# ALMOST COMPLEX SURFACES IN THE NEARLY KAEHLER S⁶

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ABSTRACT: It is shown that a compact almost complex surface in $S^6$ is either totally geodesic or the minimum of its Gaussian curvature is less than or equal to 1/3.

KEY WORDS AND PHRASES. Almost complex surfaces, nearly Kaehler structure, totally geodesic submanifold, Gaussian curvature.

1991 AMS SUBJECT CLASSIFICATION CODE. 53C40

## 1. INTRODUCTION.

The six dimensional sphere $S^6$ has almost complex structure $J$ which is nearly Kaehler, that is, it satisfies $(\bar{\nabla}_X J)(X) = 0$, where $\bar{\nabla}$ is the Riemannian connection on $S^6$ corresponding to the usual metric $g$ on $S^6$. Sekigawa [1] has studied almost complex surfaces in $S^6$ and has shown that if they have constant curvature $K$, then either $K = 0$, 1/6 or 1. Under the assumption that the almost complex surface $M$ in $S^6$ is compact, he has shown that if $K > 1/6$, then $K = 1$ and if $1/6 \leq K < 1$, then $K = 1/6$. Dillen et al [2-3] have improved this result by showing if $1/6 \leq K \leq 1$, then either $K = 1/6$ or $K = 1$ and if $0 \leq K \leq 1/6$, then either $K = 0$ or $K = 1/6$. However, using system of differential equations (1) (cf. [5], p. 67) one can construct examples of almost complex surfaces in $S^6$ whose Gaussian curvature takes values outside [9,1/6] or [1/6,1]. The object of the present paper is to prove the following:

THEOREM 1. Let $M$ be a compact almost complex surface in $S^6$ and $K_0$ be the minimum of the Gaussian curvature of $M$. Then either $M$ is totally geodesic or $K_0 \leq 1/3$.

## 2. MAIN RESULTS.

Let $M$ be a 2-dimensional complex submanifold of $S^6$ and $g$ be the induced metric on $M$. The Riemannian connection $\bar{\nabla}$ of $S^6$ induces the Riemannian connection $\nabla$ on $M$ and the connection $\nabla^\perp$ in the normal bundle $\nu$. We have the Gauss and Weingarten formulae

$$\bar{\nabla}_X Y = \nabla_X Y + h(X,Y), \quad \bar{\nabla}_X N = -A_N X + \nabla^\perp_X N, \quad X, Y \in \mathfrak{X}(M), \quad N \in \nu, \tag{2.1}$$

where $h$, $A_N$ are the second fundamental forms satisfying $g(h(X,Y),N) = g(A_N X, Y)$ and $\mathfrak{X}(M)$ is the Lie-algebra of vector fields on $M$. The curvature tensors $\bar{R}$, $R$ and $R^\perp$ of the connections $\bar{\nabla}$,

$\nabla$ and $\nabla^\perp$ respectively satisfy

$$R(X, Y; Z, W) = \bar{R}(X, Y; Z, W) + g(h(Y, Z), h(X, W)) - g(h(X, Z), h(Y, W)) \qquad (2.2)$$

$$\bar{R}(X, Y; N_1, N_2) = R^\perp(X, Y: N_1, N_2) - g([A_{N_1}, A_{N_2}](X), Y) \qquad (2.3)$$

$$[\bar{R}(X, Y)Z]^\perp = (\bar{\nabla}_X h)(Y, Z) - (\bar{\nabla}_Y h)(X, Z), \quad X, Y, Z, W\varepsilon\mathfrak{S}(M), \ N_1, N_2\varepsilon\nu, \qquad (2.4)$$

where $[\bar{R}(X, Y)Z]^\perp$ is the normal component of $\bar{R}(X, Y)Z$, and

$$(\bar{\nabla}_X h)(Y, Z) = \nabla_X^\perp h(Y, Z) - h(Y, \nabla_X Z).$$

The curvature tensor $\bar{R}$ of $S^6$ is given by

$$\bar{R}(X, Y; Z, W) = g(Y, Z)g(X, W) - g(X, Z)g(Y, W). \qquad (2.5)$$

LEMMA 1. Let $M$ be a 2-dimensional complex submanifold of $S^6$. Then $(\bar{\nabla}_X J)(Y) = 0$, $X, Y\varepsilon\mathfrak{S}(M)$.

PROOF. Take a unit vector field $X\varepsilon\mathfrak{S}(M)$. Then $\{X, JX\}$ is orthonormal frame on $M$. Since $S^6$ is nearly Kaehler manifold we have $(\bar{\nabla}_X J)(X) = 0$, and $(\bar{\nabla}_{JX} J)(JX) = 0$. Also

$$(\bar{\nabla}_X J)(JX) = -J(\bar{\nabla}_X J)(X) = 0 \text{ and } (\bar{\nabla}_{JX} J)(X) = -(\bar{\nabla}_X J)(JX) = 0.$$

Now for any $Y, Z\varepsilon\mathfrak{S}(M)$, we have $Y = aX + bJX$ and $Z = cX + dJX$, where $a, b, c$ and $d$ are smooth functions. We have

$$(\bar{\nabla}_Y J)(Z) = a(\bar{\nabla}_X J)(Z) + b(\bar{\nabla}_{JX} J)(Z) = -a(\bar{\nabla}_Z J)(X) - b(\bar{\nabla}_Z J)(JX)$$

$$= -ac(\bar{\nabla}_X J)(X) - ad(\bar{\nabla}_{JX} J)(X) - bc(\bar{\nabla}_X J)(JX) - bd(\bar{\nabla}_{JX} J)(JX) = 0.$$

LEMMA 2. For a 2-dimensional complex submanifold $M$ of $S^6$, the following hold

(i)      $h(X, JY) = h(JX, Y) = Jh(X, Y), \quad \nabla_X JY = J\nabla_X Y,$

(ii)     $JA_N X = A_{JN}X, \ A_N JX = -JA_N X,$

(iii)    $(\bar{\nabla}_X h)(Y, Z) = (\bar{\nabla}_X h)(JY, Z) = (\bar{\nabla}_X h)(Y, JZ),$

(iv)    $R(X, Y)JZ = JR(X, Y)Z, \quad X, Y, Z\varepsilon\mathfrak{S}(M), N\varepsilon\nu.$

PROOF. (i) follows directly from Lemma 1 and equation (2.1). The second part of (ii) follows from (i). For first part of (ii), observe that for $N\varepsilon\nu$ and $X\varepsilon\mathfrak{S}(M)$, $g((\bar{\nabla}_X J)(N), Y) = -g(N, (\bar{\nabla}_X J)(Y)) = 0$ for each $Y\varepsilon\mathfrak{S}(M)$, that is, $(\nabla_X J)(N)$ is normal to $M$. Hence expanding $(\bar{\nabla}_X J)(N)$ using (2.1) and equating the tangential parts we get the first part of (ii).

From equations (2.4) and (2.5), we get

$$(\bar{\nabla}_X h)(Y, Z) = (\bar{\nabla}_Y h)(X, Z) = (\bar{\nabla}_Z h)(X, Y), \quad X, Y, Z\varepsilon\mathfrak{S}(M). \qquad (2.6)$$

Also from (i) we have

$$(\bar{\nabla}_X h)(JY, Z) = (\bar{\nabla}_X h)(Y, JZ), \quad X, Y\varepsilon\mathfrak{S}(M). \qquad (2.7)$$

Thus from (2.6) and (2.7), we get that

$$(\bar{\nabla}_X h)(JY, Z) = (\bar{\nabla}_X h)(Y, JZ) = (\bar{\nabla}_Y h)(X, JZ) = (\bar{\nabla}_Y h)(JX, Z) = (\bar{\nabla}_{JX} h)(Y, Z),$$

this together with (2.7) proves (iii). The proof of (iv) follows from second part of (i).

The second covariant derivative of the second fundamental form is defined as

$$(\bar{\nabla}^2 h)(X, Y, Z, W) = \nabla^\perp {}_X(\bar{\nabla} h)(Y, Z, W) - (\bar{\nabla} h)(\nabla_X Y, Z, W)$$

$$- (\bar{\nabla} h)(Y, \nabla_X Z, W) - (\bar{\nabla} h)(Y, Z, \nabla_X W),$$

where $(\bar{\nabla} h)(X, Y, Z) = (\bar{\nabla}_X h)(Y, Z)$, $X, Y, Z, W \varepsilon \mathfrak{S}(M)$.

Let $\Pi : UM \to M$ and $UM_p$ be the unit tangent bundle of $M$ and its fiber over $p\varepsilon M$ respectively. Define the function $f : UM \to R$ by $f(U) = \| h(U, U) \|^2$.

For $U\varepsilon UM_p$, let $\sigma_U(t)$ be the geodesic in $M$ given by the initial conditions $\sigma_U(0) = p$, $\dot{\sigma}_U(0) = U$. By parallel translating a $V\varepsilon UM_p$ along $\sigma_U(t)$, we obtain a vector field $V_U(t)$. We have the following Lemma (cf. [5]).

LEMMA 3. For the function $f_U(t) = f(V_U(t))$, we have

(i) $\dfrac{d}{dt} f_U(t) = 2g((\bar{\nabla} h)(\dot{\sigma}_U, V_U, V_U), h(V_U, V_U))(t)$,

(ii) $\dfrac{d^2}{dt^2} f_U(0) = 2g((\bar{\nabla}^2)(U, U, V, V), h(V, V)) + 2\| (\bar{\nabla} h)(U, V, V) \|^2$.

3. PROOF OF THE THEOREM 1. Since $UM$ is compact, the function $f$ attains maximum at some $V\varepsilon UM$. From (i) of Lemma 2, $\| h(V, V) \|^2 = \| h(JV, JV) \|^2$ and thus we have $\dfrac{d^2}{dt^2} f_V(0) \leq 0$ and $\dfrac{d^2}{dt^2} f_{JV}(0) \leq 0$. Using (iii) of Lemma 2 in (2.8) we get that

$$(\bar{\nabla}^2 h)(JV, JV, V, V) = (\bar{\nabla}^2 h)(JV, V, JV, V).$$

The above equation together with the Ricci identity gives

$$(\bar{\nabla}^2 h)(JV, JV, V, V) - (\bar{\nabla}^2 h)(JV, V, JV, V).$$

$$= (\bar{\nabla}^2 h)(JV, V, JV, V) - (\bar{\nabla}^2 h)(V, JV, JV, V)$$

$$= R^\perp(JV, V)h(JV, V) - h(R(JV, V)JV, V) - h(JV, R(JV, V)V).$$

Taking inner product with $h(V, V)$ and using (iv) of Lemma 2, we get

$$g((\bar{\nabla}^2 h)(JV, JV, V, V) - (\bar{\nabla}^2 h)(V, JV, JV, V), h(V, V)) \tag{3.1}$$

$$= R^\perp(JV, V; h(JV, V), h(V, V)) - 2g(h(R(JV, V)JV, V), h(V, V)).$$

Now using (i) of Lemma 2, we find that $g(h(U, U), h(U, JU)) = 0$, that is, $g(A_{h(U, U)}U_U), JU) = 0$ for all $U\varepsilon UM_p$. Since $\dim M = 2$, it follows that $A_{h(U, U)}U_U) = \lambda U$. To find $\lambda$, we take inner inner product with $U$ and obtain $\lambda = \| h(U, U) \|^2$. Thus, $A_h U_{(U, U)} = \| h(U, U) \|^2 U$. From equations (2.2) and (2.5) we obtain

$$R(X, Y)Z = g(Y, Z)X - g(X, Z)Y + A_{h(Y, Z)}X - A_{h(X, Z)}Y,$$

which gives

$$R(JV, V)JV = -V + A_{h(V, JV}JV) - A_{h(JV, V}JV) = -V + 2A_{h(V, V}V) = -V + 2\| h(V, V) \|^2 V.$$
$$\tag{3.2}$$

Also from (2.3) and (2.5) we get

$$R^\perp(JV, V, h(JV, V), h(V, V)) = g([A_{h(JV, V)}, A_{h(V, V)}](JV), V)$$

$$= -2g(A_{h(V, V)}V, A_{h(V, V}V))$$

$$= -2\| h(V, V) \|^4.$$

Substituting (3.2) and (3.3) in (3.1) we get

$$g((\bar{\nabla}^2 h)(JV, JV, V, V) - (\bar{\nabla}^2 h)(V, JV, JV, V), h(V, V)) = 2f(V)(1 - 3f(V)). \qquad (3.4)$$

From (iii) of Lemma 2, it follows that

$$(\bar{\nabla} h)(JV, JV, V) = (\bar{\nabla} h)(J^2 V, V, V) = -(\bar{\nabla} h)(V, V, V),$$

this together with $\nabla_X JY = J \nabla_X Y$ of (i) in Lemma 2, gives

$$(\nabla^2 h)(V, JV, JV, V) = -(\bar{\nabla}^2 h)(V, V, V, V).$$

Using this and (ii) of Lemma 3 in (3.4), we obtain

$$\frac{d^2}{dt^2} f_V(0) + \frac{d^2}{dt^2} f_{JV}(0) = 2f(V)(1 - 3f(V)) + 2 \| (\bar{\nabla} h)(V, V, V) \|^2 + 2 \| (\bar{\nabla} h)(JV, V, V) \|^2 \leq 0$$

Thus either $f(V) = 0$, that is, $M$ is totally geodesic or $1/3 \leq f(V)$. Since an orthonormal frame of $M$ is of the form $(U, JU)$, the Gaussian curvature $K$ of $M$ is given by

$$K = 1 + g(h(U, U), h(JU, JU)) - g(h(U, JU), h(U, JU)) = 1 - 2 \| h(U, U) \|^2.$$

Thus $K: UM \rightarrow R$, is a smooth function, and $UM$ being compact, $K$ attains its minimum $K_0 = \min K$ and we have $K_0 = 1 - 2 \max \| h(U, U) \|^2$, from which for the case $1/3 \leq f(V)$, we get $K_0 \leq 1/3$. This completes the proof of the Theorem.

As a direct consequence of our Theorem we have

COROLLARY. Let $M$ be a compact almost complex surface in $S^6$. If the Gaussian curvature $K$ of $M$ satisfies $K > 1/3$, then $M$ is totally geodesic.

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