta \le \int_{-\pi}^{\pi} \Phi\left(\arg \frac{1+\beta re^{i\theta}}{1-\beta\lambda re^{i\theta}}\right) d\theta. \tag{4.5}$$

For a strictly convex function Φ, the equality holds only for $f(z) = zp_{\beta,\lambda}(xz)$, $|x| = 1$. From (4.5) we obtain the rotation theorem of $S_\lambda(\beta)$.

COROLLARY 4.1. Let $f(z) \in S_\lambda(\beta)$, then for $|z| = r < 1$ we have the following sharp inequality:

$$\left|\arg \frac{f(z)}{z}\right| \le \arcsin \frac{\beta(1+\lambda)r}{1+\lambda\beta^2 r^2}.$$

PROOF. If we take

$$\Phi(t) = \begin{cases} t^{2n} & , \quad t \geq 0 \\ 0 & , \quad t < 0 \end{cases}$$

in (4.5), we have

$$\int_o^{2\pi} \left| arg^+ \frac{f(re^{i\theta})}{re^{i\theta}} \right|^{2n} d\theta \leq \int_o^{2\pi} \left| arg^+ \frac{1+\beta re^{i\theta}}{1-\beta\lambda re^{i\theta}} \right|^{2n} d\theta.$$

Take the $2n$-th root in both sides of this inequality and let $n \to \infty$, we get

$$\max_{-\pi \leq \theta \leq \pi} arg^+ \frac{f(re^{i\theta})}{re^{i\theta}} \leq \max_{-\pi \leq \theta \leq \pi} arg^+ \frac{1+\beta re^{i\theta}}{1-\beta\lambda re^{i\theta}} = arcsin \frac{\beta(1+\lambda)r}{1+\lambda\beta^2 r^2}.$$

This implies

$$arg^+ \frac{f(re^{i\theta})}{re^{i\theta}} \leq arcsin \frac{\beta(1+\lambda)r}{1+\lambda\beta^2 r^2}.$$

Similarly, we have

$$arg^- \frac{f(re^{i\theta})}{re^{i\theta}} \leq arcsin \frac{\beta(1+\lambda)r}{1+\lambda\beta^2 r^2}.$$

So for any $f(z) \in S_\lambda(\beta)$, we have

$$\left| arg \frac{f(z)}{z} \right| \leq arcsin \frac{\beta(1+\lambda)r}{1+\lambda\beta^2 r^2} \quad (|z| = r < 1),$$

where they equality holds only for $f(z) = zp_{\beta,\lambda}(xz)$, $|x| = 1$. The proof of Corollary 4.1 is complete.

From the univalence of $f(z)$ we know $\frac{f(z)}{z} \neq 0$, so we can define a single-valued and analytic branch of $log \frac{f(z)}{z}$. Let

$$g(z) = log \frac{f(z)}{z} = \sum_{n=1}^{\infty} \gamma_n z^n,$$

then we have:

COROLLARY 4.2. Let $f(z) \in S_\lambda(\beta)$, then we have

$$\sum_{n=1}^{\infty} |\gamma_n|^2 \leq \sum_{n=1}^{\infty} \frac{(\beta^n\lambda^n + (-1)^{n-1}\beta^n)^2}{n^2}, \tag{4.6}$$

where the inequality holds only for $f(z) = zp_{\beta,\lambda}(xz)$, $|x| = 1$.

PROOF. Let

$$G(z) = log \frac{1+\beta z}{1-\beta\lambda z} = \sum_{n=1}^{\infty} \frac{\beta^n\lambda^n + (-1)^{n-1}\beta^n}{n} z^n.$$

Take $\Phi(t) = t^2$ in (4.4), we have

$$\frac{1}{2\pi} \int_o^{2\pi} |g(re^{i\theta})|^2 d\theta \leq \frac{1}{2\pi} \int_o^{2\pi} |G(re^{i\theta})|^2 d\theta,$$

that is,

$$\sum_{n=1}^{\infty} |\lambda_n|^2 r^{2n} \leq \sum_{n=1}^{\infty} \frac{(\beta^n\lambda^n + (-1)^{n-1}\beta^n)^2}{n^2} r^{2n},$$

let $r \to 1$, we obtain the inequality we need to prove.

REMARK. Let $\lambda = \beta = 1$ in Corollary 4.2, that is $f(z) \in S_1(1)$, i.e., $Re(f(z)/z) > 0$. Inequality (4.6) becomes

$$\sum_{n=1}^{\infty} |\gamma_n|^2 \leq \sum_{n=1}^{\infty} \frac{4}{(2n-1)^2} = \frac{\pi^2}{2}.$$

This inequality is sharp.

Finally, we consider the initial coefficients of $f(z) \in S_\lambda(\beta)$.

LEMMA 4.1. If $f'(0) = F'(0) = 1$ and they satisfy the following equality

$$\frac{(1-\beta)-(1+\beta\lambda)f(z)/z}{(1-\beta\lambda)f(z)/z-(1+\beta)} = \frac{F(z)}{z}, \tag{4.7}$$

then $f(z) \in S_\lambda(\beta)$ if and only if $F(z) \in S_1(1)$, i.e., $Re(F(z)/z) > 0$.

PROOF. Let $f(z) \in S_\lambda(\beta)$, then $p(z) = \frac{f(z)}{z} \prec \frac{1+\beta z}{1-\beta\lambda z} = p_{\beta,\lambda}(z)$, so $p(D) \subset p_{\beta,\lambda}(D) = D_1$ where D_1 is a disk which diameter is $(\frac{1-\beta}{1+\beta\lambda}, \frac{1+\beta}{1-\beta\lambda})$, thus

$$(1-\beta\lambda)p(z)-(1+\beta) \neq 0.$$

From this we know $F(z)$ is analytic in D. And $F(0) = 0$ because of $p(0) = 1$. On the other hand, the function

$$\frac{(1-\beta)-(1+\beta\lambda)w}{(1-\beta\lambda)w-(1+\beta)}$$

maps D_1 onto the right half plane, so we have $Re(F(z)/z) > 0$ $(z \in D)$, i.e., $F(z) \in S_1(1)$.

We can prove the opposite result similarly.

THEOREM 4.3. Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in S_\lambda(\beta)$, then for real number μ we have the sharp estimates:

$$|a_2| \le \beta(1+\lambda) \tag{4.8}$$

$$|a_3 - \mu a_2^2| \le \begin{cases} \beta(1+\lambda)(\beta\lambda - \mu(\beta+\beta\lambda)), & \mu \le -\frac{1-\beta\lambda}{\beta+\beta\lambda}, & (4.9) \\ \beta(1+\lambda), & -\frac{1-\beta\lambda}{\beta+\beta\lambda} < \mu < \frac{1+\beta\lambda}{\beta+\beta\lambda}, & (4.10) \\ \beta(1+\lambda)(\mu(\beta+\beta\lambda)-\beta\lambda), & \mu > \frac{1+\beta\lambda}{\beta+\beta\lambda}. & (4.11) \end{cases}$$

PROOF. Because $f(z) \in S_\lambda(\beta)$, then $F(z)$ defined by (4.7) belongs to $S_1(1)$, i.e., $Re(F(z)/z) > 0$, so there exists an analytic function $p(z)$ satisfying $p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n$, $Re\,p(z) > 0$ such that

$$\frac{F(z)}{z} = p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n.$$

Substituting it into (4.7) and comparing the coefficients of both sides of (4.7), we have

$$a_2 = \tfrac{1}{2}(\beta+\beta\lambda)p_1, \quad a_3 - \mu a_2^2 = \tfrac{1}{2}(\beta+\beta\lambda)\left\{ p_2 - \tfrac{1}{2}((1-\beta\lambda)+\mu(\beta+\beta\lambda))\, p_1^2 \right\}.$$

It is well known that $|p_n| \le 2$ $(n = 1, 2, \cdots\cdots)$

$$|p_2 - \mu p_1^2| \le \begin{cases} 2(1-2\mu), & \mu \le 0 \\ 2, & 0 < \mu < 1 \\ 2(2\mu-1), & \mu \ge 1 \end{cases}.$$

So we proved the results. Its easy to know the function $f(z) = \frac{z(1+\beta\varepsilon z)}{1-\beta\lambda\varepsilon z}, (|\varepsilon| = 1)$ attains the equalities in (4.8), (4.9) and (4.11), and the function $f(z) = \frac{z(1+\beta\varepsilon z^2)}{1-\beta\lambda\varepsilon z^2}, (|\varepsilon| = 1)$ attains the inequality in (4.10).

REFERENCES

1. OWA, S., A remark on certain classes of analytic functions, Math. Japon. 28 (1983), 15-20.
2. BAERSTEIN, A., Integral means, univalent functions and circular symmetrization, Acta Math. 133 (1974), 139-169.
3. CHEN, H., Integral means of the subordinate class, Acta Math. Sinica 6 (1990), 739-756.
4. POMMERENKE, CH., Univalent Functions, Göttingen, 1975.

Journal of Applied Mathematics and Decision Sciences

Special Issue on Intelligent Computational Methods for Financial Engineering

Call for Papers

As a multidisciplinary field, financial engineering is becoming increasingly important in today's economic and financial world, especially in areas such as portfolio management, asset valuation and prediction, fraud detection, and credit risk management. For example, in a credit risk context, the recently approved Basel II guidelines advise financial institutions to build comprehensible credit risk models in order to optimize their capital allocation policy. Computational methods are being intensively studied and applied to improve the quality of the financial decisions that need to be made. Until now, computational methods and models are central to the analysis of economic and financial decisions.

However, more and more researchers have found that the financial environment is not ruled by mathematical distributions or statistical models. In such situations, some attempts have also been made to develop financial engineering models using intelligent computing approaches. For example, an artificial neural network (ANN) is a nonparametric estimation technique which does not make any distributional assumptions regarding the underlying asset. Instead, ANN approach develops a model using sets of unknown parameters and lets the optimization routine seek the best fitting parameters to obtain the desired results. The main aim of this special issue is not to merely illustrate the superior performance of a new intelligent computational method, but also to demonstrate how it can be used effectively in a financial engineering environment to improve and facilitate financial decision making. In this sense, the submissions should especially address how the results of estimated computational models (e.g., ANN, support vector machines, evolutionary algorithm, and fuzzy models) can be used to develop intelligent, easy-to-use, and/or comprehensible computational systems (e.g., decision support systems, agent-based system, and web-based systems)

This special issue will include (but not be limited to) the following topics:

- **Computational methods**: artificial intelligence, neural networks, evolutionary algorithms, fuzzy inference, hybrid learning, ensemble learning, cooperative learning, multiagent learning
- **Application fields**: asset valuation and prediction, asset allocation and portfolio selection, bankruptcy prediction, fraud detection, credit risk management
- **Implementation aspects**: decision support systems, expert systems, information systems, intelligent agents, web service, monitoring, deployment, implementation

Authors should follow the Journal of Applied Mathematics and Decision Sciences manuscript format described at the journal site http://www.hindawi.com/journals/jamds/. Prospective authors should submit an electronic copy of their complete manuscript through the journal Manuscript Tracking System at http://mts.hindawi.com/, according to the following timetable:

Manuscript Due	December 1, 2008
First Round of Reviews	March 1, 2009
Publication Date	June 1, 2009

Guest Editors

Lean Yu, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, China; Department of Management Sciences, City University of Hong Kong, Tat Chee Avenue, Kowloon, Hong Kong; yulean@amss.ac.cn

Shouyang Wang, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, China; sywang@amss.ac.cn

K. K. Lai, Department of Management Sciences, City University of Hong Kong, Tat Chee Avenue, Kowloon, Hong Kong; mskklai@cityu.edu.hk