ON THE DEGREE OF APPROXIMATION OF THE
HERMITE AND HERMITE-FEJER INTERPOLATION

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ABSTRACT.   Here we find the order of convergence of the Hermite and Hermite-Fejér interpolation polynomials constructed on the zeros of $(1 - x^2)P_n(x)$ where $P_n(x)$ is the Legendre polynomial of degree n with normalization $P_n(1) = 1$.

KEY WORDS AND PHRASES.   Zeros, modulus of continuity, interior points, interpolation process, best approximation.

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1.   INTRODUCTION.

Let

$$-1 = x_{n+1} < x_n < \ldots < x_1 < x_0 = 1 \tag{1.1}$$

be the $n + 2$ distinct zeros of $(1 - x^2)P_n(x)$ where $P_n(x)$ is the Legendre polynomial of degree n with normalization $P_n(1) = 1$.   Let f be a given function on $[-1,1]$.   Let $Q_n(f,x)$ be the unique polynomial of degree $\leq 2n + 1$ such that

$$Q_n(f,x_k) = f(x_k), \quad Q_n(f, \pm 1) = f(\pm 1), \quad Q'_n(f,x_k) = 0, \quad k = 1,2,\ldots,n. \tag{1.2}$$

Then it is known [11] that

$$Q_n(f,x) = f(-1)\frac{1-x}{2}P_n^2(x) + f(1)\frac{1+x}{2}P_n^2(x) + \sum_{k=1}^{n} f(x_k)\frac{1-x^2}{1-x_k^2} 1_k^2(x) \tag{1.3}$$

where

$$1_k(x) = \frac{P_n(x)}{(x - x_k)P'_n(x_k)} \tag{1.4}$$

is the fundamental polynomial of the Lagrange interpolation.   According to a well-known result of Szasz [11]

$$\lim_{n\to\infty} Q_n(f,x) = f(x),$$

uniformly on $[-1,1]$, for every f continuous there.   A quantitative version of Szasz's result was given by Prasad and Saxena [8] who showed that

$$|Q_n(f,x) - f(x)| \leq c_1 n^{-1} \sum_{k=1}^{n} w_f\left(\frac{\sqrt{1-x^2}}{k} + \frac{1}{k^2}\right) \tag{1.5}$$

where w_f is the modulus of continuity of f on $[-1,1]$ on c_1 (later on $c_2, c_3...$) is a positive absolute constant.   Prasad and Varma [9] further improved (1.5) by proving that

$$|Q_n(f,x) - f(x)| \leq c_2 n^{-1} \sum_{k=1}^{n} w_f\left(\frac{\sqrt{1-x^2}}{k}\right). \tag{1.6}$$

Our aim is here to consider the interpolation process which requires that the derivative of the polynomial vanishes not only at the interior points $x_k, k = 1,2,...,n$, but also at the end points -1 and 1.   As we will see, the situation is quite different in this case. Let us denote by $R_n(f,x)$ the unique polynomial of degree $\leq 2n+3$ satisfying the conditions

$$R_n(f,x_k) = f(x_k), \quad R_n(f,\pm 1) = f(\pm 1), \quad R'_n(f,x_k) = 0, \quad R'_n(f,\pm 1) = 0, \quad k = 1,2,...,n. \tag{1.7}$$

Then from (1.2), (1.3) and (1.7) it follows that

$$R_n(f,x) = Q_n(f,x) + (1-x)\left[\frac{(1+x)P_n(x)}{2}\right]^2 Q'_n(f,1)$$

$$- (1+x)\left[\frac{(1-x)P_n(x)}{2}\right]^2 Q'_n(f,-1). \tag{1.8}$$

Bojanic, Varma and Vertesi [3] proved the following:

THEOREM A.   Let $f \in C[-1,1]$.   In the case when $\alpha \in [-\frac{1}{2}, \frac{1}{2})$ the necessary and sufficient conditions for

$$\lim_{n \to \infty} \| R_n^{(\alpha,\alpha)}(f,x) - f(x) \| = 0 \tag{1.9}$$

is given by

$$\lim_{n \to \infty} \int_{-1}^{1} \left(R_n^{(\alpha,\alpha)}(f,x) - f(x) \right) dx = 0$$

and

$$\lim_{n \to \infty} \int_{-1}^{1} x \left(R_n^{(\alpha,\alpha)}(f,x) - f(x) \right) dx = 0.$$

If $\alpha \in [\frac{1}{2}, 2)$, (1.9) holds true for arbitrary $f \in C[-1,1]$ (i.e., no conditions are needed).

In the case $\alpha \in [p-1,p)$, $p \geq 3$ (p integer) the necessary and sufficient conditions for the validity of (1.9) is given by

$$\left[R_n^{(\alpha,\alpha)}(f,x) \right]_{x = \pm 1}^{(r)} = o(n^{2r}), r = 1,2,...,p-1.$$

Here $R_n^{(\alpha,\alpha)}(f,x)$ is the polynomial of degree $\leq 2n+3$ satisfying the interpolatory conditions (1.7) on the zeros of ultraspherical polynomial.

For $R_n(f,x)$ satisfying the interpolatory conditions (1.7) on the zeros of $(1-x^2)P_n(x)$ we prove the following:

THEOREM 1.   Let $f \in C[-1,1]$ and let $R_n(f,x)$ be the Hermite-Fejer interpolation polynomial of degree $\leq 2n + 3$ defined by (1.7). Then for all $x \in [-1,1]$

$$| R_n(f,x) - f(x) | \leq \frac{c_3}{n} \sum_{k=1}^{n} w_f \left(\frac{\sqrt{1-x^2}}{k} \right) + c_4 \sqrt{1-x^2} \left[\frac{1}{n} + \sum_{k=1}^{n} w_f \left(\frac{1}{k^2} \right) \right]. \tag{1.10}$$

From (1.10) it is evident that the sequence $\{R_n(f,x)\}$ converges to $f(x)$ at the end points -1 and 1. Also, if $f \in \text{Lip}\sigma$, $1/2 < \sigma < 1$, then from (1.10) it follows that

$$| R_n(f,x) - f(x) | \leq \frac{c_5}{n} \sum_{k=1}^{n} \frac{(1-x^2)^{\sigma/2}}{k^\sigma} + c_6 \sqrt{1-x^2} n^{1-2\sigma}$$

$$\leq c_7 (1-x^2)^{\sigma/2} n^{-\sigma} + c_6 \sqrt{1-x^2} \, n^{1-2\sigma}$$

$$\leq c_8 (1-x^2)^{\frac{\sigma}{2}} n^{1-2\sigma}$$

Thus, $\{R_n(f,x)\}$ converges to $f(x)$ for $-1 \leq x \leq 1$ if $f \in \text{Lip}\sigma$, $\frac{1}{2} < \sigma < 1$. Next, we show that there is a $f \in C[-1,1]$ for which $\{R_n(f,x)\}$ diverges at $x = 0$. More precisely we prove the following:

THEOREM 2. The Hermite-Fejér interpolation process $\{R_n(f,x)\}$ for the function $f(x) = -(1-x^2)^\sigma$, $0 < \sigma \leq \frac{1}{2}$ with the nodes (1.1) diverges at the point $x = 0$.

This result is similar to a result of Berman [1] who proved that the Hermite-Fejér interpolation process $\{H_n(f,x)\}$ constructed for $f(x) = |x|$ with

$$x_k = \cos \frac{2k-1}{2n} \pi, \quad k = 1,2,...,n, x_0 = 1, \quad x_{n+1} = -1; \quad n = 1,2,...,$$

diverges at $x = 0$.

Let $F_n(f,x)$ be the unique polynomial of degree $\leq 2n + 1$ satisfying the conditions

$$F_n(f,-1) = f(-1), \quad F_n(f,1) = f(1),$$

$$F_n(f,x_k) = f(x_k), \quad F'_n(f,x_k) = f'(x_k), \quad k = 1,2,...n, \tag{1.11}$$

where $x'_k s$ are given by (1.1). Then we see that

$$F_n(f,x) = Q_n(f,x) + \sum_{k=1}^{n} f'(x_k) \frac{1-x^2}{1-x_k^2} (x - x_k) 1_k^2(x) \tag{1.12}$$

$$= \sum_{k=0}^{n+1} f(x_k) h_k(x) + \sum_{k=1}^{n} f'(x_k) \sigma_k(x)$$

where

$$h_0(x) = \frac{1+x}{2} P_n^2(x), \quad h_{n+1}(x) = \frac{1-x}{2} P_n^2(x) \tag{1.13}$$

and

$$h_k(x) = \frac{1-x^2}{1-x_k^2} 1_k^2(x), \quad \sigma_k(x) = (x-x_k) h_k(x), \quad k = 1,2,...,n. \tag{1.14}$$

For the polynomials $F_n(f,x)$ we prove the following:

THEOREM 3. Let f be defined and have a continuous derivative f' on $[-1,1]$. Then for the Hermite interpolation polynomial $F_n(f,x)$ of degree $\leq 2n+1$ defined by (1.11) we have for all $x \in [-1,1]$,

$$| F_n(f,x) - f(x) | \leq c_9 \frac{\log n}{n} E_{2n}(f')$$

where $E_{2n}(f')$ is the best approximation of $f'(x)$ by polynomials of degree at most $2n$.

Now, if we compare the zeros of $(1-x^2)P_n(x)$ with the zeros of $(1-x^2)T_n(x)$, the nth degree Tchebycheff polynomial of the first kind, we find that they are equally good as far as the convergence and the order of convergence of the Hermite and the Hermite Fejér interpolation is concerned.

2. PRELIMINARIES. In this section we state a few known results which we shall use later on.

From [5] we have for $-1 \leq x \leq 1$,

$$P_n^2(x) + \sum_{k=1}^{n} h_k(x) = 1. \tag{2.1}$$

Further, due to Fejér [7] we know that

$$\sum_{k=1}^{n} \frac{1}{(1-x_k^2)[P_n'(x_k)]^2} = 1. \tag{2.2}$$

Also, from Szegö [12] we have

$$(1-x^2)^{1/4} | P_n(x) | \leq \left(\frac{2}{\pi}\right)^{1/2} n^{-1/2}, \quad -1 \leq x \leq 1, \tag{2.3}$$

$$(1-x^2) > \left(k - \tfrac{1}{2}\right)^2 \left(n + \tfrac{1}{2}\right)^{-2}, \quad k = 1,2,...,[\tfrac{n}{2}], \tag{2.4}$$

$$(1-x_k^2) > \left(n - k + \tfrac{1}{2}\right)^2 \left(n + \tfrac{1}{2}\right)^{-2}, \quad k = [\tfrac{n}{2}]+1,...,n, \tag{2.5}$$

$$| P_n'(x_k) | \sim k^{-3/2} n^2, \quad k = 1,2,...[\tfrac{n}{2}], \tag{2.6}$$

$$| P_n'(x_k) | \sim (n+1-k)^{-3/2} n^2, \quad k = [\tfrac{n}{2}]+1,...,n, \tag{2.7}$$

$$| P_n(0) | = \frac{1 \cdot 3 \cdot 5 \cdots (n-1)}{2 \cdot 4 \cdots (n-2)n} \tag{2.8}$$
$$> \tfrac{1}{3} n^{-1/2}$$

and

$$\frac{(k-1/2)\pi}{n+1/2} < \Theta_k < \frac{k\pi}{n+1/2}, \quad k = 1,2,...,n, \quad x = \cos\Theta \text{ and } x_k = \cos\Theta_k. \tag{2.9}$$

Further, from [9] we also have for $-1 \leq x \leq 1$ and $k = 1,2,...,n$,

$$\frac{(1-x^2)^{1/4} | P_n(x) |}{(1-x_k^2)^{3/4} | P_n'(x_k) |} \leq \frac{c_{10}}{n}, \; n \geq 2, \tag{2.10}$$

and

$$\frac{(1-x^2)^{1/4}}{(1-x_k^2)^{1/4}} | 1_k(x) | \leq c_{11}. \tag{2.11}$$

Next, from [6] it follows that for $-1 \leq x \leq 1$,

$$| \, 1_k(x) \, | \leq c_{12}, \quad k = 1, 2, ..., n. \tag{2.12}$$

3. SOME LEMMAS. In this section we state and prove a few lemmas which will enable us to prove the theorems.

First, we assume x_j to be that zero of $P_n(x)$ which is nearest to x. From (2.9) and using the fact that x_j is the nearest zero to x we get

$$\frac{1}{\sin\dfrac{|\Theta - \Theta_k|}{2}} \leq \frac{2n+1}{2i-1}, \quad k \neq j, \quad k = j \pm i. \tag{3.1}$$

Also, we note that

$$\sin \Theta \leq \sin \Theta + \sin \Theta_k \leq 2 \sin \left(\frac{\Theta + \Theta_k}{2} \right). \tag{3.2}$$

LEMMA 1. If the polynomials $Q_n(f,x)$ and $R_n(f,x)$ are defined by (1.2) and (1.7), respectively, then

$$R_n(f,x) = Q_n(f,x) + \frac{x(1-x^2)}{4} P_n^2(x) [f(1) - f(-1)]$$

$$+ \frac{1}{2}(1-x^2)(1+x) P_n^2(x) \sum_{k=1}^{n} \frac{[f(1) - f(x_k)]}{(1-x_k^2)(1-x_k)^2 [P_n'(x_k)]^2}$$

$$+ \frac{1}{2}(1-x^2)(1-x) P_n^2(x) \sum_{k=1}^{n} \frac{[f(-1) - f(x_k)]}{(1-x_k^2)(1+x_k)^2 [P_n'(x_k)]^2}$$

PROOF. From (2.1) it follows that

$$P_n'(1) = \sum_{k=1}^{n} \frac{1}{(1-x_k^2)(1-x_k)^2 [P_n'(x_k)]^2} \, . \tag{3.3}$$

Also, it is easy to see that

$$P_n(-1)P_n'(-1) = -P_n'(1) \, . \tag{3.4}$$

Consequently, from (1.3, (3.3) and (3.4) we obtain

$$Q_n'(f,1) = \frac{1}{2}[f(1) - f(-1)] + 2 \sum_{k=1}^{n} \frac{[f(1) - f(x_k)]}{(1-x_k^2)(1-x_k)^2 [P_n'(x_k)]^2} \, . \tag{3.5}$$

and

$$Q_n'(f,-1) = \frac{1}{2}[f(1) - f(-1)] - 2 \sum_{k=1}^{n} \frac{[f(-1) - f(x_k)]}{(1-x_k^2)(1+x_k)^2 [P_n'(x_k)]^2} \, . \tag{3.6}$$

Substitution into (1.8) yields the desired result.

LEMMA 2. Let f be a continuous function on $[-1,1]$ and let x_k, $k = 1, 2, ..., n$, be the zeros of $P_n(x)$, the nth degree Legendre polynomial, then

$$| \sum_{k=1}^{n} \frac{[f(\pm 1) - f(x_k)]}{(1-x_k^2)(1 \mp x_k)^2 [P_n'(x_k)]^2} | \leq c_{13} \, n \sum_{k=1}^{n} w_f \left(\frac{1}{k^2} \right) \, .$$

PROOF. It is clearly sufficient to consider one choice of signs. Let us put $m = [\frac{n}{2}]$ and

consider

$$| \Delta_n(f) | = \left| \sum_{k=1}^{n} \frac{[f(1) - f(x_k)]}{(1 - x_k^2)(1 - x_k)^2 [P'_n(x_k)]^2} \right|$$

$$\leq \sum_{k=1}^{m} \frac{|f(1) - f(x_k)|}{(1 - x_k^2)(1 - x_k)^2 [P'_n(x_k)]^2}$$

$$+ \sum_{k=m+1}^{n} \frac{|f(1) - f(x_k)|}{(1 - x_k^2)(1 - x_k)^2 [P'_n(x_k)]^2}$$

$$= I_1 + I_2 . \tag{3.7}$$

First, we estimate I_2. On using (2.2) we obtain

$$I_2 \leq w_f(2) \sum_{k=m+1}^{n} \frac{1}{(1 - x_k^2)[P'_n(x_k)]^2} \tag{3.8}$$

$$\leq w_f(2) \sum_{k=1}^{n} \frac{1}{(1 - x_k^2)[P'_n(x_k)]^2}$$

$$\leq w_f(2) .$$

Next, we consider

$$I_1 = \sum_{k=1}^{m} \frac{1}{(1 - x_k^2)(1 - x_k)^2 [P'_n(x_k)]^2}$$

$$\leq \sum_{k=1}^{m} \frac{w_f(1 - x_k)}{1 - x_k^2} \left[\frac{1}{(1 - x_k) P'_n(x_k)} \right]^2 . \tag{3.9}$$

Since

$$1 - x_k = 1 - \cos \Theta_k = 2\sin^2 \frac{\Theta_k}{2}$$

hence on using (2.9) we see that

$$\frac{k^2}{2(n+1)^2} \leq 1 - x_k \leq \frac{20k^2}{(n+1)^2} . \tag{3.10}$$

Thus from (3.9), (3.10), (2.4) and (2.6) it follows that

$$I_1 \leq c_{14} n^2 \sum_{k=1}^{n} \frac{1}{k^3} w_f \left(\frac{k^2}{(n+1)^2} \right) . \tag{3.11}$$

Consequently, from (3.7), (3.8) and (3.11) we obtain

$$| \Delta_n(f) | \leq c_{15} n^2 \sum_{k=1}^{n} \frac{1}{k^3} w_f \left(\frac{k^2}{(n+1)^2} \right) . \tag{3.12}$$

Now following the same lines as in [3] we get

$$\sum_{k=1}^{n} \frac{1}{k^3} w_f \left(\frac{k^2}{(n+1)^2} \right) \leq \frac{2}{n+1} \sum_{k=1}^{n} w_f \left(\frac{1}{k^2} \right) . \tag{3.13}$$

Hence from (3.12) and (3.13) we conclude that

$$|\Delta_n(f)| \leq c_{16} \, n \sum_{k=1}^{n} w_f\left(\frac{1}{k^2}\right).$$

LEMMA 3. If $-1 \leq x \leq 1$ then

$$\sum_{k=1}^{n} \frac{|\sigma_k(x)|}{\sqrt{1-x_k^2}} \leq c_{17} \frac{\log n}{n}.$$

PROOF. From (1.14) we have

$$\sum_{k=1}^{n} \frac{|\sigma_k(x)|}{\sqrt{1-x_k^2}} = \sum_{k=1}^{n} \frac{(1-x^2)\,|x-x_k|}{(1-x_k^2)^{3/2}} \, 1_k^2(x) \tag{3.14}$$

$$= \frac{(1-x^2)\,|x-x_j|}{(1-x_j^2)^{3/2}} \, 1_j^2(x) + \sum_{k \neq j} \frac{(1-x^2)\,|x-x_k|}{(1-x_k^2)^{3/2}} \, 1_k^2(x) = J_1 + J_2.$$

If x_j is the zero of $P_n(x)$ which is nearest to x then one can easily see that

$$\frac{(1-x^2)^{1/2}}{(1-x_j^2)^{1/2}} \leq c_{18}. \tag{3.15}$$

Now from (2.10), (2.11), (2.12) and (3.15) it follows that

$$J_1 = \frac{(1-x^2)\,|x-x_j|}{(1-x_k^2)^{3/2}} \, 1_j^2(x) \tag{3.16}$$

$$= \left[\frac{(1-x^2)^{1/4}\,|P_n(x)\}}{(1-x_j^2)^{3/4}\,|P'_n(x_j)|}\right] \left[\frac{(1-x^2)^{1/4}}{(1-x_j^2)^{1/4}}\,|1_j(x)|\right] \left[\frac{(1-x^2)^{1/2}}{(1-x_j^2)^{1/2}}\right]$$

$$\leq c_{19} n^{-1}.$$

Next, on using (2.10), (3.2) and (3.1) we obtain

$$J_2 = \sum_{k \neq j} \frac{(1-x^2)\,|x-x_k|}{(1-x_k^2)^{3/2}} \, 1_k^2(x) \tag{3.17}$$

$$\leq c_{20} n^{-2} \sum_{k \neq j} \frac{(1-x^2)^{1/2}}{|x-x_k|}$$

$$\leq c_{20} n^{-2} \sum_{k \neq j} \frac{1}{\sin\frac{|\Theta-\Theta_k|}{2}}$$

$$\leq c_{21} \frac{\log n}{n}.$$

Consequently from (3.14), (3.16) and (3.17) it follows that

$$\sum_{k=1}^{n} \frac{|\sigma_k(x)|}{\sqrt{1-x_k^2}} \leq c_{22} \frac{\log n}{n}$$

which yields the lemma.

4.   PROOF OF THEOREM 1. Theorem 1 is now a simple consequence of Lemma 1 and Lemma 2. Due to Lemma 1 we have for $-1 \leq x \leq 1$,

$$| R_n(f,x) - f(x) | \leq | Q_n(f,x) - f(x) | + (1 - x^2) P_n^2(x)[\, | f(1) | + | f(-1) | \,]$$

$$+ (1 - x^2) P_n^2(x) | \sum_{k=1}^{n} \frac{[f(1) - f(x_k)]}{(1 - x_k^2)(1 - x_k)^2 [P_n'(x_k)]^2} |$$

$$+ (1 - x^2) P_n^2(x) | \sum_{k=1}^{n} \frac{[f(-1) - f(x_k)]}{(1 - x_k^2)(1 + x_k)^2 [P_n'(x_k)]^2} | \; .$$

Using (1.6), the inequality (2.3), and Lemma 2, we find that

$$| R_n(f,x) - f(x) \leq \frac{c_2}{n} \sum_{k=1}^{n} w_f\left(\frac{\sqrt{1-x^2}}{k}\right) + \frac{\sqrt{1-x^2}[\, | f(1) | + | f(-1) | \,]}{n}$$

$$+ c_{23}\sqrt{1-x^2} \sum_{k=1}^{n} w_f\left(\frac{1}{k^2}\right)$$

and Theorem 1 follows.

5.   PROOF OF THEOREM 2. Let $f(x) = -(1 - x^2)^\sigma$, $0 < \sigma \leq 1/2$, then we get from (1.3) for $-1 \leq x \leq 1$,

$$Q_n(f,x) = \sum_{k=1}^{n} f(x_k) \frac{1 - x^2}{1 - x_k^2}\left[\frac{P_n(x)}{(x - x_k) P_n'(x_k)}\right]^2 .$$

Since f is even and x_k are symmetrically situated around 0, $Q_n(f,x)$ is even. Thus $Q_n'(f,x)$ is odd so that $Q_n'(f,-1) = -Q_n'(f,1)$ and we have

$$R_n(f,x) = Q_n(f,x) + 1/2\,(1 - x^2)\, P_n^2(x)\, Q_n'(f,1) . \tag{5.1}$$

But

$$Q_n'(f,1) = 2 \sum_{k=1}^{n} (1 - x_k^2)^{\sigma - 1}\left[\frac{1}{(1 - x_k) P_n'(x_k)}\right]^2$$

$$\geq 2 \sum_{k=1}^{m} (1 - x_k^2)^{\sigma - 1}\left[\frac{1}{(1 - x_k) P_n'(x_k)}\right]^2, \text{ where } m = [\tfrac{n}{2}],$$

$$\geq 2 \sum_{k=1}^{m} (1 - x_k^2)^{\sigma - 3}[P_n'(x_k)]^{-2} . \tag{5.2}$$

For $k = 1, 2, ..., m, 0 < \Theta_k = \cos^{-1} x_k < \frac{\pi}{2}$. Thus, $1 - x_k^2 = \sin^2\Theta_k \sim \Theta_k^2 \sim \frac{k^2}{n^2}$. Hence on using (2.6) we have from (5.2),

$$Q_n'(f,1) \geq c_{24} n^{2 - 2\sigma} \sum_{k=1}^{m} \frac{1}{k^{3 - 2\alpha}} \geq c_{24} n^{2\sigma - 2} . \tag{5.3}$$

Consequently from (5.1), (5.3) and (2.8) we obtain

$$\limsup_{n \to \infty} | R_n(f,0) - Q_n(f,0) | \geq c_{24} \limsup_{n \to \infty} n^{1 - 2\sigma} .$$

Since $Q_n(f,0) \to f(0) = -1$, $R_n(f,0) \not\to f(0)$ if $0 < \sigma \leq 1/2$.

6.   PROOF OF THEOREM 3. One can easily see that

$$F_n(f,x) - f(x) = F_n(f,x) - F_n(G_{2n+1},x) + G_{2n+1}(x) - f(x), \qquad (6.1)$$

where $G_{2n+1}(x)$ is the polynomial of the best approximation of $f(x)$ and $F_n(f,x)$ is given by (1.12). Thus from (6.1) we obtain

$$|F_n(f,x) - f(x)| \le |F_n(f,x) - F_n(G_{2n+1},x)| + |G_{2n+1}(x) - f(x)| = v_1 + u_2 . \qquad (6.2)$$

Now, from the definition of $G_{2n+1}(x)$ it follows that for $-1 \le x \le 1$,

$$|G_{2n+1}(f,x) - f(x)| \le E_{2n+1}(f), \qquad (6.3)$$

where $E_{2n+1}(f)$ is the best approximation of $f(x)$. So owing to (6.3) we have for $-1 \le x \le 1$,

$$u_2 \le E_{2n+1}(f) . \qquad (6.4)$$

Next, we consider

$$u_1 = |F_n(f,x) - F_n(G_{2n+1},x)|$$

$$\le \sum_{k=0}^{n+1} |f(x_k) - G_{2n+1}(x_k)| \, h_k(x)$$

$$+ \sum_{k=1}^{n} |f'(x_k) - G'_{2n+1}(x_k)| \, |\sigma_k(x)|$$

$$= u_1^* + u_2^* . \qquad (6.5)$$

Again due to (6.3) and (2.1) we have for $-1 \le x \le 1$,

$$u_1^* \le E_{2n+1}(f) \sum_{k=0}^{n+1} h_k(x)$$

$$\le E_{2n+1}(f) . \qquad (6.6)$$

Further, on using a theorem of J. Czipszer and G. Freud [4] and Corollary 1.44 of T.J. Rivlin [10], p. 23, it follows that

$$\sqrt{1 - x_k^2} \, |f'(x_k) - G'_{2n+1}(x_k)| \le 40 E_{2n}(f') . \qquad (6.7)$$

Hence (6.7) and Lemma 3 yield

$$u_2^* \le 40 E_{2n}(f') \sum_{k=1}^{n} \frac{|\sigma_k(x)|}{\sqrt{1 - x_k^2}}$$

$$\le c_{25} \frac{\log n}{n} E_{2n}(f') . \qquad (6.8)$$

Consequently from (6.2), (6.4), (6.5), (6.6) and (6.8) we obtain for $-1 \le x \le 1$,

$$|F_n(f,x) - f(x)| \le 2 E_{2n+1}(f) + c_{25} \frac{\log n}{n} E_{2n}(f') . \qquad (6.9)$$

But due to Rivlin [10], p.23, we have

$$E_{2n+1}(f) \le \frac{6}{2n+1} E_{2n}(f') . \qquad (6.10)$$

Hence , from (6.9) and (6.10) the theorem follows.

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