

ON THE STRUCTURE OF SELF ADJOINT ALGEBRA OF FINITE STRICT MULTIPLICITY

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1. INTRODUCTION.

Throughout this paper X is a complex Hilbert space. For any subset M of X , $B(M)$ denotes the algebra of all bounded linear operators on M . An algebra \mathcal{A} means a strongly closed subalgebra of $B(X)$ containing the identity element I . \mathcal{A} is said to be an algebra of finite strict multiplicity (a.f.s.m.), if there exists a finite subset $\Gamma = \{x_1, x_2, \dots, x_n\}$ of X such that $\mathcal{A}(\Gamma) = \{A_1x_1 + A_2x_2 + \dots + A_nx_n, A_i \in \mathcal{A}\} = X$. In this case we denote the algebra by $(\mathcal{A}, \{x_i\}_{i=1}^n)$. If $n = 1$, i.e., if there exists a vector x_0 such that $\mathcal{A}x_0 = \{Ax_0 : A \in \mathcal{A}\} = X$, then \mathcal{A} is said to be a strictly cyclic algebra. In this case vector x_0 is called a strictly cyclic vector for \mathcal{A} . Algebra \mathcal{A} is said to be self-adjoint, if $A^* \in \mathcal{A}$, whenever A is in \mathcal{A} . For any subset \mathfrak{B} of $B(X)$, the commutant of \mathfrak{B} , denoted by \mathfrak{B}' , is the collection of all operators in $B(X)$ that commute with \mathfrak{B} .

A closed linear subspace M of X reduces the subset \mathfrak{B} of $B(X)$, if the projection of X onto M is in \mathfrak{B}' . A collection $\{M_j\}$ of closed linear subspaces of X is said to be an orthogonal decomposition of X , if the M_j 's are pair-wise orthogonal and span X . Correspondingly, a collection $\{P_j\}$ of projections, is said to be a resolution of identity, if the collection $\{P_j(X)\}$ of ranges of P_j forms an orthogonal decomposition of X .

Strictly cyclic operator algebras have been studied by Lambert [1], [2], M. Embry [3], [4], [5], Bolstein [6] and others. The study of strictly cyclic algebras was extended to that of algebras of finite strict multiplicity by Herrero in [7], [8]. This paper aims at studying the structure of the commutant of an a.f.s.m., particularly a self-adjoint a.f.s.m. in terms of its reducing subspaces. By [5], the commutant of a self-adjoint strictly cyclic algebra cannot have any infinite collection of pair-wise orthogonal projections. [9, Theorem 2] paves the way for the following:

2. MAIN RESULTS.

THEOREM 1. If $(\mathcal{A}, \{x_i\}_{i=1}^n)$ is an a.f.s.m. on X , then each collection of mutually-orthogonal projections in \mathcal{A}' is finite.

PROOF. Let $\{P_j\}$ be a collection of mutually-orthogonal projections in \mathcal{A}' . We may assume $\{P_j\}$ to be countable. Let $Q_n = \sum_{j=1}^n P_j$ and $Q = \sum_{j=1}^{\infty} P_j$. Q_n converges strongly to Q . By [9, Theorem 2], Q_n converges uniformly to Q . As $Q - Q_n$ is a projection, its norm is zero or one. Since $\|Q_n - Q\|$ can be made arbitrarily small, there exists m such that $\|Q_n - Q\| = 0$ for all $n \geq m+1$. This implies that the collection $\{P_j\} = \{P_j\}_{j=1}^m$ is finite.

COROLLARY 2. Let $(\mathcal{A}, \{x_i\}_{i=1}^n)$ be an a.f.s.m. on X . Any operator in \mathcal{A}' with residual spectrum empty is of finite spectrum.

PROOF. Let E in \mathcal{A}' have residual spectrum empty. By [8], E has no continuous spectrum. Therefore, spectrum of E consists entirely of point spectrum. By Theorem 1, E has only finite number of distinct eigenspaces. So spectrum of E is finite.

Our next theorem generalizes [5, Theorem 3] to a self-adjoint a.f.s.m.

THEOREM 3. Let \mathcal{A} be a self-adjoint a.f.s.m. on X . Then there exists a finite orthogonal decomposition $\{M_k\}$ of X such that each M_k reduces \mathcal{A} , and $\mathcal{A}|_{M_k}$ is strongly dense in $B(M_k)$.

PROOF. If X and $\{0\}$ are the only reducing subspaces of \mathcal{A} , then by [8], \mathcal{A} is strongly dense in $B(X)$. As such the trivial decomposition $\{X\}$ of X satisfies the requirements of the theorem.

Let $\{M_k\}_{k=1}^p$ be a collection of mutually orthogonal subspaces of X such that each M_k reduces \mathcal{A} , and $\mathcal{A}|_{M_k}$ is strongly dense in $B(M_k)$. If these M_k 's span X , the theorem follows. Otherwise,

consider $A_1 = \mathcal{A}|_{[\sum_{i=1}^p M_i]^\perp}$. Let P be orthogonal projection of X onto $M = [\sum_{i=1}^p M_i]^\perp$. P is in \mathcal{A}' and $(\mathcal{A}, \{Px_i\}_{i=1}^n)$ is an a.f.s.m. on M . If \mathcal{A}_1 has only trivial reducing subspace, then again, by [8], \mathcal{A}_1 is strongly dense in $B(M)$ and the construction is complete. Otherwise, \mathcal{A}_1 has a non-trivial reducing subspace. This implies \mathcal{A}_1 has a minimal reducing subspace, say M_{p+1} . By [8], $\mathcal{A}_1|_{M_{p+1}}$ is strongly dense in $B(M_{p+1})$. Thus, M_1, M_2, \dots, M_{p+1} are pair-wise orthogonal subspaces for \mathcal{A} , and $\mathcal{A}|_{M_k}$ is strongly dense in $B(M_k)$ for $k = 1, 2, \dots, p+1$. By Theorem 1, the

collection terminates with a finite number of pair-wise orthogonal reducing subspaces.

Our next theorem depicts the structure of commutant of a self-adjoint a.f.s.m. The theorem and its consequences can be proved by following the technique used by Embry in [5]. So we omit the proofs.

THEOREM 4. Let $(\mathcal{A}, \{x_i\}_{i=1}^n)$ be a self-adjoint a.f.s.m., $\{M_k\}$ a decomposition of X as required in Theorem 3 and P_k the orthogonal projection of X onto M_k . Then $\mathcal{A}' = \sum P_j \mathcal{A}' P_k$ and $P_j \mathcal{A}' P_k$ is of dimension one or zero for each value of j and k . In particular, \mathcal{A}' is finite-dimensional.

COROLLARY 5. If \mathcal{A} is self-adjoint a.f.s.m. with an abelian commutant, then $\mathcal{A}' = \{\sum_{j=1}^n \lambda_j P_j; \lambda_j \text{ is complex}\}$, wherein $\{P_j\}$ are projections as required in Theorem 4. In particular, $\{P_j\}$ consists of normal operators with finite spectra.

COROLLARY 6. Let N be a normal operator with $\{N\}'$ as an a.f.s.m. Then there exist orthogonal projections P_1, P_2, \dots, P_n such that $\{N\}'' = \{\sum_{j=1}^n \lambda_j P_j; \lambda_j \text{ complex}\}$.

COROLLARY 7. The decomposition $\{M_k\}$ in Theorem 4 is unique, if and only if, \mathcal{A}' is abelian.

If \mathcal{A} is any a.f.s.m. on X , then $\mathcal{A} = \mathcal{A}_1 \oplus \mathcal{A}_2$, where \mathcal{A}_1 is a self-adjoint a.f.s.m., and \mathcal{A}_2 is an a.f.s.m. having no reducing subspaces on which it is self-adjoint.

An operator T in $B(X)$ is said to be of finite strict multiplicity, if the weakly closed algebra $\mathcal{A}(T)$ generated by T and I is of finite strict multiplicity. Our next theorem extends [10, Theorem 6] proved by Barnes.

THEOREM 8. For an operator T in $B(X)$, let $\{x_1, x_2, \dots, x_n\}$ be a subset of X such that $(\mathcal{A}(T), \{x_i\}_{i=1}^n)$ is an a.f.s.m. on X . Then there exists a finite mutually-orthogonal collection of subspaces $\{X_1, X_2, \dots, X_n\}$ of X satisfying the following

- (i) Each X_i reduces T
- (ii) $X = X_1 \oplus X_2 \oplus \dots \oplus X_k$ and thus $T = T_1 \oplus T_2 \oplus \dots \oplus T_k$, where $T_i = T|_{X_i}$.
- (iii) Each T_j is an irreducible operator of finite strict multiplicity on X_j .

PROOF. Let \mathfrak{B} be a closed subalgebra of $B(X)$ generated by T , T^* and I . Define a positive functional f on \mathfrak{B} by $f: \mathfrak{B} \rightarrow \mathbb{C}$ as $f(S) = (Sx_o, x_o)$, where $x_o = x_1 + x_2 + \dots + x_n$, S in \mathfrak{B} . Let $K_f = \{S \in \mathfrak{B}; f(S^*S) = 0\} = \{S \in \mathfrak{B}; Sx_o = 0\}$. There are two norms on the quotient space \mathfrak{B}/K_f , viz.

$$(i) \quad \|A + K_f\|_f = f(A^*A)^{1/2} = \|Ax_o\|$$

$$(ii) \quad \|A + K_f\| = \inf \{ \|A - K\| : K \in K_f \}.$$

These norms are related by $\|A + K_f\|_f \leq \|x_o\| \|A + K_f\|$ for all A in \mathfrak{B} . \mathfrak{B}/K_f is complete w.r.t. both these norms. By closed graph theorem, the norms are equivalent on \mathfrak{B}/K_f . By Halpern [11], the commutant \mathfrak{B}' of \mathfrak{B} in $B(X)$ has the following properties:

- (i) If F is a non-zero projection in \mathfrak{B}' , then F majorizes a minimal projection E in \mathfrak{B}' ;
- (ii) A maximal set of mutually orthogonal projections in \mathfrak{B} must be finite.

By (i), we can choose a non-empty maximal set of mutually orthogonal projections in \mathfrak{B}' and, by (ii), this set is finite. Let $\{E_1, E_2, \dots, E_k\}$ be this set. Let $X_j = R(E_j)$, $j = 1, 2, \dots, k$. Then $X = X_1 \oplus X_2 \oplus \dots \oplus X_k$. The collection $\{X_1, X_2, \dots, X_k\}$ reduces \mathfrak{B} ; and \mathfrak{B} acts irreducibly on each X_j , $j = 1, 2, \dots, k$. Now $x_i \in X$ implies $x_i = x_{i1} \oplus x_{i2} \oplus \dots \oplus x_{ik}$, $i = 1, 2, \dots, n$ where $x_{ij} \in X_j$ for all $j = 1, 2, \dots, k$. For $y \in X_j \subset X$, there exist operators R_1, R_2, \dots, R_n in $\mathcal{A}(T)$ such that

$$\begin{aligned} y &= R_1x_1 + R_2x_2 + \dots + R_nx_n \\ &= R_1(x_{11} \oplus x_{12} \oplus \dots \oplus x_{1k}) + R_2(x_{21} \oplus \dots \oplus x_{2k}) + \dots + R_n(x_{n1} \oplus x_{n2} \oplus \dots \oplus x_{nk}) \\ &= (R_1x_{11} + R_2x_{21} + \dots + R_nx_{n1}) \oplus (R_1x_{12} + R_2x_{22} + \dots + R_nx_{n2}) \oplus \dots \oplus \\ &\quad \oplus (R_1x_{1k} + R_2x_{2k} + \dots + R_nx_{nk}) \end{aligned}$$

As T reduces X_j , $\mathcal{A}(T)$ also reduces X_j , $j = 1, 2, \dots, k$. This implies that

$$y = R_1x_{1j} + R_2x_{2j} + \dots + R_nx_{nj} = R_1|_{X_j}x_{1j} + R_2|_{X_j}x_{2j} + \dots + R_n|_{X_j}x_{nj},$$

where $R_i|_{X_j} \in \mathcal{A}(T_j)$ for all $i = 1, 2, \dots, n$. Thus $(\mathcal{A}(T_j), x_{1j}, x_{2j}, \dots, x_{nj})$ is an a.f.s.m. on X_j .

This completes the proof of the theorem.

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