

ON k -IDEALS OF SEMIRINGS

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ABSTRACT. Certain types of ring congruences on an additive inverse semiring are characterized with the help of full k -ideals. It is also shown that the set of all full k -ideals of an additively inverse semiring in which addition is commutative forms a complete lattice which is also modular.

KEY WORDS AND PHRASES. Semiring, inverse semiring, k -ideals and ring congruence.

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1. PRELIMINARIES. A semiring is a system consisting of a non-empty set S together with two binary operations on S called addition and multiplication (denoted in the usual manner) such that

- (i) S together with addition is a semigroup;
- (ii) S together with multiplication is a semigroup; and
- (iii) $a(b+c) = ab + ac$ and $(a+b)c = ac + bc$ for all $a, b, c \in S$.

A semiring S is said to be additively commutative if $a+b = b+a$ for all $a, b \in S$. A left (right) ideal of a semiring S is non-empty subset I of S such that

- i) $a+b \in I$ for all $a, b \in I$; and
- ii) $ra \in I$ ($ar \in I$) for all $r \in S$ and $a \in I$.

An ideal of a semiring S is a non-empty subset I of S such that I is both a left and right ideal of S .

Henriksen [1] defined a more restricted class of ideals in a semiring, which he called k -ideals.

A left k -ideal I of a semiring S is a left ideal such that if $a \in I$ and $x \in S$ and if either $a+x \in I$ or $x+a \in I$, then $x \in I$.

Right k -ideal of a semiring is defined dually. A non-empty subset I of a semiring S is called a k -ideal if it is both a left k -ideal and a right k -ideal.

A semiring S is said to be additively regular if for each $a \in S$, there exists an element $b \in S$ such that $a = a + b + a$. If in addition, the element b is unique and satisfies $b = b + a + b$, then S is

called an additively inverse semiring. In an additively inverse semiring the unique inverse b of an element a is usually denoted by a' . Karvellas [2] proved the following result:

Let S be an additively inverse semiring. Then

- i) $x = (x')', (x + y)' = y' + x', (xy)' = x'y = y'x$ and $xy = x'y'$ for all $x, y \in S$.
- ii) $E^+ = \{x \in S : x + x = x\}$ is an additively commutative semilattice and an ideal of S .

2. **FULL k -IDEALS.** In this section S denotes an additively inverse semiring in which addition is commutative and E^+ denotes the set of all additive idempotents of S .

A left k -ideal A of S is said to be full if $E^+ \subseteq A$. A right k -ideal of S is defined dually.

A non-empty subset I of S is called a full k -ideal if it is both left and a right full k -ideal.

EXAMPLE 1. In a ring every ring ideal is a full k -ideal.

EXAMPLE 2. In a distributive lattice with more than two elements, a proper ideal is a k -ideal but not a full k -ideal.

EXAMPLE 3. $Z \times Z^p = \{(a, b) : a, b \text{ are integers and } b > 0\}$. Define

$$(a, b) + (c, d) = (a + c, 1.c.m \text{ of } b, d) \text{ and } (a, b)(c, d) = (ac, h.c.f. \text{ of } b, d).$$

Then $Z \times Z^p$ becomes an additively inverse semiring in which addition is commutative.

Let $A = \{(a, b) \in Z \times Z^p : a = 0, b \in Z^p\}$. Then A is a full k -ideal of $Z \times Z^p$.

LEMMA 2.1. Every k -ideal of S is an additively inverse subsemiring of S .

PROOF. Let I be a k -ideal of S . Clearly I is a subsemiring of S . Let $a \in I$. Then

$$a + (a' + a) = a \in I.$$

Since I is a k -ideal, it follows $a' + a \in I$. Again this implies that $a' \in I$. Hence the lemma.

LEMMA 2.2. Let A be an ideal of S . Then

$$\overline{A} = \{a \in S : a + x \in A \text{ for some } x \in A\} \text{ is a } k\text{-ideal of } S.$$

PROOF. Let $a, b \in \overline{A}$. The $a + x, b + y \in A$ for some $x, y \in A$. Now

$$a + x + b + y = (a + b) + (x + y) \in A.$$

As $x + y \in A, a + b \in \overline{A}$. Next let $r \in S, ra + rx = r(a + x) \in A$.

As $rx \in A, ra \in \overline{A}$. Similarly, $ar \in \overline{A}$. As a result \overline{A} is an ideal of S . Next, let c and $c + d \in \overline{A}$. Then there exists x and y in A such that $c + x \in A$ and $c + d + y \in A$.

Now

$$d + (c + x + y) = (c + d + y) + x \in A \text{ and } c + x + y \in A.$$

Hence $d \in \overline{A}$ and \overline{A} is a k -ideal of S . Since $a + a' \in A$ for all $a \in A$, it follows that $A \subseteq \overline{A}$.

COROLLARY. Let A be an ideal of S . Then $\overline{A} = A$ iff A is a k -ideal.

LEMMA 2.3. Let A and B be two full k -ideals of S , then $\overline{A + B}$ is a full k -ideal of S such that

$$A \subseteq \overline{A + B} \text{ and } B \subseteq \overline{A + B}.$$

PROOF. It can be shown that $A + B$ is an ideal of S . Then from Lemma 2.2, we find $\overline{A + B}$ is a k -ideal and $A + B \subseteq \overline{A + B}$. Now $E^+ \subseteq A, B$. Hence $E^+ \subseteq A + B \subseteq \overline{A + B}$. This implies that $\overline{A + B}$ is a full k -ideal. Let $a \in A$. Then

$$a = a + a' + a = a + (a' + a) \in A + B \text{ as } a' + a \in E^+ \subseteq B.$$

Hence $A \subseteq \overline{A + B}$ and similarly $B \subseteq \overline{A + B}$.

THEOREM 2.4. If $I(S)$ denotes the set of all full k -ideals of S , then $I(S)$ is a complete lattice which is also modular.

PROOF. We first note that $I(S)$ is a partially ordered set with respect to usual set inclusion. Let $A, B \in I(S)$. Then $A \cap B \in I(S)$ and from Lemma 2.3, $\overline{A \cap B} \in I(S)$. Define $A \wedge B = A \cap B$ and $A \vee B = \overline{A \cap B}$. Let $C \in I(S)$ such that $A, B \subseteq C$. Then $A + B \subseteq C$ and $\overline{A + B} \subseteq \overline{C}$. But $\overline{C} = C$. Hence $\overline{A + B} \subseteq C$. As a result $\overline{A + B}$ is the l.u.b. of A, B . Thus we find that $I(S)$ is a lattice. Now E^+ is an ideal of S . Hence $\overline{E}^+ \in I(S)$ and also $S \in I(S)$; consequently $I(S)$ is a complete lattice. Next suppose that $A, B, C \in I(S)$ such that

$$A \wedge B = A \wedge C \text{ and } A \vee B = A \vee C \text{ and } B \subseteq C.$$

Let $x \in C$. Then $x \in A \vee C = A \vee B = \overline{A + B}$. Hence there exists $a + b \in A + B$ such that $x + a + b = a_1 + b_1$ for some $a_1 \in A, b_1 \in B$.

Then

$$x + a + a' + b = a_1 + b_1 + a'.$$

Now $x \in C, a + a' \in C$ and $b \in B \subseteq C$. Hence $a_1 + b_1 + a' \in C$. But $b_1 \in C$. Consequently, $a_1 + a' \in C \cap A = C \cap B$. Hence $a_1 + a' \in B$. So from $x + a + b = a_1 + b_1$ we find that $x + a + a' + b = a_1 + a' + b \in B$. But $(a + a') + b \in B$ and B is a k -ideal. Hence $x \in B$ and $B = C$. This proves that $I(S)$ is a modular lattice.

3. RING CONGRUENCES.

A congruence ρ on a semiring S is called a ring congruence if the quotient semiring S/ρ is a ring.

In this section we assume S is an additively inverse semiring in which addition is commutative. We want to characterize those ring congruences on S such that $-(a\rho) = a'\rho$ where a' denotes the inverse of a in S and $-(a\rho)$ denotes the additive inverse of $a\rho$ in the ring S/ρ .

THEOREM 3.1. Let A be a full k -ideal of S . Then the relation

$$\rho_A = \{(a, b) \in S \times S : a + b' \in A\} \text{ is a ring congruence on } S \text{ such that } -(a\rho_A) = a'\rho_A.$$

PROOF. Since $a + a' \in E^+ \subseteq A$ for all $a \in S$, it follows that ρ_A is reflexive. Let $a + b' \in A$. Now from Lemma 2.1, we find that $(a + b')' \in A$. Then $b + a' = (b')' + a' = (a + b')' \in A$. Hence ρ_A is symmetric. Let $a + b' \in A$ and $b + c' \in A$. Then $a + b + b' + c' \in A$. Also $b + b' \in E^+ \subseteq A$. Since A is a k -ideal, we find that $a + c' \in A$. Hence ρ_A is an equivalence relation. Let $(a, b) \in \rho_A$ and $c \in S$. Then $a + b' \in A$. Since

$$(c + a) + (c + b)' = c + a + b' + c' = (a + b') + (c + c') \in A, ca + (cb)' = ca + cb' = c(a + b') \in A,$$

$$ac + (bc)' = ac + b'c = (a + b')c \in A,$$

it follows that ρ_A is a congruence on S . So we obtain the quotient semiring where addition and multiplication are defined by

$$a\rho_A + b\rho_A = (a + b)\rho_A \text{ and } (a\rho_A)(b\rho_A) = (ab)\rho_A.$$

Now

$$a\rho_A + b\rho_A = (a + b)\rho_A = (b + a)\rho_A = b\rho_A + a\rho_A.$$

Let $e \in E^+$ and $a \in S$. Now $(e + a) + a' = e + (a + a') \in E^+$.

We find that $(e + a)\rho_A = a\rho_A$. Then $e\rho_A + a\rho_A = a\rho_A$.

Also

$$a\rho_A + a'\rho_A = (a + a')\rho_A = e\rho_A.$$

Hence $e\rho_A$ is the zero element and $a'\rho_A$ is the negative element of $a\rho_A$ in the ring S/ρ_A .

THEOREM 3.2. Let ρ be a congruence on S such that S/ρ is a ring and $-(a\rho) = a'\rho$. Then there exists a full k -ideal A of S such that $\rho_A = \rho$.

PROOF. Let $A = \{a \in S : (a, e) \in \rho \text{ for some } e \in E^+\}$. Since ρ is reflexive, it follows that $E^+ \subseteq A$. Then $A \neq \emptyset$, since $E^+ \neq \emptyset$. Let $a, b \in A$. Then there exist $e, f \in E^+$ such that $(a, e) \in \rho$ and $(b, f) \in \rho$. Then $(a + b, e + f) \in \rho$. But $e + f \in E^+$. Hence $a + b \in A$. Again for any $r \in S$, $(ra, re) \in \rho$ and $(ar, er) \in \rho$. But re and $er \in E^+$. Hence A is an ideal of S .

Let $a + b \in A$ and $b \in A$. Then there exist $e, f \in E^+$ such that $(a + b, f) \in \rho$ and $(b, e) \in \rho$. Hence $f\rho = (a + b)\rho = a\rho + b\rho = ap + bp$. But $f\rho$ and ep are additive idempotents in the ring S/ρ . Hence $ep = f\rho$ is the zero element of S/ρ . As a result, $a\rho$ is the zero element of S/ρ . Then $a\rho = ep$. This implies $a \in A$. So we find that A is a full k -ideal of S . Consider now the congruences ρ_A and ρ . Let $(a, b) \in \rho$. Then $(a + b', b + b') \in \rho$. But $b + b' \in E^+$. Hence $a + b' \in A$ and $(a, b) \in \rho_A$. Conversely suppose that $(a, b) \in \rho_A$. Then $a + b' \in A$. Hence $(a + b', e) \in \rho$ for some $e \in E^+$. As a result, $ep = ap + b'\rho = ap - b\rho$ holds in the ring S/ρ . But ep is the zero element of S/ρ . Consequently $a\rho = b\rho$. This shows that $(a, b) \in \rho$ and hence $\rho_A = \rho$.

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