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ON Θ-C-OPEN SETS

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ABSTRACT. The properties of the collection of complements of θ-closures of sets in a topological space are investigated in this paper. A strong continuity condition is defined in terms of these sets. Some applications to H-closed spaces and Katetov spaces are given.

KEY WORDS AND PHRASES. θ-C-open, θ-C-continuous, super-continuous, H-closed space, Katetov space.
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1. PRELIMINARIES.

All spaces are topological spaces with no separation axioms assumed unless explicitly stated. Let A be a subset of a space X. The closure of A and the interior of A are denoted by Cl A and Int A, respectively. The set A is said to be regular open (regular closed) if A = Int Cl A (A = Cl Int A). The θ-closure (δ-closure) (Velicko [1]) of A is the set of all x in X such that every closed neighborhood (the interior of every closed neighborhood) of x intersects A nontrivially. The θ-closure and the δ-closure of A are denoted by Cl_θ A and Cl_δ A, respectively. The set A is called θ-closed (δ-closed) if A = Cl_θ A (A = Cl_δ A). A set A is said to be θ-open (δ-open) if its complement is θ-closed (δ-closed). For a given space X both the collection of all θ-open sets and the collection of all δ-open sets form topologies. The collection of δ-open sets is usually referred to as the semi-regular topology.

DEFINITION 1. Arya and Gupta [2]. A function $f: X \to Y$ is said to be completely continuous if for each open subset V of Y, $f^{-1}(V)$ is regular open in X.

DEFINITION 2. Munshi and Basson [3]. A function $f: X \to Y$ is said to be super-continuous if for each $x \in X$ and each open neighborhood V of f(x), there exists an open neighborhood U of of x for which f(Int Cl U) $\subseteq$ V.

DEFINITION 3. Long and Herrington [4]. A function f: X $\to$ Y is said to be strongly θ-continuous if for each x $\in$ X and each open neighborhood V of f(x), there exists an open neighborhood U of x for which f(Cl U) $\subseteq$ V.

DEFINITION 4. Porter and Tikoo [5]. A space X is said to be H-closed if X is a closed subset in every space containing X as a subspace.

DEFINITION 5. Porter and Tikoo [5]. A space is said to be Katétov if it has a coarser minimal H-closed topology or equivalently a coarser H-closed topology.

2. θ-C-OPEN SETS

We define a subset U of a space X be θ-C-open provided there exists a subset A of X for which X - U = Cl_θ A. We call a set θ-C-closed if its complement is θ-C-open or equivalently if there is a subset A of X such that the set equals Cl_θ A.

THEOREM 1. If U is θ-open, then U is θ-C-open.

PROOF. Since U is θ-open, X - U is θ-closed. Hence X - U = Cl_θ(X - U).

THEOREM 2. If U is open, then Int Cl U is θ-C-open.

PROOF. Int Cl U = X - Cl(X - Cl U). Since X - Cl U is open, Cl(X - Cl U) = Cl_θ(X - Cl U) (Velicko [1]). Hence X - Int Cl U = Cl_θ(X - Cl U).

COROLLARY. If U is regular open, then U is θ-C-open.

Since the real numbers with the usual topology contain θ-open sets that are not regular open, it follows that the real numbers contain θ-C-open sets that are not regular open.

THEOREM 3. Regular openness is equivalent to θ-C openness if and only if Cl_θ A is regular closed for every set A.

PROOF. Let X be a space. Assume regular openness is equivalent to θ-C-openness and let A $\subseteq$ X. Then X - Cl_θ A is regular open. Thus X - Cl_θ A = Int Cl(X - Cl_θ A) = Int(X - Int Cl_θ A) = X - Cl Int Cl_θ A. Therefore Cl_θ A = Cl Int Cl_θ A which implies that Cl_θ A is regular closed.

Assume Cl_θ A is regular closed for every set A. Suppose U is θ-C-open and let A $\subseteq$ X such that U = X - Cl_θ A. Then Int Cl U = Int Cl(X - Cl_θ A) = Int(X - Int Cl_θ A) = X - Cl Int Cl_θ A = = X - Cl_θ A = U. Therefore U is regular open and hence regular openness is equivalent to θ-C-openness.

THEOREM 4. If U is θ-C-open, then U is a union of regular open sets (that is δ-open).

PROOF. Let U be θ-C-open. Let x $\in$ U. Since U is θ-C-open, there exists a set A $\subseteq$ X such that U = X - Cl_θ A. Because x $\notin$ Cl_θ A, there exists an open set W for which x $\in$ W and (Cl W) $\cap$ A = $\emptyset$. Hence x $\in$ Int Cl W $\subseteq$ X - Cl_θ A = U. Thus U is a union of regular open sets.

It follows from Theorem 4 and the corollary to Theorem 2 that the θ-C-open sets form a basis for the semi-regular topology.

THEOREM 5. The intersection of two θ-C-open sets is θ-C-open.

PROOF. Let U and V be θ-C-open sets. There exist sets A and B such that $U = X - Cl_\theta A$ and $V = X - Cl_\theta B$. Then $U \cap V = (X - Cl_\theta A) \cap (X - Cl_\theta B) = X - (Cl_\theta(A) \cup Cl_\theta(B)) = X - Cl_\theta(A \cup B)$.

The following example shows that the union of two θ-C-open sets need not be θ-C-open. It follows that the θ-C-open sets do not form a topology and hence θ-C-openness is not equivalent to either δ-openness or θ-openness.

EXAMPLE 1. Let $X = \{a, b, c\}$ and $\mathfrak{T} = \{X, \emptyset, \{a\}, \{c\}, \{a, c\}\}$. The θ-C-open sets of X are X, ∅, {a}, and {c}.

3. θ-C-CONTINUITY.

We define a function $f: X \to Y$ to be θ-C-continuous if for each open subset V of Y, $f^{-1}(V)$ is θ-C-open in X. Since θ-C-open sets are open, obviously θ-C-continuity implies continuity. Since by Theorem 2 regular openness implies θ-C-openness, complete continuity implies θ-C-continuity. The identity mapping on the real numbers with the usual topology is θ-C-continuous but not completely continuous.

THEOREM 6. (Munshi and Basson [3]) A function $f: X \to Y$ is super-continuous if and only if the inverse image of each open set in Y is δ-open in X.

By Theorem 4 every θ-C-open set is δ-open. Hence θ-C-continuity implies super-continuity. The identity mapping on the space in Example 1 is super-continuous but not θ-C-continuous.

It also follows from Theorem 4 that the corresponding "local" or "pointwise" version of θ-C-continuity is equivalent to super-continuity.

THEOREM 7. A function $f: X \to Y$ is super-continuous if and only if for each $x \in X$ and each open neighborhood V of f(x), there exists a θ-C-open set $U \subseteq X$ for which $x \in U$ and $f(U) \subseteq V$.

THEOREM 8. (Long and Herrington [4]). A function $f: X \to Y$ is strongly θ-continuous if and only if the inverse image of each open set in Y is θ-open in X.

From Theorem 1 θ-openness implies θ-C-openness. Hence strong θ-continuity implies θ-C-continuity.

EXAMPLE 2. Let $X = \{a. b, c\}$, $\mathfrak{T}_1 = \{X, \emptyset, \{a\}, \{c\}, \{a, c\}\}$ and $\mathfrak{T}_2 = \{X, \emptyset, \{a\}\}$. The identity mapping $(X, \mathfrak{T}_1) \to (X, \mathfrak{T}_2)$ is θ-C-continuous but not strongly θ-continuous.

Based upon the above theorems and remarks, we have the following implications, none of which are reversible.

$$\left.\begin{array}{l}\text{complete continuity}\\ \text{strong } \theta\text{-continuity}\end{array}\right\} \Rightarrow \theta\text{-C-continuity} \Rightarrow \text{super-continuity}$$

THEOREM 9. If $f: X \to Y$ is θ-C-continuous and $Cl_\theta A$ is regular closed for every subset A of X, then f is completely continuous.

PROOF. By Theorem 3 θ-C-openness is equivalent to regular openness.

THEOREM 10. If $f: X \to Y$ is θ-C-continuous and for every subset A of X, Cl_θ A is θ-closed, then f is strongly θ-continuous.

PROOF. Follows from Theorem 8.

The following theorems and examples illustrate some of the basic properties of θ-C-continuous functions.

THEOREM 11. If $f: X \to Y$ is θ-C-continuous and $g: Y \to Z$ is continuous, then $g \circ f: X \to Z$ is θ-C-continuous.

The proof is routine.

COROLLARY. The composition of two θ-C-continuous functions is θ-C-continuous.

THEOREM 12. Let $f_\alpha: X \to Y_\alpha$ be a function for each α in A and let $f: X \to \Pi Y_\alpha$ be given by $f(x) = (f_\alpha(x))$. If f is θ-C-continuous, then f_α is θ-C-continuous for each α in A.

PROOF. For each $\alpha \in A$ denote the projection onto Y_α by p_α. Then $f_\alpha = p_\alpha \circ f$ is θ-C-continuous by Theorem 11.

The proof of the next theorem follows from Theorem 12.

THEOREM 13. Let $f: X \to Y$ be a function and let $g: X \to X \times Y$ given by $g(x) = (x, f(x))$ be its graph function. If g is θ-C-continuous, then f is θ-C-continuous.

The following example shows that the converse of Theorem 13 does not hold.

EXAMPLE 3. Let $X = \{a, b, c\}$ and $\mathfrak{T} = \{X, \emptyset, \{a\}, \{c\}, \{a, c\}\}$. Define $f: X \to X$ by $f(a) = f(b) = f(c) = a$. Then f is θ-C-continuous, but its graph function is not θ-C-continuous since $g^{-1}(\{(a, a), (c, a)\}) = \{a, c\}$ which is not θ-C-open.

The proof of the following theorem is straightforward and is omitted.

THEOREM 14. A function $f: X \to Y$ is θ-C-continuous if and only if for each closed subset F of Y, there exists a subset A of X for which $f^{-1}(F) = Cl_\theta A$.

THEOREM 15. If the functions $f, g: X \to Y$ are θ-C-continuous and Y is Hausdorff, then the set $A = \{x : f(x) \neq g(x)\}$ is a union of θ-C-open sets.

PROOF. Let $x \in A$. Since $f(x) \neq g(x)$ and Y is Hausdorff, there exist disjoint open sets V and W containing f(x) and g(x), respectively. Then $f^{-1}(V)$ and $g^{-1}(W)$ are θ-C-open. By Theorem 5 $f^{-1}(V) \cap g^{-1}(W)$ is θ-C-open. Obviously $x \in f^{-1}(V) \cap g^{-1}(W) \subseteq A$.

COROLLARY. If the functions $f, g: X \to Y$ are θ-C-continuous, then the set $B = \{x : f(x) = g(x)\}$ is δ-closed.

PROOF. By Theorem 15 X - B is a union of θ-C-open sets and by Theorem 4 each θ-C-open set is a union of regular open sets.

For a function $f: X \to Y$ the graph of f, denoted by G(f), is the subset $\{(x, f(x)) : x \in X\}$ of the product space $X \times Y$.

THEOREM 16. If $f: X \to Y$ is θ-C-continuous and Y is Hausdorff, then $X \times Y - G(f)$ is a union of sets of the form $A \times B$ where A is θ-C-open and B is open.

PROOF. Let $(x, y) \in X \times Y - G(f)$. There exist disjoint open sets V and W for which $f(x) \in V$ and $y \in W$. Then $f^{-1}(V)$ is θ-C-open and $(x, y) \in f^{-1}(V) \times W \subseteq X \times Y - G(f)$.

The following theorem is easily proved.

THEOREM 17. If $f: X \to Y$ is a θ-C-continuous injection and Y is Hausdorff, then points in X can be separated by θ-C-open sets.

4. APPLICATIONS TO H-CLOSED SPACES AND KATETOV SPACES

In this section all spaces are assumed to be Hausdorff.

Since H-closed spaces and Katetov spaces are related to the θ-closures of sets, there are natural relationships between these spaces and θ-C-open sets and θ-C-continuity. The following results are required.

THEOREM 18. (Porter and Tikoo [5]). If X is an H-closed space and $A \subseteq X$, then $Cl_\theta A$ is Katetov.

THEOREM 19. (Porter and Tikoo [5]). An H-closed space in which every closed set is the θ-closure of some set is compact.

The next result follows immediately from Theorems 14 and 18.

Theorem 20. If X is H-closed and $f: X \to Y$ is θ-C-continuous, then for each closed subset F of Y, $f^{-1}(F)$ is a Katetov subspace of X.

As a consequence of Theorem 19 we have the following result.

Theorem 21. If X is H-closed and every open set is θ-C-open, then X is compact.

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REFERENCES

1. Velicko, N.V. H-closed topological spaces, Amer. Math. Soc. Transl. 78 (1968), 103-118
2. Arya, S.P. and Gupta, R. On strongly continuous mappings, Kyungpook Math J. 14 (1974), 131-143.
3. Munshi, B.M. and Basson, D.S Super-continuous mappings, Indian J. Pure Appl. Math. 13 (1982), 229-236.
4. Long, P.E. and Herrington, L.L. Strongly θ-continuous functions, J. Korean Math. Soc. 18 (1981), 21-28.
5. Porter, J. and Tikoo, M. On Katetov spaces, Canad. Math. Bull. 32 (1989), 425-433.

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