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A FOCAL BOUNDARY VALUE PROBLEM FOR DIFFERENCE EQUATIONS

CATHRYN DENNY and DARREL HANKERSON

Department of Algebra, Combinatorics, and Analysis
Auburn University
Auburn, Alabama 36849-5307

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ABSTRACT. The eigenvalue problem in difference equations, $(-1)^{n-k}\Delta^n y(t) = \lambda \sum_{i=0}^{k-1} p_i(t)\Delta^i y(t)$, with $\Delta^i y(0) = 0$, $0 \le i < k$, $\Delta^{k+i} y(T+1) = 0$, $0 \le i < n-k$, is examined. Under suitable conditions on the coefficients p_i, it is shown that the smallest positive eigenvalue is a decreasing function of T. As a consequence, results concerning the first focal point for the boundary value problem with $\lambda = 1$ are obtained.

KEY WORDS AND PHRASES. Difference equation, eigenvalue, boundary value problem, focal point, Green's function.
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1. INTRODUCTION

Let k and n be integers with $1 \le k < n$. For functions y defined on an interval of integers, define the difference operator Δ by $\Delta^0 y = y$, $\Delta y(t) = y(t+1) - y(t)$, and $\Delta^i y = \Delta(\Delta^{i-1} y)$ for $i \ge 1$. We shall be concerned first with the eigenvalue problem for difference equations

$$(-1)^{n-k}\Delta^n y(t) = \lambda \sum_{i=0}^{k-1} p_i(t)\Delta^i y(t), \quad 0 \le t \le T, \tag{1.1}$$

$$\begin{aligned} &\Delta^i y(0) = 0, \quad 0 \le i \le k-1, \\ &\Delta^{k+i} y(T+1) = 0, \quad 0 \le i \le n-k-1, \end{aligned} \tag{1.2}$$

where T is a nonnegative integer. Throughout this paper, the interval notation in expressions such as (1.1) denote intervals of integers; for example, $[0,T] = \{0,1,\ldots,T\}$. Under suitable conditions on the coefficients p_i, we show that the smallest positive eigenvalue is a decreasing function of T.

Next, we will consider the boundary value problem (1.2),

$$(-1)^{n-k}\Delta^n y(t) = \sum_{i=0}^{k-1} p_i(t)\Delta^i y(t), \quad 0 \le t \le T. \tag{1.3}$$

If there is a nontrivial solution of (1.3), (1.2), then T is said to be a $(k, n-k)$-*focal point* of (1.3). The smallest such T is called the *first* $(k, n-k)$-*focal point*, or, more briefly, the first focal point. The results concerning the monotonicity of the eigenvalue will be used to investigate relationships between the existence of first $(k, n-k)$-focal points of (1.3) and the existence of solutions of (1.3), (1.2) that are positive with respect to a cone in a suitable Banach space.

It can be shown that the Green's function $G_T(t,s)$ for the focal boundary value problem

$$\begin{aligned} &(-1)^{n-k}\Delta^n y(t) = \delta_{ts}, \\ &\Delta^i y(0) = 0, \quad 0 \le i \le k-1, \\ &\Delta^{k+i} y(T+1) = 0, \quad 0 \le i \le n-k-1, \end{aligned} \tag{1.4}$$

exists. Extensive discussions concerning Green's functions for difference equations can be found in Hartman [1] and Kelley and Peterson [2]; see also [3]. In particular, if $y(t)$ is a solution of (1.3), (1.2) on $[0, T+n]$, then $y(t)$ solves the equation

$$y(t) = \sum_{s=0}^{T} G_T(t,s) \sum_{i=0}^{k-1} p_i(s)\Delta^i y(s), \quad t \in [0, T+n].$$

As a consequence, if sign conditions on $G_T(t,s)$ are known and certain positivity conditions are placed on the p_i's, then questions concerning the eigenvalue of (1.1), (1.2) and the existence of focal points for (1.3), (1.2) can be examined in terms of a family of linear operators that depend on T.

Many authors have applied the theory of cones in a Banach space and positive operators either to demonstrate the existence of smallest positive eigenvalues, to compare these eigenvalues, or to establish the existence of first conjugate points or first focal points of boundary value problems for linear equations; see, for example, Eloe and Henderson [4, 5], Gentry and Travis [6], Hankerson and Henderson [7], Hankerson and Peterson [8, 9], Keener and Travis [10], Tomastik [11, 12], and Travis [13]. We also mention papers by Eloe [14] and Eloe and Henderson [15] which examine criteria for disfocality of difference equations, two papers by Henderson [16, 17] on focal boundary value problems for nonlinear difference equations, and a paper by Henderson and Lee [18] on continuous dependence and differentiation of solutions of difference equations. Much of our motivation for this study are the works of Keener and Travis [19], Schmitt and Smith [20], and Tomastik [21].

In section 2, we include preliminary notation, and fundamental results from the theory of cones in a Banach space. In section 3, we show, under suitable assumptions on the coefficients p_i, that the smallest positive eigenvalue of (1.1), (1.2) decreases with T. This will lead to results concerning the first focal point of (1.3), (1.2).

2. PRELIMINARIES

In this section, we give definitions and auxiliary results from cone theory. Much of the discussion in this section involving the theory of cones in a Banach space arises from results in Krasnosel'skiĭ's book [22]. Other good references include Kreĭn and Rutman [23], and Deimling [24].

Let B be a Banach space. A closed subset K of B is said to be a *cone* provided: (i) if $u, v \in K$ then $\alpha u + \beta v \in K$ for all $\alpha, \beta \ge 0$, and (ii) if $u, -u \in K$ then $u = 0$. A cone K is said to be *reproducing* provided every $x \in B$ can be written as $x = u - v$ for some $u, v \in K$. If K is a cone and $u, v \in B$, then we write $u \le v$ (with respect to K) provided $v - u \in K$. If L and M are linear operators on B, then $L \le M$ (with respect to K) provided $Ly \le My$ for all $y \in K$. Finally, given a bounded linear operator L on B, we say that L is *positive* if $L(K) \subseteq K$, and we say that L is *u_0-positive* if given any nonzero $u \in K$, there exist $k_1, k_2 > 0$, such that $k_1 u_0 \le Lu \le k_2 u_0$. If $L : B \to B$ is a bounded linear operator, we shall use $r(L)$ to denote the spectral radius of L.

THEOREM 2.1 *Let L be a positive compact linear operator with respect to a reproducing cone K and let $r(L) > 0$. Then $r(L)$ is an eigenvalue of L with corresponding eigenvector in K.*

THEOREM 2.2 *If L and M are compact linear positive operators such that $L \leq M$, then $r(L) \leq r(M)$.*

THEOREM 2.3 *Let L be a compact linear positive operator, and suppose $Lx \geq \mu x$ for some $\mu > 0$ and $x \in B$ with $-x \notin K$ and $x = u - v$ for some $u, v \in K$. Then L has an eigenvector $x_0 \in K$ corresponding to an eigenvalue $\lambda_0 \geq \mu$.*

Theorems 2.5, 2.10, 2.11, and 2.13 of Krasnosel'skiĭ give the following theorem.

THEOREM 2.4 *Let K be a reproducing cone. If L is a compact u_0-positive linear operator then L has an essentially unique eigenvector in K and the corresponding eigenvalue is simple, positive, and larger than the modulus of any other eigenvalue.*

The following theorem appears in Keener and Travis [19, Theorem 2.3] and is a generalization of Travis [13, Theorem 2.3].

THEOREM 2.5 *Let L and M be bounded linear operators and assume that at least one of the operators is u_0-positive. Assume $L \leq M$ and there exist nonzero vectors $u_1, u_2, \in K$ and scalars $\lambda_1, \lambda_2 > 0$ such that $Lu_1 \geq \lambda_1 u_1$ and $Mu_2 \leq \lambda_2 u_2$. Then $\lambda_1 \leq \lambda_2$. If $\lambda_1 = \lambda_2$ then u_1 is a scalar multiple of u_2.*

3. EIGENVALUES AND FOCAL POINTS

Our main objective in this section is to describe how the smallest positive eigenvalue of (1.1), (1.2) changes with T. We will transform these questions about the eigenvalue into questions about the spectral radius of certain operators on a Banach space, and then apply the cone theory.

First, let $(c)^{(i)}$ denote the factorial polynomial defined by $(c)^{(i)} = c(c-1)\cdots(c-i+1)$. If $G_T(t,s)$ is the Green's function for (1.4), then

$$\Delta^j G_T(t,s) = \sum_{\tau=0}^{\min\{t-1,s\}} \frac{(t-\tau-1)^{(k-j-1)}}{(k-j-1)!} \frac{(-\tau+s+n-k-1)^{(n-k-1)}}{(n-k-1)!}$$

for $j < k$, and

$$\Delta^{k+j} G_T(t,s) = \begin{cases} \frac{(-1)^j (s+n-k-1-(t+j))^{(n-k-1-j)}}{(n-k-1-j)!}, & t \leq s \\ 0, & t > s \end{cases}$$

for $0 \leq j \leq n-k-1$. The Green's function can be found with the aid of [25, Lemma 1].

Next, let $B = \{y : [0,\infty) \to \mathbf{R} \mid y \text{ is bounded and } \Delta^i y(0) = 0,\ 0 \leq i < k\}$. Then B is a Banach space under the sup norm. Define $L_T : B \to B$ by

$$L_T y(t) = \begin{cases} \sum_{s=0}^{T} G_T(t,s) \sum_{i=0}^{k-1} p_i(s) \Delta^i y(s), & 0 \leq t \leq T+n, \\ 0, & T+n < t. \end{cases}$$

We assume that $p_i(t) \geq 0$ for $t \geq 0$, $0 \leq i \leq k-1$.

Remark. Note that $L_T y$ depends only on the values of y on $[0, T+k-1]$. Hence, if $T_1 \geq T$ and $B_{T_1} = \{y : [0, T_1+k-1] \to \mathbb{R} \mid \Delta^i y(0) = 0,\ 0 \leq i \leq k-1\}$, then we can regard L_T as a map $B_{T_1} \to B$. We can then define the map $M_T : B_{T_1} \to B_{T_1}$ by $M_T z = L_T z|_{[0,T_1+k-1]}$.

Now, if (λ, z) is an eigenpair for M_T, then we can extend z to a function in B by setting $y(t) = z(t)$ for $0 \leq t \leq T_1 + k - 1$ and $\lambda y(t) = L_T z(t)$ for $t \geq T_1 + k$; then (λ, y) is an eigenpair for L_T. Conversely, if (λ, y) is an eigenpair for L_T with $\lambda \neq 0$, then $z = y|_{[0,T_1+k-1]} \neq 0$ (otherwise $L_T y = 0$ and $\lambda = 0$) and (λ, z) is an eigenpair for M_T.

In addition, for any function $x(t)$ defined on $[0, T+k-1]$, the expression $L_T x$ makes sense and we will allow this slight abuse of notation. Then if $y(t) = L_T x(t)$, it follows that $y(t)$ is a solution of the boundary value problem (1.2),

$$(-1)^{n-k}\Delta^n y(t) = \sum_{i=0}^{k-1} p_i(t)\Delta^i x(t).$$

If (λ, x) is an eigenpair for L_T with $\lambda \neq 0$, then $(1/\lambda, x)$ is an eigenpair for (1.1), (1.2), and conversely. Note also that $\lambda = 0$ is not an eigenvalue of (1.1), (1.2).

To begin with, we wish to examine, under suitable conditions, what happens to the spectral radius of L_T as T increases. Our first result will play a key role in subsequent work.

THEOREM 3.1 *The spectral radius $r(L_T)$ is a nondecreasing function of T. Morever, if $p_{i_0}(T) > 0$ for some $T \geq k - i_0$, then $r(L_{T-1}) < r(L_T)$.*

PROOF. Define the cone K_T in B_T by

$$K_T = \{y \in B_T \mid \Delta^i y(t) \geq 0,\ t \in [0, T+k-1-i],\ 0 \leq i \leq k-1\}.$$

Then $K_T^\circ = \{y \in B_T \mid \Delta^i y(t) > 0,\ t \in [k-i, T+k-1-i],\ 0 \leq i \leq k-1\}$, and K_T is reproducing.

We will begin by showing $r(L_T)$ is nondecreasing; that is, we will show $r(L_{T-1}) \leq r(L_T)$. Let $x \in K_T$ and regard L_{T-1} as an operator on B_T. Then

$$\begin{aligned}
\Delta^k L_{T-1}x(t) &= \sum_{s=t}^{T-1} \frac{(s+n-k-1-t)^{(n-k-1)}}{(n-k-1)!} \sum_{i=0}^{k-1} p_i(s)\Delta^i x(s) \\
&\leq \sum_{s=t}^{T} \frac{(s+n-k-1-t)^{(n-k-1)}}{(n-k-1)!} \sum_{i=0}^{k-1} p_i(s)\Delta^i x(s) \\
&= \Delta^k L_T x(t), \quad 0 \leq t \leq T-1.
\end{aligned}$$

We can repeatedly sum both sides of $\Delta^k L_{T-1}x(t) \leq \Delta^k L_T x(t)$ and use the boundary conditions to show that $\Delta^i L_{T-1}x(t) \leq \Delta^i L_T x(t)$ for $0 \leq t \leq T+k-1-i$, $0 \leq i \leq k-1$. Then $L_{T-1} \leq L_T$ with respect to K_T and by Theorem 2.2, $r(L_{T-1}) \leq r(L_T)$.

Now suppose that $p_{i_0}(T) > 0$ for some $T \geq k - i_0$. We first show that $r(L_T) > 0$. Let $u(t) = t^{(k)} \in K_T^\circ$. Note that

$$\Delta^j L_T u(t) = \sum_{s=0}^{T} \Delta^j G_T(t,s) \sum_{i=0}^{k-1} p_i(s)\Delta^i u(s) \geq \Delta^j G_T(t,T) p_{i_0}(T)\Delta^{i_0} u(T) > 0$$

for $k-j \le t \le T+k-j-1$ and $0 \le j \le k-1$. Thus, $L_T u \in K_T^\circ$. It follows that there exists an $\epsilon > 0$ such that $L_T u \ge \epsilon u$, and by Theorem 2.3 we have $r(L_T) \ge \epsilon > 0$.

Finally, we will show $r(L_{T-1}) < r(L_T)$. Assume $r(L_{T-1}) > 0$. By Theorem 2.1, $r(L_{T-1})$ is an eigenvalue of L_{T-1} with corresponding eigenvector $y \in K_{T-1}$. We can use $r(L_{T-1})y = L_{T-1}y$ to extend y to a function in B_T. We claim that $y \in K_T^\circ$. To see this, since y is an eigenvector corresponding to $r(L_{T-1})$, there exists $s_0 \in [0, T-1]$ such that $\sum_{i=0}^{k-1} p_i(s_0)\Delta^i y(s_0) > 0$. Then

$$\begin{aligned} r(L_{T-1})\Delta^j y(t) &= \sum_{s=0}^{T-1} \Delta^j G_{T-1}(t,s) \sum_{i=0}^{k-1} p_i(s)\Delta^i y(s) \\ &\ge \Delta^j G_{T-1}(t,s_0) \sum_{i=0}^{k-1} p_i(s_0)\Delta^i y(s_0) > 0, \end{aligned}$$

for $t \in [k-j, T+k-1-j]$, $0 \le j \le k-1$. It follows that $y \in K_T^\circ$. Hence $p_{i_0}(T)\Delta^{i_0} y(T) > 0$ and

$$\begin{aligned} \Delta^k L_{T-1} y(t) &= \sum_{s=0}^{T-1} \Delta^k G_{T-1}(t,s) \sum_{i=0}^{k-1} p_i(s)\Delta^i y(s) \\ &< \sum_{s=0}^{T} \Delta^k G_T(t,s) \sum_{i=0}^{k-1} p_i(s)\Delta^i y(s) \\ &= \Delta^k L_T y(t), \quad 0 \le t \le T-1. \end{aligned}$$

Using the boundary conditions at 0, we obtain $L_T y - L_{T-1} y \in K_T^\circ$. Hence, there exists an $\epsilon > 0$ such that $L_T y - L_{T-1} y \ge \epsilon y$, or $L_T y \ge \epsilon y + L_{T-1} y = (\epsilon + r(L_{T-1}))y$. By Theorem 2.3, $r(L_T) > r(L_{T-1})$. □

We obtain a corresponding result for the smallest positive eigenvalue, λ_0, of the eigenvalue problem (1.1), (1.2).

COROLLARY 3.2 *Assume $p_{i_0}(T_0) > 0$ for some i_0, T_0 such that $T_0 \ge k - i_0$. Then the smallest positive eigenvalue, $\lambda_0(T)$, for the eigenvalue problem* (1.1), (1.2) *decreases for $T > T_0$. If, in addition, there exist i_1, T_1 such that $T_1 \ge \min\{k-i_1, T_0+1\}$ and $p_{i_1}(T_1) > 0$, then $\lambda_0(T_1) < \lambda_0(T_0)$.*

PROOF. By Theorem 3.1, $r(L_{T_0}) > 0$ and $r(L_{T_0})$ is an eigenvalue of L_{T_0}. From the correspondence between eigenpairs of the eigenvalue problem and L_{T_0}, we see that $1/r(L_{T_0})$ is the smallest positive eigenvalue of (1.1), (1.2). Since $r(L_T)$ is increasing, it follows that $\lambda_0(T) = 1/r(L_T)$ is decreasing for $T > T_0$. Finally, $r(L_{T_1-1}) < r(L_{T_1})$ by the previous theorem, and hence $\lambda_0(T_0) \ge \lambda_0(T_1 - 1) > \lambda_0(T_1)$. □

We now shall concentrate on the characterization of the first focal point of the boundary value problem and corresponding results.

THEOREM 3.3 *Assume $p_{i_0}(T) > 0$ for some $T \ge k - i_0$. Suppose that the boundary value problem* (1.3), (1.2) *has a nontrivial solution $y \in K_T$ (again, we mean $y|_{[0,T+k-1]} \in K_T$). Then T is the first focal point.*

PROOF. Our first step will be to show $r(L_T) = 1$. Now $L_T y = y$ implies $r(L_T) \ge 1$. Let $u \in K_T$ be an eigenvector corresponding to $r(L_T)$. We have shown in the proof of Theorem 3.1 that $y \in K_T^\circ$. Choose α maximal so that $y \ge \alpha u$. Then $y = L_T y \ge \alpha L_T u = \alpha r(L_T)u$. By the maximality of α, $r(L_T) \le 1$. Hence, $r(L_T) = 1$.

Finally, if the first focal point $\eta < T$, then $r(L_\eta) \geq 1$. But Theorem 3.1 shows that $r(L_\eta) < r(L_T)$, contradicting $r(L_T) = 1$. □

We are also interested in the uniqueness of the function corresponding to the first focal point. If conditions can be placed on the p_i's so that L_T is u_0-positive with respect to K_T, then Krasnosel'skiĭ's Theorem 2.4 can be applied. In this direction, then, suppose that $p_{k-1}(t) > 0$ for $t \in [0,T]$. Since K_T° is nonempty, K_T is reproducing. To show L_T is u_0-positive, it is sufficient to show $L_T(K_T \backslash \{0\}) \subseteq K_T^\circ$.

Let $z \in K_T \backslash \{0\}$. First $\Delta^{k-1} z(t) > 0$ for some $t \in [0,T]$; otherwise, z would be the trivial solution. Then for some $s_0 \in [0,T]$,

$$\Delta^j L_T z(t) = \sum_{s=0}^{T} \Delta^j G_T(t,s) \sum_{i=0}^{k-1} p_i(s) \Delta^i z(s) \geq \Delta^j G_T(t,s_0) \sum_{i=0}^{k-1} p_i(s_0) \Delta^i z(s_0) > 0,$$

for $t \in [k-j, T+k-1-j]$, $0 \leq j \leq k-1$. Therefore, $L_T z \in K_T^\circ$ for all $z \in K_T \backslash \{0\}$, and L_T is u_0-positive.

Now suppose, in addition, that (1.3), (1.2) has a nontrivial solution $y \in K_T$. Note that by Theorem 3.3, T is the first focal point. An application of Theorem 2.4 shows that y is unique up to scalar multiple.

Finally, the requirement that p_{k-1} be strictly positive can be replaced by other similar conditions. For example, if p_{k-1} is identically zero, we could require $p_{k-2} > 0$ and change our interval to $[0, T+k-2]$ in the definition of B_T.

However, under weaker conditions we can be certain that nontrivial solutions in K_T are actually in K_T°. This condition is the key to show uniqueness.

THEOREM 3.4 *Let $p_{i_0}(T) > 0$ for some $T \geq k - i_0$ and let $y \in K_T$ be a nontrivial solution of* (1.3), (1.2). *Then y is unique up to scalar multiple.*

PROOF. Since $y \in K_T$ is a solution to the boundary value problem (1.3), (1.2), then $y = L_T y$. Let z also be a solution of the boundary value problem (1.3), (1.2) and assume $-z \notin K_T$. Choose α maximal so that $y \geq \alpha z$. We know that $\alpha > 0$ since $y \in K_T^\circ$ by earlier arguments. Suppose $y - \alpha z \neq 0$. From previous work we know that $L_T y - \alpha L_T z \in K_T^\circ$. Then there is an $\epsilon > 0$ so that $L_T y - \alpha L_T z \geq \epsilon z$. Hence $y - \alpha z \geq \epsilon z$ implies $y \geq (\alpha + \epsilon) z$, which contradicts the maximality of α. □

The next two theorems will examine conditions which guarantee that the first focal point of equation (1.3) is greater than or equal to T.

THEOREM 3.5 *Assume that $p_{i_0}(T) > 0$ for some $T \geq k - i_0$. If the smallest positive eigenvalue $\lambda_0(T)$ of* (1.1), (1.2) *satisfies $\lambda_0(T) \geq 1$, then the first focal point $\eta \geq T$.*

PROOF. Suppose, on the contrary, that $\eta < T$. By Corollary 3.2, $\lambda_0(\eta) > \lambda_0(T)$. Then $\lambda_0(\eta) > 1$ since $\lambda_0(T) \geq 1$. This says $r(L_\eta) = 1/\lambda_0(\eta) < 1$. But for z, a solution of the boundary value problem, $L_\eta z = z$ implies $r(L_\eta) \geq 1$, which is a contradiction. Therefore, $\eta \geq T$. □

THEOREM 3.6 *Let v be a nontrivial function in K_T such that*

$$\sum_{s=0}^{T} G_T(t,s) \sum_{i=0}^{k-1} p_i(s) \Delta^i v(s) \leq v(t)$$

with respect to K_T, where $p_{k-1}(t) > 0$ for all $t \in [0,T]$. Then the first focal point $\eta \geq T$.

PROOF. Under the given hypothesis, we get that $L_T v \leq 1v$ with respect to K_T. Let $\lambda_0(T)$ be the smallest positive eigenvalue of the eigenvalue problem (1.1), (1.2), and let x be a corresponding eigenfunction in K_T. Then $r(L_T)x = L_T x = (1/\lambda_0(T))x$. By remarks preceding Theorem 3.4, L_T is u_0-positive, and it follows by Theorem 2.5 that $1/\lambda_0(T) \leq 1$. Therefore, by Theorem 3.5, the first focal point $\eta \geq T$. □

As in the discussion preceding Theorem 3.4, the result still holds if we require that the last nontrivial coefficient function be strictly positive, and then modify the Banach space accordingly. These conditions guaranteed that L_T was u_0-positive and allowed the application of Theorem 2.5. However, we can relax the requirement of u_0-positivity, provided we add the condition $L_T v \neq 0$. To see this, consider the situation in Theorem 2.5 where $L \leq M$ and there are nonzero vectors $u_1, u_2 \in K$ and scalars $\lambda_1, \lambda_2 > 0$ such that $Lu_1 \geq \lambda_1 u_1$ and $Mu_2 \leq \lambda_2 u_2$. In the proof of Theorem 2.5, u_0-positivity of L was used to show $\epsilon_0 = \sup\{\epsilon \mid L(u_2 - \epsilon u_1) \geq 0\} > 0$. In this case,

$$0 \leq L(u_2 - \epsilon_0 u_1) = Lu_2 - \epsilon_0 Lu_1 \leq Mu_2 - \epsilon_0 \lambda_1 u_1 \leq \lambda_2 u_2 - \epsilon_0 \lambda_1 u_1 = \lambda_2(u_2 - \epsilon_0 \frac{\lambda_1}{\lambda_2} u_1).$$

Hence $L(u_2 - \epsilon_0 \frac{\lambda_1}{\lambda_2} u_1) \in K_T$ implies $\lambda_1 \leq \lambda_2$ by maximality of ϵ_0.

Now suppose that $p_{i_0}(T) > 0$ for some $T \geq k - i_0$, and that there is $v \in K_T$ such that $L_T v$ is nontrivial and $L_T v \leq v$. Using techniques similar to those in the proof of Theorem 3.1, it can be shown that $L_T v \in K_T^o$. If $x \in K_T$ is an eigenfunction for (1.1), (1.2) corresponding to $\lambda_0(T)$, then $\epsilon_0 = \sup\{\epsilon \mid L_T(v - \epsilon x) \geq 0\} > 0$. Hence, $1 \leq \lambda_0(T)$, and the first focal point $\eta \geq T$ from the same arguments used at the end of the proof of Theorem 3.6.

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Guest Editors

Lean Yu, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, China; Department of Management Sciences, City University of Hong Kong, Tat Chee Avenue, Kowloon, Hong Kong; yulean@amss.ac.cn

Shouyang Wang, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, China; sywang@amss.ac.cn

K. K. Lai, Department of Management Sciences, City University of Hong Kong, Tat Chee Avenue, Kowloon, Hong Kong; mskklai@cityu.edu.hk