

APPROXIMATING FIXED POINTS OF NONEXPANSIVE AND GENERALIZED NONEXPANSIVE MAPPINGS

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ABSTRACT.

In this paper we consider a mapping S of the form

$$S = \alpha_0 I + \alpha_1 T + \alpha_2 T^2 + \dots + \alpha_k T^k,$$

where $\alpha_i \geq 0$, $\alpha_1 > 0$ with $\sum_{i=0}^k \alpha_i = 1$, and show that in a uniformly convex Banach space the Picard iterates of S converge to a fixed point of T when T is nonexpansive or generalized nonexpansive or even quasi-nonexpansive.

KEY WORDS AND PHRASES: Fixed points, nonexpansive mappings, uniformly convex Banach spaces.

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1. INTRODUCTION.

Let B be a Banach space and C a convex subset of B . A mapping $T: C \rightarrow C$ is said to be nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. A mapping $T: C \rightarrow C$ is said to be quasi-nonexpansive if T has a fixed point p such that $\|Tx - p\| \leq \|x - p\|$ for all $x \in C$. The concept of quasi-nonexpansiveness is more general than that of nonexpansiveness. Indeed, a nonexpansive mapping with at least one fixed point is quasi-nonexpansive, but there exists quasi-nonexpansive mappings which are not nonexpansive. See, for example, Petryshyn and Williamson [7].

If T is nonexpansive, then the Picard iterates of T may not converge and, even if they do converge, they may not converge to a fixed point of T . However, to circumvent the difficulty, one may consider the mapping

$$T_\lambda = (1 - \lambda)I + \lambda T, \quad (1.1)$$

where I is the identity mapping and $0 < \lambda < 1$, and show that the Picard iterates of T_λ converge to a fixed point of T under certain restrictions, see [3,5,10]. Generalizing the idea Kirk [6] has introduced a mapping S given by

$$S = \alpha_0 I + \alpha_1 T + \alpha_2 T^2 + \dots + \alpha_k T^k \quad (1.2)$$

where $\alpha_i \geq 0$, $\alpha_1 > 0$ with $\sum_{i=0}^k \alpha_i = 1$, and has shown that the Picard iterates of S converge to a fixed point of T under conditions similar to those imposed in connection with the convergence of Picard iterates of T_λ .

Our purpose here is two-fold. First we show that the Picard iterates of S converge to a fixed point of T under conditions weaker than those imposed by Kirk [6]. Secondly, we establish that the Picard iterates of S converge to a fixed point of T even when T is generalized nonexpansive, i.e., when T satisfies

$$\|Tx - Ty\| \leq a\|x - y\| + b\{\|x - Tx\| + \|y - Ty\|\} + c\{\|y - Tx\| + \|x - Ty\|\} \quad (1.3)$$

for all $x, y \in C$, where $a, b, c \geq 0$ with $a + 2b + 2c \leq 1$. Then the analysis has been extended to a more general mapping resulting in generalization of some results obtained by Ray and Rhoades [9].

2. CONVERGENCE TO FIXED POINTS

It has been established by Kirk [6] that S and T have common fixed points if T is nonexpansive. Let the common fixed point set be denoted by F . Further, the set F is closed when T is nonexpansive or even when T is quasi-nonexpansive (see Dotson [2]). We now state the following conditions:

CONDITION-A. A mapping $T: C \rightarrow C$ with a nonempty fixed point set F is said to satisfy Condition-A if there is a nondecreasing function $f: [0, \infty) \rightarrow [0, \infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r \in (0, \infty)$ such that $\|x - Sx\| \geq f(d(x, F))$ for all $x \in C$, where $d(x, F) = \inf_{p \in F} \|x - p\|$.

CONDITION-B. A mapping $T: C \rightarrow C$ with a nonempty fixed point set F is said to satisfy Condition-B if there exists a number $\alpha > 0$ such that $\|x - Sx\| \geq \alpha d(x, F)$ for all $x \in C$.

It may be remarked that the mappings which satisfy Condition-B also satisfy Condition-A. However, Condition-B may be verified easily by giving examples. It may be further remarked that Conditions I and II of Senter and Dotson [11] are identical with Conditions A and B when $\alpha_2 = \alpha_3 = \dots = \alpha_k = 0$.

We now recall the following lemma due to Dotson [1]. This will be used later to establish our results.

LEMMA. If the sequences $\{s_n\}$ and $\{t_n\}$ are in the closed unit ball of a uniformly convex Banach space and $\{z_n\} = \{(1 - \alpha_n)s_n + \alpha_n t_n\}$ satisfies $\lim_{n \rightarrow \infty} \|z_n\| = 1$, where $0 < \alpha \leq \alpha_n \leq b < 1$, then $\lim_{n \rightarrow \infty} \|s_n - t_n\| = 0$.

THEOREM 1. Let C be a nonempty, closed, convex and bounded subset of a uniformly convex Banach space B and $T: C \rightarrow C$ be a nonexpansive mapping. If T satisfies Condition-A, where F is the fixed point set of T in C , then for an arbitrary $x_0 \in C$, the Picard iterates $(S^n x_0)$ converge to a member of F .

PROOF. If $x_0 \in F$, then the result is trivial. We assume that $x_0 \in C - F$. Then, setting $x_n = S^n x_0$, we have for an arbitrary $p \in F$

$$\begin{aligned} \|x_{n+1} - p\| &= \|S^{n+1} x_0 - p\| = \|Sx_n - p\| \\ &= \|\alpha_0 x_n + \alpha_1 T x_n + \alpha_2 T^2 x_n + \dots + \alpha_k T^k x_n - p\| \\ &= \|\alpha_0(x_n - p) + \alpha_1(Tx_n - p) + \alpha_2(T^2 x_n - p) + \dots + \alpha_k(T^k x_n - p)\| \\ &\leq \alpha_0\|x_n - p\| + \alpha_1\|Tx_n - p\| + \alpha_2\|T^2 x_n - p\| + \dots + \alpha_k\|T^k x_n - p\| \\ &\leq \|x_n - p\|. \end{aligned}$$

This implies that $d(x_{n+1}, F) \leq d(x_n, F)$ and hence that the sequence $\{d(x_n, F)\}$ is nonincreasing. Then $\lim_{n \rightarrow \infty} d(x_n, F)$ exists. In the sequel we shall show that this limit is zero.

Suppose that $\lim_{n \rightarrow \infty} d(x_n, F) = b > 0$. Then, for a $p \in F$, $\lim_{n \rightarrow \infty} \|x_n - p\| = b' \geq b > 0$. Choose a positive integer N such that $\|x_n - p\| \leq 2b'$ for $n \geq N$. Set $y_n^i = (T^i x_n - p)/\|x_n - p\|$ for all n and all $i = 0, 1, 2, \dots, k$ with $T^0 x_n = x_n$. Then $\|y_n^i\| \leq 1$. Further, set $z_n = \alpha_0 y_n^0 + (1 - \alpha_0)t_n$, where $t_n = \sum_{i=1}^k (\alpha_i y_n^i)/(1 - \alpha_0)$ with $\|t_n\| \leq 1$. Then $\{y_n^0\}$ and $\{T_n\}$ are in the closed unit ball. Now

$$\begin{aligned}
\|z_n\| &= \left\| \alpha_0 \frac{x_n - p}{\|x_n - p\|} + \alpha_1 \frac{Tx_n - p}{\|x_n - p\|} + \alpha_2 \frac{T^2x_n - p}{\|x_n - p\|} + \dots + \alpha_k \frac{T^kx_n - p}{\|x_n - p\|} \right\| \\
&= \frac{\|\alpha_0x_n + \alpha_1Tx_n + \alpha_2T^2x_n + \dots + \alpha_kT^kx_n - p\|}{\|x_n - p\|} = \frac{\|x_{n+1} - p\|}{\|x_n - p\|}, \tag{2.1}
\end{aligned}$$

implying $\|z_n\| \rightarrow 1$ as $n \rightarrow \infty$. But, for $n \geq N$, we have

$$\begin{aligned}
\|y_n^0 - t_n\| &= \left\| \frac{x_n - p}{\|x_n - p\|} - \frac{1}{(1 - \alpha_0)} \sum_{i=1}^k \frac{\alpha_i(T^i x_n - p)}{\|x_n - p\|} \right\| \\
&= \left\| \frac{x_n - p}{\|x_n - p\|} - \frac{Sx_n - \alpha_0x_n - (1 - \alpha_0)p}{(1 - \alpha_0)\|x_n - p\|} \right\| \\
&= \frac{\|x_n - Sx_n\|}{(1 - \alpha_0)\|x_n - p\|} \\
&\geq \frac{f(d(x_n, F))}{(1 - \alpha_0)\|x_n - p\|} \geq \frac{f(b)}{2b'(1 - \alpha_0)} > 0, \tag{2.2}
\end{aligned}$$

implying $\lim_{n \rightarrow \infty} \|y_n^0 - t_n\| \neq 0$, which contradicts the lemma. Hence $\lim_{n \rightarrow \infty} d(x_n, F) = 0$. This implies that $\{x_n\}$ converges to a member of F , since F is closed.

REMARK 1. It is obvious that the above theorem holds if Condition-B is satisfied instead of Condition-A.

REMARK 2. Condition-A is more general than the condition imposed by Kirk [6] in establishing the convergence of Picard iterates $\{S^n x_0\}$, see Senter and Dotson [11].

REMARK 3. In the above theorem the existence of a nonempty fixed point set F is not assumed and is ensured by the conditions assumed therein. However, if we assume that T has a nonempty fixed point set F , then T need not be assumed to be nonexpansive and it is enough for T to be quasi-nonexpansive. Further, C need not be bounded. Indeed, the following result holds.

THEOREM 2. Let C be a nonempty, closed and convex subset of a uniformly convex Banach space B and $T: C \rightarrow C$ be a quasi-nonexpansive mapping. If T satisfies Condition-A, where F is the fixed point set of T in C , then, for an arbitrary $x_0 \in C$, the Picard iterates $\{S^n x_0\}$ converge to a member of F .

The proof may be established exactly in the same way as in Theorem 1. It only remains to be shown here that S and T have common fixed points. A fixed point of T is obviously a fixed point of S . We now show that the converse is also so. Let p be a fixed point of S . Then from Condition-A it is obvious that $d(p, F) = 0$, implying $p \in \overline{F}$. Since T is quasi-nonexpansive, F is closed and hence $p \in F$, i.e., p is a fixed point of T . Hence the result.

Next, we show that the Picard iterates of S converge to a fixed point of T even when T is generalized nonexpansive. However, one need not assume Condition-A or Condition-B in this case. These conditions are automatically satisfied.

THEOREM 3. Let C be a nonempty, bounded, closed and convex subset of a uniformly convex Banach space B and $T: C \rightarrow C$ be a continuous mapping such that

$$\|Tx - Ty\| \leq a\|x - y\| + b\{\|x - Tx\| + \|y - Ty\|\} + c\{\|x - Ty\| + \|y - Tx\|\} \tag{2.3}$$

for all $x, y \in C$, where $a, c \geq 0$ and $b > 0$ with $a + 2b + 2c \leq 1$. Then for an arbitrary $x_0 \in C$, the Picard iterates $\{S^n x_0\}$ converges to the unique fixed point of T .

PROOF. By Theorem 2 of Goebel, Kirk and Shimi [4] the mapping T has a unique fixed point p , say. Setting $y = p$ in (2.3) we have

$$\begin{aligned}\|Tx - p\| &\leq (a + c)\|x - p\| + b\|x - Tx\| + c\|Tx - p\| \\ &\leq (a + b + c)\|x - p\| + (b + c)\|Tx - p\|,\end{aligned}$$

implying

$$\|Tx - p\| \leq \frac{a + b + c}{1 - b - c}\|x - p\| \leq \|x - p\|, \quad (2.4)$$

since $a + 2b + 2c \leq 1$. Thus T is quasi-nonexpansive. Further, it is easy to verify that

$$\|Sx - p\| \leq \|Tx - p\| \leq \|x - p\|, \quad (2.5)$$

implying S is also quasi-nonexpansive.

It is obvious that p is also a fixed point of S . We now show that S cannot have a fixed point other than p . If possible, let $q (\neq p)$ be a fixed point of S . Then

$$\begin{aligned}\|q - Tq\| &= \|Sq - Tq\| \\ &= \|\alpha_0q + \alpha_1Tq + \alpha_2T^2q + \dots + \alpha_kT^kq - Tq\| \\ &= \|\alpha_0(q - Tq) + \alpha_1(T^2q - Tq) + \dots + \alpha_k(T^kq - Tq)\| \\ &\leq \alpha_0\|q - Tq\| + \alpha_1\|T^2q - Tq\| + \dots + \alpha_k\|T^kq - Tq\| \\ &\leq \alpha_0\{\|q - p\| + \|Tq - p\|\} + \alpha_2\{\|T^2q - p\| + \|Tq - p\|\} \\ &\quad + \dots + \alpha_k\{\|T^kq - p\| + \|Tq - p\|\} \\ &\leq 2(\alpha_0 + \alpha_2 + \dots + \alpha_k)\|q - p\| - 2(1 - \alpha_1)\|q - p\|. \quad (2.6)\end{aligned}$$

Since T is generalized nonexpansive, we have

$$\begin{aligned}\|Tq - p\| &= \|Tq - Tp\| \\ &\leq a\|q - p\| + b\|Tq - q\| + c\{\|Tq - p\| + \|q - p\|\} \\ &\leq (a + 2c)\|q - p\| + b\|Tq - q\|. \quad (2.7)\end{aligned}$$

Substituting from (2.6) into (2.7) and noting that $a + 2c \leq 1 - 2b$ we obtain

$$\begin{aligned}\|Tq - p\| &\leq (1 - 2b)\|q - p\| + 2b(1 - \alpha_1)\|q - p\| \\ &= \|q - p\| - 2b\alpha_1\|q - p\| \\ &< \|q - p\| - 2b\alpha_1\|Tq - p\|,\end{aligned}$$

implying

$$\|Tq - p\| \leq \frac{1}{1 + 2b\alpha_1}\|q - p\| \quad (2.8)$$

Now from (2.8) we have

$$\|q - p\| - \|Sq - p\| \leq \|Tq - p\| \leq \frac{1}{1 + 2b\alpha_1}\|q - p\|, \quad (2.9)$$

which implies $q = p$, since $b, \alpha_1 > 0$. Thus S and T have a unique fixed point p .

Next, we show that T satisfies Condition-B. For $x \in C$ we have

$$\|Tx - p\| = \|Tx - Tp\| \leq a\|x - p\| + b\|x - Tx\| + c\{\|x - p\| + \|Tx - p\|\},$$

implying

$$\|Tx - p\| \leq \frac{a + c}{1 - c}\|x - p\| + \frac{b}{1 - c}\|x - Tx\|. \quad (2.10)$$

Now,

$$\begin{aligned}
 \|Tx - x\| &\leq \|Sx - Tx\| + \|Sx - x\| \\
 &\leq \alpha_0 \|x - Tx\| + \alpha_2 \|T^2x - Tx\| + \dots + \alpha_k \|T^kx - Tx\| + \|Sx - x\| \\
 &\leq 2(1 - \alpha_1) \|x - p\| + \|Sx - x\|. \tag{2.11}
 \end{aligned}$$

Also we observe that

$$\|Sx - p\| \leq \alpha_0 \|x - p\| + (\alpha_1 + \alpha_2 + \dots + \alpha_k) \|Tx - p\| \tag{2.12}$$

and that

$$\|Sx - x\| \geq \|x - p\| - \|Sx - p\|. \tag{2.13}$$

Now, substituting from (2.12) into (2.13) we derive

$$\begin{aligned}
 \|Sx - x\| &\geq \|x - p\| - \alpha_0 \|x - p\| - (\alpha_1 + \alpha_2 + \dots + \alpha_k) \|Tx - p\| \\
 &= (1 - \alpha_0) \{ \|x - p\| - \|Tx - p\| \},
 \end{aligned}$$

whence, using (2.10), we obtain

$$\begin{aligned}
 \|Sx - x\| &\geq (1 - \alpha_0) \left\{ \|x - p\| - \frac{a+c}{1-c} \|x - p\| - \frac{b}{1-c} \|x - Tx\| \right\} \\
 &= (1 - \alpha_0) \left\{ \frac{1-a-2c}{1-c} \|x - p\| - \frac{b}{1-c} \|x - Tx\| \right\} \\
 &\geq (1 - \alpha_0) \left\{ \frac{2b}{1-c} \|x - p\| - \frac{b}{1-c} \|x - Tx\| \right\} \\
 &= \frac{b(1 - \alpha_0)}{1 - c} \{ 2\|x - p\| - \|x - Tx\| \}. \tag{2.14}
 \end{aligned}$$

Now, substituting from (2.11) into (2.14) we get

$$\begin{aligned}
 \|Sx - x\| &\geq \frac{b(1 - \alpha_0)}{1 - c} \{ 2\|x - p\| - 2(1 - \alpha_1) \|x - p\| - \|Sx - x\| \} \\
 &\quad - \frac{b(1 - \alpha_0)}{1 - c} \{ 2\alpha_1 \|x - p\| - \|Sx - x\| \},
 \end{aligned}$$

implying

$$\|Sx - x\| \geq \alpha \|x - p\|, \tag{2.15}$$

where

$$\alpha = \frac{2b\alpha_1(1 - \alpha_0)}{1 - c + b(1 - \alpha_0)} > 0,$$

since $b, \alpha_1 > 0$. Thus T satisfies Condition-B. Hence, by Theorem 2, the result follows.

REMARK 4. It may be noted that the stipulation $\alpha_1 > 0$ in S is necessary to rule out the possibility that fixed point of S is a point at which T may be periodic.

REMARK 5. If we do not restrict $b > 0$ in Theorem 3, then the fixed point set of T is not a singleton, and Condition-A is to be imposed to ensure the convergence of $\{S^n x_0\}$.

The present analysis can be extended to a more general mapping T which satisfies

$$\|Tx - Ty\| \leq \max \{ \|x - y\|, [\|x - Tx\| + \|y - Ty\|]/2, [\|y - Tx\| + \|x - Ty\|]/2 \} \tag{2.16}$$

for all $x, y \in C$. This mapping includes nonexpansive and generalized nonexpansive mappings (see Rhoades [8]). It is easy to verify that T is quasi-nonexpansive. It has been proved by Ray and Rhoades [9] that S and T have the same fixed point set. Further, they have established the following theorem in this connection.

THEOREM 4. ([9, Theorem 2]). Let C be a nonempty, closed convex and bounded subset of a uniformly convex Banach space B and T a self-mapping of C which satisfies (2.16). If $I-S$ maps bounded closed subsets of C into closed sets of B , then, for each $x_0 \in C$, the sequence $\{S^n x_0\}$ converges to a fixed point of T in C .

However, the fact that $I-S$ maps bounded closed subsets of C into closed sets implies Condition-A (see Senter and Dotson [11]). Thus Condition-A is more general and incorporating this condition we may obtain the following as generalizations of Theorems 2 and 3 of Ray and Rhoades [9].

THEOREM 5. Let C be a nonempty, closed and convex subset of a uniformly convex Banach space B and T a self-mapping of C which satisfies (2.16). If T satisfies Condition-A, where F is the nonempty fixed point set of T in C , then, for an arbitrary $x_0 \in C$, the Picard iterates $\{S^n x_0\}$ converge to a member of F .

We may note that C need not be bounded in Theorem 5. Because we have assumed the existence of nonempty fixed point set of T and Condition-A (see [11]). But the boundedness of C cannot be omitted from the statement of Theorem 4.

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