

ON POLYNOMIAL EP_r -MATRICES

AR. MEENAKSHI and N. ANANDAM

Department of Mathematics,
Annamalai University,
Annamalainagar - 608 002,
Tamil Nadu, INDIA.

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ABSTRACT. This paper gives a characterization of EP_r - λ -matrices. Necessary and sufficient conditions are determined for (i) the Moore-Penrose inverse of an EP_r - λ -matrix to be an EP_r - λ -matrix and (ii) Moore-Penrose inverse of the product of EP_r - λ -matrices to be an EP_r - λ -matrix. Further, a condition for the generalized inverse of the product of λ -matrices to be a λ -matrix is determined.

KEY WORDS AND PHRASES: EP_r - λ -matrices, generalized inverse of a matrix.

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1. INTRODUCTION

Let $F[\lambda]^{mn}$ be the set of all $m \times n$ matrices whose elements are polynomials in λ over an arbitrary field F with an involutory automorphism $\sigma: a \leftrightarrow \bar{a}$ for $a \in F$. The elements of $F[\lambda]^{mn}$ are called λ -matrices. For $A(\lambda) = (a_{ij}(\lambda)) \in F[\lambda]^{mn}$, $A^*(\lambda) = (\bar{a}_{ji}(\lambda))$. Let $F[\lambda]^{mn}$ be the set of all $m \times n$ matrices whose elements are rational functions of the form $f(\lambda)/g(\lambda)$ where $f(\lambda), g(\lambda) \neq 0$ are polynomials in λ . For simplicity, let us denote $A(\lambda)$ by A itself.

The rank of $A \in F[\lambda]^{mn}$ is defined to be the order of its largest minor that is not equal to the zero polynomial ([2]p.259). $A \in F[\lambda]^{mn}$ is said to be an unimodular λ -matrix (or) invertible in $F[\lambda]^{mn}$ if the determinant of $A(\lambda)$, that is, $\det A(\lambda)$ is a nonzero constant. $A \in F[\lambda]^{mn}$ is said to be a regular λ -matrix if and only if it is of rank n ([2]p.259), that is, if and only if the kernel of A contains only the zero element. $A \in F[\lambda]^{mn}$ is said to be EP_r over the field $F(\lambda)$ if $\text{rk}(A) = r$ and $R(A) = R(A^*)$ where $R(A)$ and $\text{rk}(A)$ denote the range space of A and rank of A respectively [4]. We have $\{ \text{unimodular } \lambda\text{-matrices} \} \subset \{ \text{regular } \lambda\text{-matrices} \} \subset \{ EP\text{-}\lambda\text{-matrices} \}$.

Throughout this paper, let $A \in F[\lambda]^{mn}$. Let 1 be identity element of F . The Moore-Penrose inverse of A , denoted by A^+ is the unique solution of the following set of equations:

$$AXA=A \quad (1.1); \quad XAX=X \quad (1.2); \quad (AX)^*=AX \quad (1.3); \quad (XA)^*=XA \quad (1.4)$$

A^+ exists and $A^+ \in F[\lambda]^{mn}$ if and only if $\text{rk}(AA^*) = \text{rk}(A^*A) = \text{rk}(A)$ [7]. When A^+ exists, A is EP_r over $F(\lambda) \Leftrightarrow AA^+ = A^*A$. For $A \in F[\lambda]^{mn}$, a generalized inverse (or) {1} inverse is defined as a solution of the polynomial matrix equation (1.1) and a reflexive generalized inverse (or) {1,2} inverse is defined as a solution of the equations (1.1) and (1.2) and they belong to $F[\lambda]^{mn}$. The purpose of this paper is to give a characterization of an EP_r - λ -matrix. Some results on EP_r - λ -matrices having the same range space are obtained. As an application necessary and sufficient conditions are derived for $(AB)^+$ to be an EP_r - λ -matrix whenever A and B are EP_r - λ -matrices.

2. CHARACTERIZATION OF AN EP_r - λ -MATRIX

THEOREM 1. $A \in F_r^{nxn}[\lambda]$ is EP_r over the field $F(\lambda)$ if and only if there exist an nxn unimodular λ -matrix P and a $r \times r$ regular λ -matrix E such that

$$PAP^* = \begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix}$$

PROOF. By the Smith's canonical form, $A = \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix} Q$ where P and Q are unimodular- λ -matrices of order n and D is a rxr regular diagonal λ -matrix. Any {1} inverse of A is given by $A^{(1)} = Q^{-1} \begin{bmatrix} D^{-1} & R_2 \\ R_3 & R_4 \end{bmatrix} P^{-1}$ where R_2, R_3 , and R_4 are arbitrary conformable matrices over $F(\lambda)$. A is EP_r over the field $F(\lambda)$

$$\Rightarrow R(A) = R(A^*)$$

$$\Rightarrow A = AA^{*(1)}A^*$$

(By Theorem 17[3])

$$\Rightarrow \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix} QP^{*-1} = \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix} QP^{*-1} \begin{bmatrix} D^{*-1} & R_3^* \\ R_2^* & R_4^* \end{bmatrix} Q^{*-1} Q^* \begin{bmatrix} D^* & 0 \\ 0 & 0 \end{bmatrix}$$

Partitioning conformably, let, $QP^{*-1} = \begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix}$

$$\begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix} = \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix} \begin{bmatrix} D^{*-1} & R_3^* \\ R_2^* & R_4^* \end{bmatrix} \begin{bmatrix} D^* & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} DT_1 & DT_2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} DT_1 + DT_2 R_2^* D^* \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow T_2 = 0 \quad (\text{since } D \text{ is regular}).$$

Therefore $QP^{*-1} = \begin{bmatrix} T_1 & 0 \\ T_3 & T_4 \end{bmatrix}$

Hence $A = P \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} T_1 & 0 \\ T_3 & T_4 \end{bmatrix} P^* = P \begin{bmatrix} DT_1 & 0 \\ 0 & 0 \end{bmatrix} P^* = P \begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix} P^*$

where $E = DT_1$ is a $r \times r$ regular λ -matrix.

Conversely, let $PAP^* = \begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix}$ where E is a $r \times r$ regular λ -matrix.

Since E is regular, E is EP_r over $F(\lambda)$.

$$\Rightarrow R(E) = R(E^*)$$

$$\Rightarrow R(PAP^*) = R(PA^*P^*)$$

$$\Rightarrow R(A) = R(A^*)$$

$$\Rightarrow A \text{ is } EP_r \text{ over } F(\lambda). \text{ Hence the theorem.}$$

If $A \in F_r^{nxn}[\lambda]$ and is EP over the field $F(\lambda)$ then we can find nxn regular rational λ -matrices H and K such that $A^* = HA = AK$ [4]. In general the above H and K need not be unimodular λ -matrices. For example, consider $A = \begin{bmatrix} 1 & \lambda \\ Q & \lambda^2 \end{bmatrix}$. A is

EP, being a regular λ -matrix. If $A^* = HA$ then $H = A^*A^{-1}$; If $A^* = AK$ then $K = A^{-1}A^*$. Here $H = \begin{bmatrix} 1 & -1/\lambda \\ \lambda & 0 \end{bmatrix}$ and $K = \begin{bmatrix} 0 & -\lambda \\ 1/\lambda & 1 \end{bmatrix}$ are not λ -matrices.

The following theorem gives a necessary condition for H and K to be unimodular λ -matrices.

THEOREM 2. If A is an $n \times n$ EP_r- λ -matrix and A has a λ -matrix {1} inverse then there exist $n \times n$ unimodular λ -matrices H and K such that $A^* = HA = AK$.

PROOF. Let A be an $n \times n$ EP_r- λ -matrix. By Theorem 1, there exists an $n \times n$ unimodular λ -matrix P such that $PAP^* = \begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix}$ where E is a $r \times r$ regular λ -matrix. Since A has a λ -matrix {1} inverse, E^{-1} is also a λ -matrix.

$$\text{Now } A = P^{-1} \begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix} P^{-1*}$$

$$\text{Therefore } A^* = P^{-1} \begin{bmatrix} E^* & 0 \\ 0 & 0 \end{bmatrix} P^{-1*}$$

$$\begin{aligned} &= P^{-1} \begin{bmatrix} E^*E^{-1} & 0 \\ 0 & I \end{bmatrix} PP^{-1} \begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix} P^{-1*} \\ &= HA \text{ where } H = P^{-1} \begin{bmatrix} E^*E^{-1} & 0 \\ 0 & I \end{bmatrix} P \text{ is an } n \times n \text{ unimodular} \end{aligned}$$

λ -matrix. Similarly we can write $A^* = AK$ where

$$K = P^* \begin{bmatrix} E^{-1}E^* & 0 \\ 0 & I \end{bmatrix} P^{-1*} \text{ is an } n \times n \text{ unimodular } \lambda\text{-matrix.}$$

Therefore $A^* = HA = AK$.

REMARK 1. The converse of Theorem 2 need not be true. For example, consider $A = \begin{bmatrix} \lambda & 0 \\ 0 & 0 \end{bmatrix}$. Since $A^* = A$, $H = K = I_2$. A is an EP₁- λ -matrix. However A has no λ -matrix {1} inverse.

3. MOORE-PENROSE INVERSE OF AN EP_r- λ -MATRIX

The following theorem gives a set of necessary and sufficient conditions for the existence of the λ -matrix Moore-Penrose inverse of a given λ -matrix.

THEOREM 3. For $A \in F_r^{n \times n}(\lambda)$, the following statements are equivalent.

- A is EP_r, $\text{rk}(A) = \text{rk}(A^2)$ and A^*A has a λ -matrix {1} inverse.
- There exists an unimodular λ -matrix U with $A = U \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix} U^*$ where D is a $r \times r$ unimodular λ -matrix and U^*U is a diagonal block matrix.
- $A = GLG^*$ where L and G^*G are $r \times r$ unimodular λ -matrices and G is a λ -matrix.
- A^+ is a λ -matrix and EP_r.
- There exists a symmetric idempotent λ -matrix E, ($E^2 = E = E^*$) such that $AE = EA$ and $R(A) = R(E)$.

PROOF. (i) \Rightarrow (ii) Since A is an EP_r- λ -matrix over the field $F(\lambda)$ and $\text{rk}(A) = \text{rk}(A^2)$, A^+ exists, by Theorem 2.3 of [5]. By Theorem 4 in [6], A^*A has a λ -matrix {1} inverse implies that there exists an unimodular λ -matrix P with $PP^* = \begin{bmatrix} P_1 & 0 \\ 0 & P_4 \end{bmatrix}$ where P_1 is a symmetric $r \times r$ unimodular λ -matrix such that

$PA = \begin{bmatrix} W \\ 0 \end{bmatrix}$ where W is a $r \times n$ λ -matrix of rank r . Hence by Theorem 2 in [6], AA^+ is a λ -matrix and $PAA^+P^* = \begin{bmatrix} P_1 & 0 \\ 0 & 0 \end{bmatrix}$. Since A is EP_r , $AA^+ = A^+A$ and $A = AA^+A = A(AA^+)$. Therefore $A = P^{-1} \begin{bmatrix} W \\ 0 \end{bmatrix} P^{-1} \begin{bmatrix} P_1 & 0 \\ 0 & 0 \end{bmatrix} P^{-1} = P^{-1} \begin{bmatrix} W \\ 0 \end{bmatrix} \begin{bmatrix} H & 0 \end{bmatrix} P^{-1}$ where

H consists of the first r columns of P , thus H is a $n \times r$ λ -matrix of rank r .

Now $A = P^{-1} \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix} P^{-1}^* = U \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix} U^*$ where $U = P^{-1}$ and $D = WH$ is a $r \times r$ regular λ -matrix. Since A^* has a λ -matrix $\{1\}$ inverse and P is an unimodular λ -matrix, $PAA^+P^* = \begin{bmatrix} D & P_1^*D \\ 0 & 0 \end{bmatrix}$ has a λ -matrix $\{1\}$ inverse. Therefore by Theorem 1 in [6], $D P_1^* D$ is an unimodular λ -matrix which implies D is an unimodular λ -matrix. Hence $A = U \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix} U^*$ where D is a $r \times r$ unimodular λ -matrix and U^*U is a diagonal block λ -matrix. Thus (ii) holds.

(ii) \Rightarrow (iii)

Let us partition U as $U = \begin{bmatrix} U_1 & U_2 \\ U_3 & U_4 \end{bmatrix}$ where U_1 is a $r \times r$ λ -matrix. Then $A = \begin{bmatrix} U_1 & U_2 \\ U_3 & U_4 \end{bmatrix} \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} U_1^* & U_3^* \\ U_2^* & U_4^* \end{bmatrix} = \begin{bmatrix} U_1 \\ U_3 \end{bmatrix} D \begin{bmatrix} U_1^* & U_3^* \\ U_2^* & U_4^* \end{bmatrix} = GLG^*$

where $L = D$ and $G = \begin{bmatrix} U_1 \\ U_3 \end{bmatrix}$ are λ -matrices.

Since U^*U is a diagonal block λ -matrix, $G^*G = U_1^*U_1 + U_3^*U_3$ and L are $r \times r$ unimodular λ -matrices. Thus (iii) holds.

(iii) \Rightarrow (iv)

Since $A = GLG^*$, L and G^*G are unimodular λ -matrices. One can verify that $A^+ = G(G^*G)^{-1}L^{-1}(G^*G)^{-1}G^*$.

Now $AA^+ = GLG^*G(G^*G)^{-1}L^{-1}(G^*G)^{-1}G^* = G(G^*G)^{-1}G^* = A^+A$ implies that A^+ is EP_r . Since L and G^*G are unimodular, L^{-1} and $(G^*G)^{-1}$ are λ -matrices, and G is a λ -matrix. Therefore A^+ is a λ -matrix. Thus (iv) holds.

(iv) \Rightarrow (v)

Proof is analogous to that of (ii) \Rightarrow (iii) of Theorem 2.3 [5].

(v) \Rightarrow (i)

Since E is a symmetric idempotent λ -matrix with $R(A) = R(E)$ and $AE = EA$, by Theorem 2.3 in [5] we have A is EP_r and $\text{rk}(A) = \text{rk}(A^2) \Rightarrow A^+$ exists. Since $E^+ = E$ and $R(A) = R(E) \Rightarrow AA^+ = EE^+ = E$. Now $AE = EA = (AA^+)A = A$. Let e_j and a_j denote the j th columns of E and A respectively. Then $AE = A \Rightarrow Ae_j = a_j$, since e_j is a λ -matrix, the equation $Ax = a_j$ where a_j is a λ -matrix, has a λ -matrix solution. Hence by Theorem 1 in [6] it follows that A has a λ -matrix $\{1\}$ inverse. Further $AA^+ = E$ is also a λ -matrix. Hence by Theorem 4 in [6] we see that A^+A has a λ -matrix $\{1\}$ inverse. Thus (i) holds. Hence the theorem.

REMARK 2. The condition (i) in Theorem 3 cannot be weakened which can be seen by the following examples.

EXAMPLE 1. Consider the matrix $A = \begin{bmatrix} \lambda & \lambda \\ 2\lambda^2 & 2\lambda^2 \\ 2\lambda^2 & 2\lambda^2 \end{bmatrix}$. A is EP₁ and $\text{rk}(A) = \text{rk}(A^2) = 1$. $A^* A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ has no λ -matrix {1} inverse (since the invariant polynomial of $A^* A$ is λ^2 which is not the identity of F). For this A , $A^+ = \frac{1}{4\lambda} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ is not a λ -matrix. Thus the theorem fails.

EXAMPLE 2. Consider the matrix $A = \begin{bmatrix} \lambda & 2\lambda \\ 2\lambda & 4\lambda \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$ over GF(5). A is EP₁. Since $A^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $\text{rk}(A) \neq \text{rk}(A^2)$, $A^* A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ has a λ -matrix {1} inverse (since any conformable λ -matrix is a λ -matrix {1} inverse). For this A , A^+ does not exist. Thus the theorem fails.

REMARKS 3. From Theorem 3, it is clear that if E is a symmetric idempotent λ -matrix, and A is a λ -matrix such that $R(E) = R(A)$ then A is EP $\Leftrightarrow AE = EA \Leftrightarrow A^+$ is a λ -matrix and EP.

We can show that the set of all EP_r- λ -matrices with common range space as that of given symmetric idempotent λ -matrix forms a group, analogous to that of the Theorem 2.1 in [5].

COROLLARY 1. Let $E = E^* = E^2 \in F[\lambda]^{DxD}$. Then $H(E) = \{A \in F[\lambda]^{DxD} : A \text{ is EP}_r \text{ over } F(\lambda) \text{ and } R(A) = R(E)\}$ is a maximal subgroup of $F[\lambda]^{DxD}$ containing E as identity.

PROOF. This can be proved similar to that of Theorem 2.1 of [5] by applying Theorem 3.

4. APPLICATION

In general, if A and B are λ -matrices, having λ -matrix {1} inverses, it is not necessary that AB has a λ -matrix {1} inverse.

EXAMPLE 3. Consider $A = \begin{bmatrix} 1 & \lambda \\ \lambda & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 2\lambda & 0 \end{bmatrix}$. Here $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ is one of the λ -matrix {1} inverse for both A and B . But $AB = \begin{bmatrix} 1+2\lambda^2 & 0 \\ \lambda+2\lambda^3 & 0 \end{bmatrix}$. Since the invariant polynomial of AB is $1+2\lambda^2 \neq 1$, AB has no λ -matrix {1} inverse.

The following theorem leads to the existence of λ -matrix {1} inverse of the product AB .

THEOREM 4. Let $A, B \in F[\lambda]^{DxD}$. If $A^2 = A$ and B has λ -matrix {1} inverse and $R(A) \subseteq R(B)$ then AB has a λ -matrix {1} inverse.

PROOF. Suppose $ABx = b$, where b is a λ -matrix, is a consistent system. Then $b \in R(AB) \subseteq R(A) \subseteq R(B)$ and therefore $Bz_0 = b$. Since B has a λ -matrix {1} inverse, by Theorem 1 in [6] we get z_0 is a λ -matrix. Since A is idempotent, so in particular A is a {1} inverse of A and $b \in R(A)$, we have $Ab = b$. Now $ABz_0 = Ab = b$. Thus $ABx = b$ has a λ -matrix solution. Hence by Theorem 1 in [6], AB has a λ -matrix {1} inverse. Hence the theorem.

The converse of Theorem 4 need not be true which can be seen by the following example.

EXAMPLE 4. Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$; $B = \begin{bmatrix} 1 & 1 \\ \lambda & \lambda \end{bmatrix}$; $AB = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$. Here $A^2 = A$ and $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ is a λ -matrix {1} inverse for both AB and B . However

$R(A) \neq R(B)$. Hence the converse is not true.

Next we shall discuss the necessary and sufficient condition for the Moore-Penrose inverse of the product of EP_r - λ -matrices to be an EP_r - λ -matrix.

THEOREM 5. Let A and B be EP_r - λ -matrices. Then A^*A has a λ -matrix $\{1\}$ inverse, $\text{rk}(A) = \text{rk}(A^2)$ and $R(A) = R(B)$ if and only if AB is EP_r and $(AB)^+ = B^+A^+$ is a λ -matrix.

PROOF. Since A and B are EP_r with $R(A) = R(B)$ and $\text{rk}(A) = \text{rk}(A^2)$, by a Theorem of Katz [1], AB is EP_r . Since A is a EP_r - λ -matrix, $\text{rk}(A) = \text{rk}(A^2)$ and A^*A has a λ -matrix $\{1\}$ inverse, by Theorem 3, A^+ is a λ -matrix and there exists a symmetric idempotent λ -matrix E such that $R(A) = R(E)$. Hence $AA^+ = AA^2 = E$. Since A and B are EP_r and $R(A) = R(B)$, we have $AA^+ = BB^+ = E = A^+A = B^+B$. Therefore $BE = EB$ and $R(B) = R(E)$. Again from Theorem 3, for the EP_r - λ -matrix B , we see that B^+ is a λ -matrix. Since A and B are EP_r with $R(A) = R(B)$, we can verify that $(AB)^+ = B^+A^+$. Since B^+ and A^+ are λ -matrices, it follows that $(AB)^+$ is a λ -matrix.

Conversely, if $(AB)^+$ is a λ -matrix and AB is EP_r then $(AB)^+$ is an EP_r - λ -matrix. Therefore by Theorem 3, there exists a symmetric idempotent λ -matrix E such that $R(AB) = R(E)$ and $(AB)^+ (AB)^+ = E = (AB)^+ (AB)$. Since $\text{rk}(AB) = \text{rk}(A) = r$ and $R(AB) \subseteq R(A)$, we get $R(A) = R(E)$. Since A is EP_r , by Remark 3, it follows that A^+ is a EP_r - λ -matrix. Now by Theorem 3, A^*A has a λ -matrix $\{1\}$ inverse and $\text{rk}(A) = \text{rk}(A^2)$. Since AB and B are EP_r , $R(E) = R(AB) = R((AB)^+) \subseteq R(B^+) = R(B)$ and $\text{rk}(AB) = \text{rk}(B)$ implies $R(B) = R(E)$. Therefore $R(A) = R(B)$. Hence the theorem.

REMARK 4. The condition that both A and B are EP_r - λ -matrices, is essential in Theorem 5, is illustrated as follows:

Let $A = \begin{bmatrix} 1 & \lambda \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2\lambda \\ 0 & 0 \end{bmatrix}$. A and B are not EP_1 .
 $A^*A = \begin{bmatrix} 1 & \lambda \\ \lambda & 2 \end{bmatrix}$ has a λ -matrix $\{1\}$ inverse and $R(A) = R(B)$. But AB is not EP_1 . $(AB)^+ = \frac{1}{1+4\lambda^2} \begin{bmatrix} 1 & 0 \\ 2\lambda & 0 \end{bmatrix}$ is not a λ -matrix. Hence the claim.

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