

# ON POLYNOMIAL $EP_r$ MATRICES

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**ABSTRACT.** This paper gives a characterization of  $EP_r$ - $\lambda$ -matrices. Necessary and sufficient conditions are determined for (i) the Moore-Penrose inverse of an  $EP_r$ - $\lambda$ -matrix to be an  $EP_r$ - $\lambda$ -matrix and (ii) Moore-Penrose inverse of the product of  $EP_r$ - $\lambda$ -matrices to be an  $EP_r$ - $\lambda$ -matrix. Further, a condition for the generalized inverse of the product of  $\lambda$ -matrices to be a  $\lambda$ -matrix is determined.

**KEY WORDS AND PHRASES:**  $EP_r$ - $\lambda$ -matrices, generalized inverse of a matrix.

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## 1. INTRODUCTION

Let  $F[\lambda]$  be the set of all  $m \times n$  matrices whose elements are polynomials in  $\lambda$  over an arbitrary field  $F$  with an involutory automorphism  $\alpha: a \leftrightarrow \bar{a}$  for  $a \in F$ . The elements of  $F[\lambda]$  are called  $\lambda$ -matrices. For  $A(\lambda) = (a_{ij}(\lambda)) \in F[\lambda]$ ,  $A^*(\lambda) = (\bar{a}_{ji}(\lambda))$ . Let  $F(\lambda)$  be the set of all  $m \times n$  matrices whose elements are rational functions of the form  $f(\lambda)/g(\lambda)$  where  $f(\lambda), g(\lambda) \neq 0$  are polynomials in  $\lambda$ . For simplicity, let us denote  $A(\lambda)$  by  $A$  itself.

The rank of  $A \in F[\lambda]$  is defined to be the order of its largest minor that is not equal to the zero polynomial ([2]p.259).  $A \in F[\lambda]$  is said to be an unimodular  $\lambda$ -matrix (or) invertible in  $F[\lambda]$  if the determinant of  $A(\lambda)$ , that is,  $\det A(\lambda)$  is a nonzero constant.  $A \in F[\lambda]$  is said to be a regular  $\lambda$ -matrix if and only if it is of rank  $n$  ([2]p.259), that is, if and only if the kernel of  $A$  contains only the zero element.  $A \in F[\lambda]$  is said to be  $EP_r$  over the field  $F(\lambda)$  if  $\text{rk}(A) = r$  and  $R(A) = R(A^*)$  where  $R(A)$  and  $\text{rk}(A)$  denote the range space of  $A$  and rank of  $A$  respectively [4]. We have  $\{\text{unimodular } \lambda\text{-matrices}\} \subset \{\text{regular } \lambda\text{-matrices}\} \subset \{EP\text{-}\lambda\text{-matrices}\}$ .

Throughout this paper, let  $A \in F[\lambda]$ . Let  $1$  be identity element of  $F$ . The Moore-Penrose inverse of  $A$ , denoted by  $A^+$  is the unique solution of the following set of equations:

$$AXA = A \quad (1.1); \quad XAX = X \quad (1.2); \quad (AX)^* = AX \quad (1.3); \quad (XA)^* = XA \quad (1.4)$$

$A^+$  exists and  $A^+ \in F[\lambda]$  if and only if  $\text{rk}(AA^*) = \text{rk}(A^*A) = \text{rk}(A)$  [7]. When  $A^+$  exists,  $A$  is  $EP_r$  over  $F(\lambda) \Leftrightarrow AA^+ = A^+A$ . For  $A \in F[\lambda]$ , a generalized inverse (or)  $\{1\}$  inverse is defined as a solution of the polynomial matrix equation (1.1) and a reflexive generalized inverse (or)  $\{1,2\}$  inverse is defined as a solution of the equations (1.1) and (1.2) and they belong to  $F[\lambda]$ . The purpose of this paper is to give a characterization of an  $EP_r$ - $\lambda$ -matrix. Some results on  $EP_r$ - $\lambda$ -matrices having the same range space are obtained. As an application necessary and sufficient conditions are derived for  $(AB)^+$  to be an  $EP_r$ - $\lambda$ -matrix whenever  $A$  and  $B$  are  $EP_r$ - $\lambda$ -matrices.

2. CHARACTERIZATION OF AN  $EP_r$ - $\lambda$ -MATRIX

THEOREM 1.  $A \in F_r^{n \times n}[\lambda]$  is  $EP_r$  over the field  $F(\lambda)$  if and only if there exist an  $n \times n$  unimodular  $\lambda$ -matrix  $P$  and a  $r \times r$  regular  $\lambda$ -matrix  $E$  such that

$$PAP^* = \begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix}$$

PROOF. By the Smith's canonical form,  $A = \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix} Q$  where  $P$  and  $Q$  are unimodular- $\lambda$ -matrices of order  $n$  and  $D$  is a  $r \times r$  regular diagonal  $\lambda$ -matrix. Any {1} inverse of  $A$  is given by  $A^{(1)} = Q^{-1} \begin{bmatrix} D^{-1} & R_2 \\ R_3 & R_4 \end{bmatrix} P^{-1}$  where  $R_2, R_3$ , and  $R_4$  are arbitrary conformable matrices over  $F(\lambda)$ .  $A$  is  $EP_r$  over the field  $F(\lambda)$

$$\Rightarrow R(A) = R(A^*)$$

$$\Rightarrow A = AA^{*(1)}A^*$$

(By Theorem 17[3])

$$\Rightarrow \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix} QP^{*-1} = \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix} QP^{*-1} \begin{bmatrix} D^{*-1} & R_3^* \\ R_2^* & R_4^* \end{bmatrix} Q^{*-1} Q^* \begin{bmatrix} D^* & 0 \\ 0 & 0 \end{bmatrix}$$

$$\text{Partitioning conformably, let, } QP^{*-1} = \begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix}$$

$$\begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix} = \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix} \begin{bmatrix} D^{*-1} & R_3^* \\ R_2^* & R_4^* \end{bmatrix} \begin{bmatrix} D^* & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} DT_1 & DT_2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} DT_1 + DT_2 R_2^* D^* & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow T_2 = 0 \quad (\text{since } D \text{ is regular}).$$

$$\text{Therefore } QP^{*-1} = \begin{bmatrix} T_1 & 0 \\ T_3 & T_4 \end{bmatrix}$$

$$\text{Hence } A = P \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} T_1 & 0 \\ T_3 & T_4 \end{bmatrix} P^* = P \begin{bmatrix} DT_1 & 0 \\ 0 & 0 \end{bmatrix} P^* = P \begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix} P^*$$

where  $E = DT_1$  is a  $r \times r$  regular  $\lambda$ -matrix.

$$\text{Conversely, let } PAP^* = \begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix} \quad \text{where } E \text{ is a } r \times r \text{ regular } \lambda\text{-matrix.}$$

Since  $E$  is regular,  $E$  is  $EP_r$  over  $F(\lambda)$ .

$$\Rightarrow R(E) = R(E^*)$$

$$\Rightarrow R(PAP^*) = R(PA^*P^*)$$

$$\Rightarrow R(A) = R(A^*)$$

$$\Rightarrow A \text{ is } EP_r \text{ over } F(\lambda). \text{ Hence the theorem.}$$

If  $A \in F_r^{n \times n}[\lambda]$  and is  $EP$  over the field  $F(\lambda)$  then we can find  $n \times n$  regular rational  $\lambda$ -matrices  $H$  and  $K$  such that  $A^* = HA = AK$  [4]. In general the above  $H$  and  $K$  need not be unimodular  $\lambda$ -matrices. For example, consider  $A = \begin{bmatrix} 1 & \lambda \\ 0 & 2 \end{bmatrix}$ .  $A$  is

$EP_r$ , being a regular  $\lambda$ -matrix. If  $A^* = HA$  then  $H = A^* A^{-1}$ ; If  $A^* = AK$  then  $K = A^{-1} A^*$ . Here  $H = \begin{bmatrix} 1 & -1/\lambda \\ \lambda & 0 \end{bmatrix}$  and  $K = \begin{bmatrix} 0 & -\lambda \\ 1/\lambda & 1 \end{bmatrix}$  are not  $\lambda$ -matrices.

The following theorem gives a necessary condition for  $H$  and  $K$  to be unimodular  $\lambda$ -matrices.

**THEOREM 2.** If  $A$  is an  $n \times n$   $EP_r$ - $\lambda$ -matrix and  $A$  has a  $\lambda$ -matrix  $\{1\}$  inverse then there exist  $n \times n$  unimodular  $\lambda$ -matrices  $H$  and  $K$  such that  $A^* = HA = AK$ .

**PROOF.** Let  $A$  be an  $n \times n$   $EP_r$ - $\lambda$ -matrix. By Theorem 1, there exists an  $n \times n$  unimodular  $\lambda$ -matrix  $P$  such that  $PAP^* = \begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix}$  where  $E$  is a  $r \times r$  regular  $\lambda$ -matrix. Since  $A$  has a  $\lambda$ -matrix  $\{1\}$  inverse,  $E^{-1}$  is also a  $\lambda$ -matrix.

$$\begin{aligned} \text{Now } A &= P^{-1} \begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix} P^{-1*} \\ \text{Therefore } A^* &= P^{-1} \begin{bmatrix} E^* & 0 \\ 0 & 0 \end{bmatrix} P^{-1*} \\ &= P^{-1} \begin{bmatrix} E^* E^{-1} & 0 \\ 0 & I \end{bmatrix} PP^{-1} \begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix} P^{-1*} \\ &= HA \text{ where } H = P^{-1} \begin{bmatrix} E^* E^{-1} & 0 \\ 0 & I \end{bmatrix} P \text{ is an } n \times n \text{ unimodular} \\ &\lambda\text{-matrix. Similarly we can write } A^* = AK \text{ where} \end{aligned}$$

$$K = P^* \begin{bmatrix} E^{-1} E^* & 0 \\ 0 & I \end{bmatrix} P^{-1*} \text{ is an } n \times n \text{ unimodular } \lambda\text{-matrix.}$$

Therefore  $A^* = HA = AK$ .

**REMARK 1.** The converse of Theorem 2 need not be true. For example, consider  $A = \begin{bmatrix} \lambda & 0 \\ 0 & 0 \end{bmatrix}$ . Since  $A^* = A$ ,  $H = K = I_2$ .  $A$  is an  $EP_1$ - $\lambda$ -matrix. However  $A$  has no  $\lambda$ -matrix  $\{1\}$  inverse.

### 3. MOORE-PENROSE INVERSE OF AN $EP_r$ - $\lambda$ -MATRIX

The following theorem gives a set of necessary and sufficient conditions for the existence of the  $\lambda$ -matrix Moore-Penrose inverse of a given  $\lambda$ -matrix.

**THEOREM 3.** For  $A \in F_r^{n \times n}[\lambda]$ , the following statements are equivalent.

- i)  $A$  is  $EP_r$ ,  $\text{rk}(A) = \text{rk}(A^2)$  and  $A^* A$  has a  $\lambda$ -matrix  $\{1\}$  inverse.
- ii) There exists a unimodular  $\lambda$ -matrix  $U$  with  $A = U \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix} U^*$  where  $D$  is a  $r \times r$  unimodular  $\lambda$ -matrix and  $U^* U$  is a diagonal block matrix.
- iii)  $A = GLG^*$  where  $L$  and  $G^* G$  are  $r \times r$  unimodular  $\lambda$ -matrices and  $G$  is a  $\lambda$ -matrix.
- iv)  $A^+$  is a  $\lambda$ -matrix and  $EP_r$ .
- v) There exists a symmetric idempotent  $\lambda$ -matrix  $E$ , ( $E^2 = E = E^*$ ) such that  $AE = EA$  and  $R(A) = R(E)$ .

**PROOF.** (i)  $\Rightarrow$  (ii) Since  $A$  is an  $EP_r$ - $\lambda$ -matrix over the field  $F(\lambda)$  and  $\text{rk}(A) = \text{rk}(A^2)$ ,  $A^+$  exists, by Theorem 2.3 of [5]. By Theorem 4 in [6],  $A^* A$  has a  $\lambda$ -matrix  $\{1\}$  inverse implies that there exists a unimodular  $\lambda$ -matrix  $P$  with  $PP^* = \begin{bmatrix} P_1 & 0 \\ 0 & P_4 \end{bmatrix}$  where  $P_1$  is a symmetric  $r \times r$  unimodular  $\lambda$ -matrix such that

$PA = \begin{bmatrix} W \\ 0 \end{bmatrix}$  where  $W$  is a  $rxn$ ,  $\lambda$ -matrix of rank  $r$ . Hence by Theorem 2 in [6],  $AA^+$  is a  $\lambda$ -matrix and  $PAA^+P^* = \begin{bmatrix} P_1 & 0 \\ 0 & 0 \end{bmatrix}$ . Since  $A$  is  $EP_r$ ,  $AA^+ = A^+A$  and  $A = AA^+A = A(AA^+)$ . Therefore  $A = P^{-1} \begin{bmatrix} W \\ 0 \end{bmatrix} P^{-1} \begin{bmatrix} P_1 & 0 \\ 0 & 0 \end{bmatrix} P^{*-1}$

$$= P^{-1} \begin{bmatrix} W \\ 0 \end{bmatrix} [H \ 0] P^{*-1} \text{ where}$$

$H$  consists of the first  $r$  columns of  $P^*$ , thus  $H$  is a  $n \times r$ ,  $\lambda$ -matrix of rank  $r$ .

Now  $A = P^{-1} \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix} P^{-1*} = U \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix} U^*$  where  $U = P^{-1}$  and  $D = WH$  is a  $rxr$  regular  $\lambda$ -matrix. Since  $A^+$  has a  $\lambda$ -matrix  $\{1\}$  inverse and  $P$  is an unimodular  $\lambda$ -matrix,  $PAA^+P^* = \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix}$  has a  $\lambda$ -matrix

$\{1\}$  inverse. Therefore by Theorem 1 in [6],  $D^*P_1^{-1}D$  is an unimodular  $\lambda$ -matrix which implies  $D$  is an unimodular  $\lambda$ -matrix. Hence  $A = U \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix} U^*$  where  $D$  is a  $rxr$  unimodular  $\lambda$ -matrix and  $U^*U$  is a diagonal block  $\lambda$ -matrix.

Thus (ii) holds.

(ii)  $\Rightarrow$  (iii)

Let us partition  $U$  as  $U = \begin{bmatrix} U_1 & U_2 \\ U_3 & U_4 \end{bmatrix}$  where  $U_1$  is a  $rxr$   $\lambda$ -matrix. Then

$$A = \begin{bmatrix} U_1 & U_2 \\ U_3 & U_4 \end{bmatrix} \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} U_1^* & U_3^* \\ U_2^* & U_4^* \end{bmatrix} = \begin{bmatrix} U_1 \\ U_3 \end{bmatrix} D \begin{bmatrix} U_1^* & U_3^* \end{bmatrix} = GLG^*$$

where  $L = D$  and  $G = \begin{bmatrix} U_1 \\ U_3 \end{bmatrix}$  are  $\lambda$ -matrices.

Since  $U^*U$  is a diagonal block  $\lambda$ -matrix,  $G^*G = U_1^*U_1 + U_3^*U_3$  and  $L$  are  $rxr$  unimodular  $\lambda$ -matrices. Thus (iii) holds.

(iii)  $\Rightarrow$  (iv)

Since  $A = GLG^*$ ,  $L$  and  $G^*G$  are unimodular  $\lambda$ -matrices. One can verify that  $A^+ = G(G^*G)^{-1}L^{-1}(G^*G)^{-1}G^*$ .

Now  $AA^+ = GLG^*G(G^*G)^{-1}L^{-1}(G^*G)^{-1}G^* = G(G^*G)^{-1}G^* = A^+A$  implies that  $A^+$  is  $EP_r$ . Since  $L$  and  $G^*G$  are unimodular,  $L^{-1}$  and  $(G^*G)^{-1}$  are  $\lambda$ -matrices, and  $G$  is a  $\lambda$ -matrix. Therefore  $A^+$  is a  $\lambda$ -matrix. Thus (iv) holds.

(iv)  $\Rightarrow$  (v)

Proof is analogous to that of (ii)  $\Rightarrow$  (iii) of Theorem 2.3 [5].

(v)  $\Rightarrow$  (i)

Since  $E$  is a symmetric idempotent  $\lambda$ -matrix with  $R(A) = R(E)$  and  $AE = EA$ , by Theorem 2.3 in [5] we have  $A$  is  $EP_r$  and  $\text{rk}(A) = \text{rk}(A^2) \Rightarrow A^+$  exists. Since  $E^+ = E$  and  $R(A) = R(E) \Rightarrow AA^+ = EE^+ = E$ . Now  $AE = EA = (AA^+)A = A$ . Let  $e_j$  and  $a_j$  denote the  $j$ th columns of  $E$  and  $A$  respectively. Then  $AE = A \Rightarrow Ae_j = a_j$ , since  $e_j$  is a  $\lambda$ -matrix, the equation  $Ax = a_j$  where  $a_j$  is a  $\lambda$ -matrix, has a  $\lambda$ -matrix solution. Hence by Theorem 1 in [6] it follows that  $A$  has a  $\lambda$ -matrix  $\{1\}$  inverse. Further  $AA^+ = E$  is also a  $\lambda$ -matrix. Hence by Theorem 4 in [6] we see that  $A^*A$  has a  $\lambda$ -matrix  $\{1\}$  inverse. Thus (i) holds. Hence the theorem.

REMARK 2. The condition (i) in Theorem 3 cannot be weakened which can be seen by the following examples.

EXAMPLE 1. Consider the matrix  $A = \begin{bmatrix} \lambda & \lambda \\ \lambda & \lambda \end{bmatrix}$ .  $A$  is  $EP_1$  and  $\text{rk}(A) = \text{rk}(A^2) = 1$ .  $A^*A = \begin{bmatrix} 2\lambda^2 & 2\lambda^2 \\ 2\lambda^2 & 2\lambda^2 \end{bmatrix}$  has no  $\lambda$ -matrix  $\{1\}$  inverse (since the invariant polynomial of  $A^*A$  is  $\lambda^2$  which is not the identity of  $F$ ). For this  $A$ ,  $A^+ = \frac{1}{4\lambda} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$  is not a  $\lambda$ -matrix. Thus the theorem falls.

EXAMPLE 2. Consider the matrix  $A = \begin{bmatrix} \lambda & 2\lambda \\ 2\lambda & 4\lambda \end{bmatrix}$  over  $GF(5)$ .  $A$  is  $EP_1$ . Since  $A^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ ,  $\text{rk}(A) \neq \text{rk}(A^2)$ ,  $A^*A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$  has a  $\lambda$ -matrix

$\{1\}$  inverse (since any conformable  $\lambda$ -matrix is a  $\lambda$ -matrix  $\{1\}$  inverse). For this  $A$ ,  $A^+$  does not exist. Thus the theorem falls.

REMARKS 3. From Theorem 3, it is clear that if  $E$  is a symmetric idempotent  $\lambda$ -matrix, and  $A$  is a  $\lambda$ -matrix such that  $R(E) = R(A)$  then  $A$  is  $EP \Leftrightarrow AE = EA \Leftrightarrow A^+$  is a  $\lambda$ -matrix and  $EP$ .

We can show that the set of all  $EP_r$ - $\lambda$ -matrices with common range space as that of given symmetric idempotent  $\lambda$ -matrix forms a group, analogous to that of the Theorem 2.1 in [5].

COROLLARY 1. Let  $E = E^* = E^2 \in F[\lambda]^{n \times n}$ . Then  $H(E) = \{A \in F[\lambda]^{n \times n} : A \text{ is } EP_r \text{ over } F(\lambda) \text{ and } R(A) = R(E)\}$  is a maximal subgroup of  $F[\lambda]^{n \times n}$  containing  $E$  as identity.

PROOF. This can be proved similar to that of Theorem 2.1 of [5] by applying Theorem 3.

#### 4. APPLICATION

In general, if  $A$  and  $B$  are  $\lambda$ -matrices, having  $\lambda$ -matrix  $\{1\}$  inverses, it is not necessary that  $AB$  has a  $\lambda$ -matrix  $\{1\}$  inverse.

EXAMPLE 3. Consider  $A = \begin{bmatrix} 1 & \lambda \\ \lambda & \lambda^2 \end{bmatrix}$  and  $B = \begin{bmatrix} 1 & 0 \\ 2\lambda & 0 \end{bmatrix}$ . Here  $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$  is one of the  $\lambda$ -matrix  $\{1\}$  inverse for both  $A$  and  $B$ . But  $AB = \begin{bmatrix} 1+2\lambda^2 & 0 \\ \lambda+2\lambda^3 & 0 \end{bmatrix}$ . Since the invariant polynomial of  $AB$  is  $1+2\lambda^2 \neq 1$ ,  $AB$  has no  $\lambda$ -matrix  $\{1\}$  inverse.

The following theorem leads to the existence of  $\lambda$ -matrix  $\{1\}$  inverse of the product  $AB$ .

THEOREM 4. Let  $A, B \in F[\lambda]^{n \times n}$ . If  $A^2 = A$  and  $B$  has  $\lambda$ -matrix  $\{1\}$  inverse and  $R(A) \subseteq R(B)$  then  $AB$  has a  $\lambda$ -matrix  $\{1\}$  inverse.

PROOF. Suppose  $ABx = b$ , where  $b$  is a  $\lambda$ -matrix, is a consistent system. Then  $b \in R(AB) \subseteq R(A) \subseteq R(B)$  and therefore  $Bz_0 = b$ . Since  $B$  has a  $\lambda$ -matrix  $\{1\}$  inverse, by Theorem 1 in [6] we get  $z_0$  is a  $\lambda$ -matrix. Since  $A$  is idempotent, so in particular  $A$  is a  $\{1\}$  inverse of  $A$  and  $b \in R(A)$ , we have  $Ab = b$ . Now  $ABz_0 = Ab = b$ . Thus  $ABx = b$  has a  $\lambda$ -matrix solution. Hence by Theorem 1 in [6],  $AB$  has a  $\lambda$ -matrix  $\{1\}$  inverse. Hence the theorem.

The converse of Theorem 4 need not be true which can be seen by the following example.

EXAMPLE 4. Let  $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ ;  $B = \begin{bmatrix} 1 & 1 \\ \lambda & \lambda \end{bmatrix}$ ;  $AB = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ . Here  $A^2 = A$  and  $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$  is a  $\lambda$ -matrix  $\{1\}$  inverse for both  $AB$  and  $B$ . However

$R(A) \not\subseteq R(B)$ . Hence the converse is not true.

Next we shall discuss the necessary and sufficient condition for the Moore-Penrose inverse of the product of  $EP_r$ - $\lambda$ -matrices to be an  $EP_r$ - $\lambda$ -matrix.

**THEOREM 5.** Let  $A$  and  $B$  be  $EP_r$ - $\lambda$ -matrices. Then  $A^*A$  has a  $\lambda$ -matrix  $\{1\}$  inverse,  $\text{rk}(A) = \text{rk}(A^2)$  and  $R(A) = R(B)$  if and only if  $AB$  is  $EP_r$  and  $(AB)^+ = B^+A^+$  is a  $\lambda$ -matrix.

**PROOF.** Since  $A$  and  $B$  are  $EP_r$  with  $R(A) = R(B)$  and  $\text{rk}(A) = \text{rk}(A^2)$ , by a Theorem of Katz [1],  $AB$  is  $EP_r$ . Since  $A$  is a  $EP_r$ - $\lambda$ -matrix,  $\text{rk}(A) = \text{rk}(A^2)$  and  $A^*A$  has a  $\lambda$ -matrix  $\{1\}$  inverse, by Theorem 3,  $A^+$  is a  $\lambda$ -matrix and there exists a symmetric idempotent  $\lambda$ -matrix  $E$  such that  $R(A) = R(E)$ . Hence  $AA^+ = AA^+ = E$ . Since  $A$  and  $B$  are  $EP_r$  and  $R(A) = R(B)$ , we have  $AA^+ = BB^+ = E = A^+A = B^+B$ . Therefore  $BE = EB$  and  $R(B) = R(E)$ . Again from Theorem 3, for the  $EP_r$ - $\lambda$ -matrix  $B$ , we see that  $B^+$  is a  $\lambda$ -matrix. Since  $A$  and  $B$  are  $EP_r$  with  $R(A) = R(B)$ , we can verify that  $(AB)^+ = B^+A^+$ . Since  $B^+$  and  $A^+$  are  $\lambda$ -matrices, it follows that  $(AB)^+$  is a  $\lambda$ -matrix.

Conversely, if  $(AB)^+$  is a  $\lambda$ -matrix and  $AB$  is  $EP_r$  then  $(AB)^+$  is an  $EP_r$ - $\lambda$ -matrix. Therefore by Theorem 3, there exists a symmetric idempotent  $\lambda$ -matrix  $E$  such that  $R(AB) = R(E)$  and  $(AB)(AB)^+ = E = (AB)^+(AB)$ . Since  $\text{rk}(AB) = \text{rk}(A) = r$  and  $R(AB) \subseteq R(A)$ , we get  $R(A) = R(E)$ . Since  $A$  is  $EP_r$ , by Remark 3, it follows that  $A^+$  is a  $EP_r$ - $\lambda$ -matrix. Now by Theorem 3,  $A^*A$  has a  $\lambda$ -matrix  $\{1\}$  inverse and  $\text{rk}(A) = \text{rk}(A^2)$ . Since  $AB$  and  $B$  are  $EP_r$ ,  $R(E) = R(AB) = R((AB)^+) \subseteq R(B^+) = R(B)$  and  $\text{rk}(AB) = \text{rk}(B)$  implies  $R(B) = R(E)$ . Therefore  $R(A) = R(B)$ . Hence the theorem.

**REMARK 4.** The condition that both  $A$  and  $B$  are  $EP_r$ - $\lambda$ -matrices, is essential in Theorem 5, is illustrated as follows:

Let  $A = \begin{bmatrix} 1 & \lambda \\ 0 & 0 \end{bmatrix}$  and  $B = \begin{bmatrix} 1 & 2\lambda \\ 0 & 0 \end{bmatrix}$ .  $A$  and  $B$  are not  $EP_1$ .  
 $A^*A = \begin{bmatrix} 1 & \lambda \\ \lambda & \lambda^2 \end{bmatrix}$  has a  $\lambda$ -matrix  $\{1\}$  inverse and  $R(A) = R(B)$ . But  $AB$  is not  $EP_1$ .  
 $(AB)^+ = \frac{1}{1+4\lambda^2} \begin{bmatrix} 1 & 0 \\ 2\lambda & 0 \end{bmatrix}$  is not a  $\lambda$ -matrix. Hence the claim.

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#### REFERENCES

1. KATZ, I.J. Theorems on Products of  $EP_r$  Matrices II, Lin. Alg. Appl. **10** (1975), 37-40.
2. LANCASTER, P. and TISMENETSKY, M. Theory of Matrices, 2nd ed., Academic Press, 1985.
3. MARSAGLIA, G. and STYAN, G.P.H. Equalities and inequalities for ranks of matrices, Lin. and Multi. Alg. **2** (1974), 269-292.
4. MARTIN PEARL, On Normal and  $EP_r$  Matrices, Mich. Math. J. **6** (1959), 1-5.
5. MEENAKSHI, AR. On  $EP_r$  Matrices with Entries from an Arbitrary Field, Lin. and Multi. Alg. **9** (1980), 159-164.
6. MEENAKSHI, AR. and ANANDAM, N. Polynomial Generalized Inverses of Polynomial Matrices, (submitted).
7. PEARL, M.H. Generalized Inverses of Matrices with Entries Taken from an Arbitrary Field, Lin. Alg. Appl. **1** (1968), 571-587.

## Special Issue on Time-Dependent Billiards

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