

A PROOF OF COMPLETENESS OF THE GREEN-LAMÉ TYPE SOLUTION IN THERMOELASTICITY

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(Received December 26, 1991)

ABSTRACT. A proof of completeness of the Green-Lamé type solution for the unified governing field equations of conventional and generalized thermoelasticity theories is given.

KEY WORDS AND PHRASES. Thermoelasticity, Generalized Thermoelasticity, Green-Lamé solution, completeness of solution.

1980 AMS SUBJECT CLASSIFICATION CODE. 73U.

1. INTRODUCTION

In [1], the author presented three complete solutions for the following system of coupled partial differential equations which may be interpreted as a unified system of governing field equations of the conventional and generalized models of the linear thermoelasticity theory of homogeneous and isotropic materials:

$$\left\{ \begin{array}{l} \left(c^2 \nabla^2 - \frac{\partial^2}{\partial t^2} \right) \underline{u} + (1 - c^2) \nabla \operatorname{div} \underline{u} - \left(1 + \alpha \frac{\partial}{\partial t} \right) \nabla \theta + \underline{F} = \underline{0} \\ \left(\nabla^2 - \frac{\partial}{\partial t} - \beta \frac{\partial^2}{\partial t^2} \right) \theta - \left(1 + \gamma \frac{\partial}{\partial t} \right) \left[\epsilon \frac{\partial}{\partial t} (\operatorname{div} \underline{u}) - h \right] = 0 \end{array} \right. \quad (1.1 \text{ a,b})$$

The notation employed in these equations and those to follow are as explained in [1].

One of the three solutions of the system (1.1) presented in [1] is analogous to the Green-Lamé solution in classical elastodynamics [2]; this solution is described by the following relations:

$$\underline{u} = \left(1 + \alpha \frac{\partial}{\partial t} \right) (\nabla \phi + \operatorname{curl} \underline{\psi}) \quad (1.2)$$

$$\theta = D_1 \phi - f \quad (1.3)$$

$$D_5 \phi = D_3 f - \left(1 + \gamma \frac{\partial}{\partial t} \right) h \quad (1.4)$$

$$D_2 \underline{\psi} = \underline{g} \quad (1.5)$$

$$\underline{F} = - \left(1 + \alpha \frac{\partial}{\partial t} \right) (\nabla f + \operatorname{curl} \underline{g}) \quad (1.6)$$

That is, if the known function \underline{F} is represented by the relation (1.6) (by virtue of the Helmholtz resolution

of a vector field), then a solution $\{\underline{u}, \theta\}$ for the system (1.1) is given by the representations (1.2) and (1.3) where ϕ and $\underline{\psi}$ are arbitrary scalar and vector functions (respectively) obeying the partial differential equations (1.4) and (1.5). Here D_1 , D_2 , D_3 and D_5 are partial differential operators defined by [1]:

$$D_1 = \nabla^2 - \frac{\partial^2}{\partial t^2} \quad (1.7)$$

$$D_2 = c^2 \nabla^2 - \frac{\partial^2}{\partial t^2} = D_1 - (1 - c^2) \nabla^2 \quad (1.8)$$

$$D_3 = \nabla^2 - \frac{\partial}{\partial t} - \beta \frac{\partial^2}{\partial t^2} \quad (1.9)$$

$$D_5 = D_3 D_1 - \epsilon \nabla^2 \left(1 + \alpha \frac{\partial}{\partial t} \right) \left(1 + \gamma \frac{\partial}{\partial t} \right) \quad (1.10)$$

It was also shown in [1] that the solution described above is complete in the sense that if the known function \underline{F} is represented as in (1.6), then every solution $\{\underline{u}, \theta\}$ of the system (1.1) admits a representation given by the relations (1.2) and (1.3) with ϕ and $\underline{\psi}$ obeying the equations (1.4) and (1.5).

The proof of completeness suggested in [1] was an extension of the proof given in [2] in the context of classical elastodynamics. This proof makes the hypothesis that in the representation (1.6) for \underline{F} the function \underline{g} is divergence-free (that is, $\text{div } \underline{g} = 0$) and infers that $\underline{\psi}$ also has to be divergence-free.

The object of the present Note is to give a proof of the completeness of the Green-Lamé type solution that does not make the hypothesis that $\text{div } \underline{g} = 0$ and consequently does not infer that $\text{div } \underline{\psi} = 0$. This proof is motivated by the work of Long [3] in classical elastodynamics and is analogous to that given in [4] in the context of the theory of elastic materials with voids.

2. PROOF OF COMPLETENESS

Consider any solution $\{\underline{u}, \theta\}$ of the system (1.1). By virtue of the Helmholtz representation of a vector field, \underline{u} may be expressed as

$$\underline{u} = \left(1 + \alpha \frac{\partial}{\partial t} \right) (\nabla p + \text{curl } \underline{q}) \quad (2.1)$$

for some scalar field p and a vector field \underline{q} .

Substituting for \underline{u} from (2.1) into equation (1.1a), we get the equation

$$\left(1 + \alpha \frac{\partial}{\partial t} \right) [\nabla \{D_1 p - (\theta + f)\} + \text{curl} \{D_2 \underline{q} - \underline{g}\}] \quad (2.2)$$

Here, we have made use of the representation (1.6) for \underline{F} and the relations (1.7) and (1.8).

For $\alpha = 0$, equation (2.2) gives

$$\nabla \{D_1 p - (\theta + f)\} = \text{curl} \{\underline{g} - D_2 \underline{q}\} \quad (2.3)$$

For $\alpha \neq 0$, equation (2.2) yields equation (2.3) provided

$$[\nabla \{D_1 p - (\theta + f)\} + \text{curl} \{D_2 \underline{q} - \underline{g}\}]_{t=0} = 0.$$

This condition may be taken to be valid when \underline{u} and θ obey homogeneous initial conditions.

Taking the divergence of both sides of (2.3) and noting that $\text{div } \nabla = \nabla^2$ and $\text{div } \text{curl}$ is the zero

operator, we get the equation

$$\nabla^2 \{D_1 p - (\theta + f)\} = 0. \quad (2.4)$$

This equation implies that [4, Appendix]

$$p = \phi + \phi_0 \quad (2.5)$$

where

$$D_1 \phi = \theta + f \quad (2.6)$$

$$\nabla^2 \phi_0 = 0. \quad (2.7)$$

Taking the *curl curl* of both sides of (2.3) and noting that *curl* ∇ is the zero operator and *curl curl* = $\nabla \text{div} - \nabla^2$, we obtain the equation

$$\nabla^2 \text{curl}(D_2 q - g) = 0. \quad (2.8)$$

This equation implies that [4, Appendix]

$$q = \psi_0 + \psi_1 \quad (2.9)$$

where

$$\nabla^2(\text{curl } \psi_0) = 0 \quad (2.10)$$

$$D_2 \psi_1 = g \quad (2.11)$$

Substituting for p and q from (2.5) and (2.9) into the expression (2.3) and using (2.6) and (2.11) we obtain the relation

$$\nabla(D_1 \phi_0) + \text{curl}(D_2 \psi_0) = 0.$$

Using the relations (1.7), (1.8), (2.7) and (2.10), this yields

$$\frac{\partial^2}{\partial t^2} \{\nabla \phi_0 + \text{curl } \psi_0\} = 0$$

from which it follows that

$$\nabla \phi_0 + \text{curl } \psi_0 = t \psi_2 + \psi_3 \quad (2.12)$$

where ψ_2 and ψ_3 are independent of t .

Taking the divergence of (2.12) and using (2.7), we get

$$t \text{div } \psi_2 + \text{div } \psi_3 = 0.$$

Since this holds for any t , we should have $\text{div } \psi_2 = 0$ and $\text{div } \psi_3 = 0$ from which it follows that

$$\psi_2 = \text{curl } \xi_2, \quad \psi_3 = \text{curl } \xi_3 \quad (2.13)$$

for some ξ_2, ξ_3 .

Taking the Laplacian of (2.12) and using (2.7) and (2.10), we get

$$t \nabla^2 \psi_2 + \nabla^2 \psi_3 = 0$$

which on using (2.13) yields

$$t \text{curl}(\nabla^2 \xi_2) + \text{curl}(\nabla^2 \xi_3) = 0.$$

Since this holds for any t , we should have $\text{curl}(\nabla^2 \xi_2) = 0$ and $\text{curl}(\nabla^2 \xi_3) = 0$ from which it follows that

$$\nabla^2 \xi_2 = \nabla \phi_2, \quad \nabla^2 \xi_3 = \nabla \phi_3 \quad (2.14)$$

for some ϕ_2 and ϕ_3 .

We now define the function $\psi = \psi(P, t)$ by

$$\psi = \psi_1 + (t \xi_2 + \xi_3) + \frac{1}{4\pi} \nabla \int_D \frac{\Phi(Q, t - R/c)}{R} dV \quad (2.15)$$

where

$$\Phi = t \phi_2 + \phi_3 \quad (2.16)$$

and R is the distance from the field point P to a point Q , the integration (over D) being w.r.t. Q .

From (2.15) we get

$$\text{curl } \psi = \text{curl}(\psi_1 + t \xi_2 + \xi_3). \quad (2.17)$$

Substituting for p and q from (2.5) and (2.9) in the right-hand side of (2.1) and using (2.12), (2.13) and (2.17), we obtain

$$\underline{u} = \left(1 + \alpha \frac{\partial}{\partial t}\right) (\nabla \phi + \text{curl } \psi).$$

This is the desired representation (1.2) for \underline{u} . The desired representation (1.3) for θ is given by (2.6).

Substituting for \underline{u} and θ from (1.2) and (1.3) into equation (1.1b) and using (1.9) and (1.10), we obtain the equation

$$D_5 \phi - D_3 f + \left(1 + \gamma \frac{\partial}{\partial t}\right) h = 0.$$

This is precisely the desired governing equation (1.4) for ϕ .

With the aid of the identity [1]

$$\left(\nabla^2 - \frac{1}{c^2} \frac{\partial}{\partial t^2}\right) \int_D \frac{\Phi(Q, t - R/c)}{R} dV = -4\pi \Phi$$

and the relations (1.8), (2.14) and (2.16), expression (2.15) yields $D_2 \psi = D_2 \psi_1$. Using the relation (2.11), we now find that ψ obeys the equation $D_2 \psi = g$, which is the desired governing equation (1.5) for ψ .

Thus, we have shown that, given any solution $\{\underline{u}, \theta\}$ for the system (1.1), one can construct functions ϕ and ψ such that \underline{u} and θ can be represented by the relations (1.2) and (1.3) with ϕ and ψ obeying the equations (1.4) and (1.5).

This completes the proof of completeness of the Green-Lamé type solution for the system (1.1). Note that no where in the proof it has been assumed that $\text{div } g = 0$ and inferred that $\text{div } \psi$ has to be zero.

ACKNOWLEDGEMENT. This work is supported by the U. S. Government Fulbright Grant #15068 under the Indo-American Fellowship Program. The author is thankful to Bangalore University, Bangalore and the University Grants Commission, New Delhi for nominating him for the Fellowship and the Indo-U.S. subcommission on education and culture for awarding the Fellowship. His thanks are also due to Professor Lokenath Debnath for the facilities.

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