

**A PROOF OF COMPLETENESS OF THE GREEN-LAMÉ
TYPE SOLUTION IN THERMOELASTICITY**

D.S. CHANDRASEKHARAIAH

Department of Mathematics
University of Central Florida
Orlando, Florida 32816

(Received December 26, 1991)

ABSTRACT. A proof of completeness of the Green-Lamé type solution for the unified governing field equations of conventional and generalized thermoelasticity theories is given.

KEY WORDS AND PHRASES. Thermoelasticity, Generalized Thermoelasticity, Green-Lamé solution, completeness of solution.

1980 AMS SUBJECT CLASSIFICATION CODE. 73U.

1. INTRODUCTION

In [1], the author presented three complete solutions for the following system of coupled partial differential equations which may be interpreted as a unified system of governing field equations of the conventional and generalized models of the linear thermoelasticity theory of homogeneous and isotropic materials:

$$\left. \begin{aligned} & \left(c^2 \nabla^2 - \frac{\partial^2}{\partial t^2} \right) \underline{u} + (1 - c^2) \nabla \operatorname{div} \underline{u} - \left(1 + \alpha \frac{\partial}{\partial t} \right) \nabla \theta + \underline{F} = \underline{0} \\ & \left(\nabla^2 - \frac{\partial}{\partial t} - \beta \frac{\partial^2}{\partial t^2} \right) \theta - \left(1 + \gamma \frac{\partial}{\partial t} \right) \left[\epsilon \frac{\partial}{\partial t} (\operatorname{div} \underline{u}) - h \right] = 0 \end{aligned} \right\} \quad (1.1 \text{ a,b})$$

The notation employed in these equations and those to follow are as explained in [1].

One of the three solutions of the system (1.1) presented in [1] is analogous to the Green-Lamé solution in classical elastodynamics [2]; this solution is described by the following relations:

$$\underline{u} = \left(1 + \alpha \frac{\partial}{\partial t} \right) (\nabla \phi + \operatorname{curl} \underline{\psi}) \quad (1.2)$$

$$\theta = D_1 \phi - f \quad (1.3)$$

$$D_5 \phi = D_3 f - \left(1 + \gamma \frac{\partial}{\partial t} \right) h \quad (1.4)$$

$$D_2 \underline{\psi} = \underline{g} \quad (1.5)$$

$$\underline{F} = - \left(1 + \alpha \frac{\partial}{\partial t} \right) (\nabla f + \operatorname{curl} \underline{g}) \quad (1.6)$$

That is, if the known function \underline{F} is represented by the relation (1.6) (by virtue of the Helmholtz resolution

of a vector field), then a solution $\{\underline{u}, \theta\}$ for the system (1.1) is given by the representations (1.2) and (1.3) where ϕ and $\underline{\psi}$ are arbitrary scalar and vector functions (respectively) obeying the partial differential equations (1.4) and (1.5). Here D_1 , D_2 , D_3 and D_5 are partial differential operators defined by [1]:

$$D_1 = \nabla^2 - \frac{\partial^2}{\partial t^2} \quad (1.7)$$

$$D_2 = c^2 \nabla^2 - \frac{\partial^2}{\partial t^2} = D_1 - (1 - c^2) \nabla^2 \quad (1.8)$$

$$D_3 = \nabla^2 - \frac{\partial}{\partial t} - \beta \frac{\partial^2}{\partial t^2} \quad (1.9)$$

$$D_5 = D_3 D_1 - \varepsilon \nabla^2 \frac{\partial}{\partial t} \left(1 + \alpha \frac{\partial}{\partial t} \right) \left(1 + \gamma \frac{\partial}{\partial t} \right) \quad (1.10)$$

It was also shown in [1] that the solution described above is complete in the sense that if the known function F is represented as in (1.6), then every solution $\{\underline{u}, \theta\}$ of the system (1.1) admits a representation given by the relations (1.2) and (1.3) with ϕ and $\underline{\psi}$ obeying the equations (1.4) and (1.5).

The proof of completeness suggested in [1] was an extension of the proof given in [2] in the context of classical elastodynamics. This proof makes the hypothesis that in the representation (1.6) for F the function g is divergence-free (that is, $\operatorname{div} g = 0$) and infers that $\underline{\psi}$ also has to be divergence-free.

The object of the present Note is to give a proof of the completeness of the Green-Lamé type solution that does not make the hypothesis that $\operatorname{div} g = 0$ and consequently does not infer that $\operatorname{div} \underline{\psi} = 0$.

This proof is motivated by the work of Long [3] in classical elastodynamics and is analogous to that given in [4] in the context of the theory of elastic materials with voids.

2. PROOF OF COMPLETENESS

Consider any solution $\{\underline{u}, \theta\}$ of the system (1.1). By virtue of the Helmholtz representation of a vector field, \underline{u} may be expressed as

$$\underline{u} = \left(1 + \alpha \frac{\partial}{\partial t} \right) (\nabla p + \operatorname{curl} \underline{q}) \quad (2.1)$$

for some scalar field p and a vector field \underline{q} .

Substituting for \underline{u} from (2.1) into equation (1.1a), we get the equation

$$\left(1 + \alpha \frac{\partial}{\partial t} \right) \nabla \{ D_1 p - (\theta + f) \} + \operatorname{curl} \{ D_2 \underline{q} - \underline{g} \} \quad (2.2)$$

Here, we have made use of the representation (1.6) for F and the relations (1.7) and (1.8).

For $\alpha = 0$, equation (2.2) gives

$$\nabla \{ D_1 p - (\theta + f) \} = \operatorname{curl} \{ \underline{g} - D_2 \underline{q} \} \quad (2.3)$$

For $\alpha \neq 0$, equation (2.2) yields equation (2.3) provided

$$[\nabla \{ D_1 p - (\theta + f) \} + \operatorname{curl} \{ D_2 \underline{q} - \underline{g} \}]_{t=0} = 0.$$

This condition may be taken to be valid when \underline{u} and θ obey homogeneous initial conditions.

Taking the divergence of both sides of (2.3) and noting that $\operatorname{div} \nabla = \nabla^2$ and $\operatorname{div} \operatorname{curl}$ is the zero

operator, we get the equation

$$\nabla^2 \{D_1 p - (\theta + f)\} = 0. \quad (2.4)$$

This equation implies that [4, Appendix]

$$p = \phi + \phi_0 \quad (2.5)$$

where

$$D_1 \phi = \theta + f \quad (2.6)$$

$$\nabla^2 \phi_0 = 0. \quad (2.7)$$

Taking the *curl* *curl* of both sides of (2.3) and noting that *curl* ∇ is the zero operator and *curl* *curl* = $\nabla \operatorname{div} - \nabla^2$, we obtain the equation

$$\nabla^2 \operatorname{curl} (D_2 \underline{q} - \underline{g}) = \underline{0}. \quad (2.8)$$

This equation implies that [4, Appendix]

$$\underline{q} = \underline{\psi}_0 + \underline{\psi}_1 \quad (2.9)$$

where

$$\nabla^2 (\operatorname{curl} \underline{\psi}_0) = \underline{0} \quad (2.10)$$

$$D_2 \underline{\psi}_1 = \underline{g} \quad (2.11)$$

Substituting for p and \underline{q} from (2.5) and (2.9) into the expression (2.3) and using (2.6) and (2.11) we obtain the relation

$$\nabla (D_1 \phi_0) + \operatorname{curl} (D_2 \underline{\psi}_0) = \underline{0}.$$

Using the relations (1.7), (1.8), (2.7) and (2.10), this yields

$$\frac{\partial^2}{\partial t^2} \{ \nabla \phi_0 + \operatorname{curl} \underline{\psi}_0 \} = \underline{0}$$

from which it follows that

$$\nabla \phi_0 + \operatorname{curl} \underline{\psi}_0 = t \underline{\psi}_2 + \underline{\psi}_3 \quad (2.12)$$

where $\underline{\psi}_2$ and $\underline{\psi}_3$ are independent of t .

Taking the divergence of (2.12) and using (2.7), we get

$$t \operatorname{div} \underline{\psi}_2 + \operatorname{div} \underline{\psi}_3 = 0.$$

Since this holds for any t , we should have $\operatorname{div} \underline{\psi}_2 = 0$ and $\operatorname{div} \underline{\psi}_3 = 0$ from which it follows that

$$\underline{\psi}_2 = \operatorname{curl} \underline{\xi}_2, \quad \underline{\psi}_3 = \operatorname{curl} \underline{\xi}_3 \quad (2.13)$$

for some $\underline{\xi}_2, \underline{\xi}_3$.

Taking the Laplacian of (2.12) and using (2.7) and (2.10), we get

$$t \nabla^2 \underline{\psi}_2 + \nabla^2 \underline{\psi}_3 = \underline{0}$$

which on using (2.13) yields

$$t \operatorname{curl} (\nabla^2 \underline{\xi}_2) + \operatorname{curl} (\nabla^2 \underline{\xi}_3) = \underline{0}.$$

Since this holds for any t , we should have $\text{curl}(\nabla^2 \xi_2) = 0$ and $\text{curl}(\nabla^2 \xi_3) = 0$ from which it follows that

$$\nabla^2 \xi_2 = \nabla \phi_2, \quad \nabla^2 \xi_3 = \nabla \phi_3 \quad (2.14)$$

for some ϕ_2 and ϕ_3 .

We now define the function $\psi = \psi(P, t)$ by

$$\psi = \psi_1 + (t \xi_2 + \xi_3) + \frac{1}{4\pi} \nabla \int_D \frac{\Phi(Q, t - R/c)}{R} dV \quad (2.15)$$

where

$$\Phi = t \phi_2 + \phi_3 \quad (2.16)$$

and R is the distance from the field point P to a point Q , the integration (over D) being w.r.t. Q .

From (2.15) we get

$$\text{curl } \psi = \text{curl}(\psi_1 + t \xi_2 + \xi_3). \quad (2.17)$$

Substituting for p and q from (2.5) and (2.9) in the right-hand side of (2.1) and using (2.12), (2.13) and (2.17), we obtain

$$\underline{u} = \left(1 + \alpha \frac{\partial}{\partial t}\right) (\nabla \phi + \text{curl } \psi).$$

This is the desired representation (1.2) for \underline{u} . The desired representation (1.3) for θ is given by (2.6).

Substituting for \underline{u} and θ from (1.2) and (1.3) into equation (1.1b) and using (1.9) and (1.10), we obtain the equation

$$D_5 \phi - D_3 f + \left(1 + \gamma \frac{\partial}{\partial t}\right) h = 0.$$

This is precisely the desired governing equation (1.4) for ϕ .

With the aid of the identity [1]

$$\left(\nabla^2 - \frac{1}{c^2} \frac{\partial}{\partial t^2}\right) \int_D \frac{\Phi(Q, t - R/c)}{R} dV = -4\pi \Phi$$

and the relations (1.8), (2.14) and (2.16), expression (2.15) yields $D_2 \psi = D_2 \psi_1$. Using the relation (2.11), we now find that ψ obeys the equation $D_2 \psi = g$, which is the desired governing equation (1.5) for ψ .

Thus, we have shown that, given any solution $\{\underline{u}, \theta\}$ for the system (1.1), one can construct functions ϕ and ψ such that \underline{u} and θ can be represented by the relations (1.2) and (1.3) with ϕ and ψ obeying the equations (1.4) and (1.5).

This completes the proof of completeness of the Green-Lamé type solution for the system (1.1). Note that no where in the proof it has been assumed that $\text{div } g = 0$ and inferred that $\text{div } \psi$ has to be zero.

ACKNOWLEDGEMENT. This work is supported by the U. S. Government Fulbright Grant #15068 under the Indo-American Fellowship Program. The author is thankful to Bangalore University, Bangalore and the University Grants Commission, New Delhi for nominating him for the Fellowship and the Indo-U.S. subcommission on education and culture for awarding the Fellowship. His thanks are also due to Professor Lokenath Debnath for the facilities.

REFERENCES

1. CHANDRASEKHARAIAH, D. S.: Complete solutions of a coupled system of partial differential equations arising in Thermoelasticity, Quart. Appl. Math. XLV (1987) 471-480.
2. GURTIN, M. E.: The linear theory of elasticity, Encyclopedia of Physics, Vol. VI a/2, Springer-Verlag, New York (1972), p. 234.
3. LONG, C. F.: On the completeness of the Lamé potentials, Acta Mechanica 3 (1967) 371-375.
4. CHANDRASEKHARAIAH, D. S.: Complete solutions in the theory of elastic materials with voids, Quart. J. Mech. Appl. Math. 37 (1987) 401-416.

Permanent address of the author:

Department of Mathematics, Bangalore University, Central College Campus, Bangalore 560 001, India.

Special Issue on Modeling Experimental Nonlinear Dynamics and Chaotic Scenarios

Call for Papers

Thinking about nonlinearity in engineering areas, up to the 70s, was focused on intentionally built nonlinear parts in order to improve the operational characteristics of a device or system. Keying, saturation, hysteretic phenomena, and dead zones were added to existing devices increasing their behavior diversity and precision. In this context, an intrinsic nonlinearity was treated just as a linear approximation, around equilibrium points.

Inspired on the rediscovering of the richness of nonlinear and chaotic phenomena, engineers started using analytical tools from "Qualitative Theory of Differential Equations," allowing more precise analysis and synthesis, in order to produce new vital products and services. Bifurcation theory, dynamical systems and chaos started to be part of the mandatory set of tools for design engineers.

This proposed special edition of the *Mathematical Problems in Engineering* aims to provide a picture of the importance of the bifurcation theory, relating it with nonlinear and chaotic dynamics for natural and engineered systems. Ideas of how this dynamics can be captured through precisely tailored real and numerical experiments and understanding by the combination of specific tools that associate dynamical system theory and geometric tools in a very clever, sophisticated, and at the same time simple and unique analytical environment are the subject of this issue, allowing new methods to design high-precision devices and equipment.

Authors should follow the Mathematical Problems in Engineering manuscript format described at <http://www.hindawi.com/journals/mpe/>. Prospective authors should submit an electronic copy of their complete manuscript through the journal Manuscript Tracking System at <http://mts.hindawi.com/> according to the following timetable:

Manuscript Due	December 1, 2008
First Round of Reviews	March 1, 2009
Publication Date	June 1, 2009

Guest Editors

José Roberto Castilho Piqueira, Telecommunication and Control Engineering Department, Polytechnic School, The University of São Paulo, 05508-970 São Paulo, Brazil; piqueira@lac.usp.br

Elbert E. Neher Macau, Laboratório Associado de Matemática Aplicada e Computação (LAC), Instituto Nacional de Pesquisas Espaciais (INPE), São José dos Campos, 12227-010 São Paulo, Brazil ; elbert@lac.inpe.br

Celso Grebogi, Center for Applied Dynamics Research, King's College, University of Aberdeen, Aberdeen AB24 3UE, UK; grebogi@abdn.ac.uk