

A NOTE ON A FUNCTIONAL INEQUALITY

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ABSTRACT. We prove: If r_1, \dots, r_k are (fixed) positive real numbers with $\prod_{j=1}^k r_j > 1$, then the only entire solutions $\varphi: \mathbb{C} \rightarrow \mathbb{C}$ of the functional inequality

$$\prod_{j=1}^k |\varphi(r_j z)| \geq \left(\prod_{j=1}^k r_j\right) |\varphi(z)|^k$$

are $\varphi(z) = cz^n$, where c is a complex number and n is a positive integer.

KEY WORDS AND PHRASES. Functional inequality, entire functions.

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1. INTRODUCTION.

Inspired by a problem of H. Haruki, who asked for all entire solutions of

$$|\varphi(z+w)|^2 + |\varphi(z-w)|^2 + 2|\varphi(0)|^2 \geq 2|\varphi(z)|^2 + 2|\varphi(w)|^2, \quad (1.1)$$

J. Walorski [1] proved in 1987 the following interesting proposition:

Let $r > 1$ be a (fixed) real number. Then the only entire solutions $\varphi: \mathbb{C} \rightarrow \mathbb{C}$ of the functional inequality

$$|\varphi(rz)| \geq r|\varphi(z)|$$

are

$$\varphi(z) = cz^n, \quad (1.2)$$

where $c \in \mathbb{C}$ and $n \in \mathbb{N}$.

As an application of this theorem, Walorski showed that the only entire functions $\varphi: \mathbb{C} \rightarrow \mathbb{C}$ satisfying (1.1) and $\varphi(0) = 0$ are the monomials (1.2). The aim of this note is to prove an extension of Walorski's result by using a method which is (slightly) different from the two approaches presented in [1].

2. MAIN RESULTS.

Theorem. Let r_1, \dots, r_k be (fixed) positive real numbers with $\prod_{j=1}^k r_j > 1$. Then the only entire

solutions $\varphi: \mathbb{C} \rightarrow \mathbb{C}$ of

$$\prod_{j=1}^k |\varphi(r_j z)| \geq \left(\prod_{j=1}^k r_j\right) |\varphi(z)|^k \quad (2.1)$$

are the functions $\varphi(z) + cz^n$, where c is a complex number and n is a positive integer.

PROOF. Simple calculations reveal that the functions $\varphi(z) = cz^n$ ($c \in \mathbb{C}$, $n \in \mathbb{N}$) satisfy (2.1). Next we assume that φ is an entire solution of inequality (2.1).

Because of $\prod_{j=1}^k r_j > 1$ we conclude from (2.1) with $z = 0$ that φ has at 0 a zero. Let n be the order of this zero; we define

$$f(z) = \varphi(z)/z^n, \quad (2.2)$$

then f is an entire function with $f(0) \neq 0$. From (2.1) we obtain

$$\prod_{j=1}^k |f(r_j z)| \geq \left(\prod_{j=1}^k r_j^{1-n}\right) |f(z)|^k. \quad (2.3)$$

We suppose that f has a zero z_0 . By induction it follows from (2.3) that z_0/r_1^m is a root of f for all non-negative integers m . From the identity theorem we conclude $f(z) \equiv 0$ which contradicts the condition $f(0) \neq 0$. Hence f has no zero which implies that the function

$$g(z) = \frac{f(z)^k}{\prod_{j=1}^k f(r_j z)} \quad (2.4)$$

is entire. From (2.3) we conclude

$$|g(z)| \leq \prod_{j=1}^k r_j^{n-1} \quad \text{for all } z \in \mathbb{C},$$

and Liouville's theorem implies that g is a constant. Therefore we have

$$f(z)^k = K \prod_{j=1}^k f(r_j z), \quad K \in \mathbb{C}. \quad (2.5)$$

Since $f(0) \neq 0$ we get from (2.5): $K = 1$;

hence

$$f(z)^k = \prod_{j=1}^k f(r_j z). \quad (2.6)$$

Differentiation leads to

$$k \frac{f'(z)}{f(z)} = \sum_{j=1}^k r_j \frac{f'(r_j z)}{f(r_j z)}. \quad (2.7)$$

Setting

$$\frac{f'(z)}{f(z)} = \sum_{m=0}^{\infty} a_m z^m \quad (2.8)$$

we obtain from (2.7) and (2.8):

$$\sum_{m=0}^{\infty} k a_m z^m = \sum_{m=0}^{\infty} \left(a_m \sum_{j=1}^k r_j^{m+1}\right) z^m, \quad (2.9)$$

and comparing the coefficients of z^m yields for all $m \geq 0$:

$$ka_m = a_m \sum_{j=1}^k r_j^{m+1}. \quad (2.10)$$

We assume that there exists an integer $m_0 \geq 0$ such that $a_{m_0} \neq 0$, then we get from the arithmetic mean-geometric mean inequality and from (2.10):

$$\left[\sum_{j=1}^k r_j^{m_0+1} \right]^{1/k} \leq \frac{1}{k} \sum_{j=1}^k r_j^{m_0+1} = 1,$$

which contradicts the assumption $\sum_{j=1}^k r_j > 1$. Hence, $a_m = 0$ for all $m > 0$. This implies that f is a constant, say $c \in \mathbb{C}$, and therefore we obtain $\varphi(z) = cz^n$.

It is natural to look for all entire functions $\varphi: \mathbb{C} \rightarrow \mathbb{C}$ which satisfy the following additive counterpart of inequality (2.1):

$$\left(\sum_{j=1}^k \varphi(r_j z) \right) \geq \sum_{j=1}^k r_j |\varphi(z)|, \quad (2.11)$$

where r_1, \dots, r_k are (fixed) positive real numbers with $\sum_{j=1}^k r_j > k$. The monomials $\varphi(z) = cz^n$ ($c \in \mathbb{C}, n \in \mathbb{N}$) are solutions of (2.11). Indeed, inequality (2.11) with $\varphi(z) = cz^n$ reduces to

$$\sum_{j=1}^k r_j^n \geq \sum_{j=1}^k r_j, \quad (2.12)$$

which follows immediately from Jensen's inequality and the assumption $\sum_{j=1}^k r_j > k$. By an argumentation similar to the one we have used to establish the theorem it can be shown that the functions $\varphi(z) = cz^n$ ($c \in \mathbb{C}, n \in \mathbb{N}$) are the only entire solutions of (2.11). This provides another extension of Walorski's result.

If the expression on the left-hand side of (2.11) will be replaced by $\sum_{j=1}^k |\varphi(r_j z)|$, then we conclude from the triangle inequality that $\varphi(z) = cz^n$ ($c \in \mathbb{C}, n \in \mathbb{N}$) also solve

$$\sum_{j=1}^k |\varphi(r_j z)| \geq \sum_{j=1}^k r_j |\varphi(z)|, \quad (2.13)$$

where r_1, \dots, r_k are (fixed) positive real numbers with $\sum_{j=1}^k r_j > k$. We finish by asking: Are there more solutions of (2.13) (if $k > 1$)?

REFERENCE

1. WALORSKI, J., On a functional inequality, *Aequationes Math.* 32 (1987), 213-215.

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