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A CHARACTERIZATION OF THE HALL PLANES BY PLANAR AND NONPLANAR INVOLUTIONS

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ABSTRACT. In this article, the Hall planes of even order q^2 are characterized as translation planes of even order q^2 admitting a Baer group of order q and at least q+1 nontrivial elations.

KEY WORDS AND PHRASES. Translation plane, Baer groups, elations.
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1. INTRODUCTION AND BACKGROUND.

Let Σ denote an affine Desarguesian plane of order q^2 coordinatized by a field F isomorphic to $GF(q^2)$. Let $\mathcal{N}$ denote the net defined on the points of Σ whose lines have slopes in $GF(q) \cup (\infty)$. Let σ denote the involution defined by $(x,y) \longrightarrow (x^q,y^q)$ where $x,y \in F$. Let $\hat{F}^*$ denote the kernel homology group of Σ defined by $(x,y) \longrightarrow (ax,ay)$ where $|a| = q+1$, $a \in F$.

Now derive $\mathcal{N}$ to obtain the Hall plane $\bar{\Sigma}$ of order q^2. Then the involutions in $\langle\sigma\rangle\hat{F}^*$ are central collineations in $\bar{\Sigma}$.

If $\mathcal{E}$ denotes an elation group fixing $\mathcal{a} = (0,0)$ with axis $\mathcal{L}$ in $\mathcal{N}$ which acts regularly on the remaining lines of $\mathcal{N}$ incident with $\mathcal{a}$ then $\mathcal{E}$ becomes a collineation group of $\bar{\Sigma}$ of order q which fixes a Baer subplane pointwise.

In [3] and [4], Foulser and Johnson classify the translation planes of order q^2 that admit SL(2,q). In particular, if $q^2 > 16$, the Hall planes are precisely the translation planes admitting SL(2,q) where the Sylow p–subgroups for $q = p^r$ fix Baer subplanes pointwise.

So, the Hall planes of order q^2 admit a Baer group of order q and at least 1+q involutary central collineations.

In this article, we consider translation planes of order q^2 that admit a Baer group of order q and $\geq 1+q$ involutory central collineations. For q odd, it turns out that there are other (i.e. non Hall) translation planes possessing this configuration of groups. For example, the translation planes π corresponding to the Fisher flock of a quadratic cone in PG(3,q) for

$q \equiv 3 \mod 4$ derive planes $\bar{\pi}$ admitting such groups (see [5]).

However, for q even, we are able to characterize the Hall planes using these planar and non planar involutions.

Our main result is

THEOREM A. Let π be a translation plane of even order q^2 which admits a Baer collineation group $\mathcal{B}$ of order q and at least 1+q nontrivial elations (all groups are assumed to be in the translation complement). Then π is the Hall plane of order q^2 and conversely, the Hall plane admits such groups.

The proof of theorem A will be given as a series of lemmas. As a preliminary to the proof, we remind the reader of some results required in the arguments.

RESULT I (JHA, JOHNSON [7] (4.1)). Let π be a translation plane of even order $q^2 \neq 64$. Assume π admits a Baer group of order q and a dihedral group of order 2(1+q) which is generated by elations with affine axes. Then π is derivable where the elation axes define a derivable partial spread.

RESULT II (FOULSER [2] THEOREM 2 AND COROLLARY 3 (2)). Let π be a translation plane of order q^2 that admits a Baer group $\mathcal{B}$ of order q. (1) Then the Baer subplane π_0 = Fix $\mathcal{B}$ pointwise fixed by $\mathcal{B}$ is Desarguesian. (2) Furthermore, if the collineation group $\mathcal{G}_{[\pi_0]}$ fixing π_0 pointwise has order > q then the net $\mathcal{N}$ defined by the lines of π_0 is a derivable net. (3) In the general case, $\mathcal{G}_{[\pi_0]}$ is a subgroup of AG(1,q), the 1–dimensional affine group over GF(q).

RESULT III (JHA, JOHNSON [7]). Let π be a translation plane of even order q^2 that admits a Baer 2–group of order $\geq 2\sqrt{q}$. Then an elation group with fixed affine axis has order ≤ 2.

RESULT IV (A MODIFIED VERSION OF THE MAIN RESULTS OF HERING [6], OSTROM [10]). Let π be a translation plane of even order. Let $\mathcal{G}$ denote the collineation group generated by all elations in the translation complement. If $\mathcal{G}$ is solvable then either $\mathcal{G}$ is an elementary abelian 2–group or has order $2 \cdot t$ where t is odd.

RESULT V (JHA–JOHNSON [8]). Let π be a translation plane of even order q^2 which admits collineation groups $\mathcal{B}_1$, $\mathcal{B}_2$ of orders $\geq 2\sqrt{q}$ such that $\mathcal{B}_i$ fixes a Baer subplane π_i i = 1,2 pointwise. If $\pi_1 \neq \pi_2$ then π is Hall or a known plane of order 16.

2. THE CHARACTERIZATION.

Assume for this section, the assumptions of Theorem A and assume π is not Hall.

(2.1) LEMMA. Result I is valid for $q^2 = 64$.

PROOF. π is a translation plane of order 64 that admits a Baer group $\mathscr{B}$ of order 8 and $\geq 1+8$ affine elations. If π is not Hall then $\mathscr{D}$ still becomes dihedral of order $2 \cdot 9$ and centralizes $\mathscr{B}$. Let $\mathscr{C}$ denote the cyclic stem of $\mathscr{D}$. Let $\mathscr{C} = \langle g \rangle$. There are $8 \cdot 7$ components of π not in $\mathscr{N}$ so that g must fix at least two of these components $\mathscr{L}_1$, $\mathscr{L}_2$. Now g leaves invariant $\pi_0 = \text{Fix}\,\mathscr{B}$, $\mathscr{L}_1$ and $\mathscr{L}_2$. Thus, g fixes ≥ 3 mutually disjoint 2m–spaces (if $q = 2^m$) over GF(2). Now the argument given by Jha–Johnson [7] for result I will be valid for $q^2 = 64$. This proves (2.1).

Now assume the order of the plane is 16. The translation planes of order 16 are either semifield planes or derived from semifield planes (see Johnson [9] and Dempwolff and Riefart [1]). In any case, the non Hall planes admitting Baer groups of order 4 do not admit ≥ 5 elations.

So, we may assume $q \neq 4$.

(2.2) LEMMA. Let $\mathscr{D}$ denote the collineation group generated by the affine elations. Then $\mathscr{D}$ is dihedral of order $2(q+1)$, acts faithfully on π_0 and centralizes $\mathscr{B}$.

PROOF. By result IV, no two of the elations can have a common axis. Hence, it follows that $\mathscr{D}$ is solvable by result IV, $|\mathscr{D}| = 2 \cdot t$ where t is odd.

By result II, $\mathscr{D}$ must normalize $\mathscr{B}$. Clearly, the elations must have axes nontrivially intersecting π_0 and leaving π_0 invariant. Since a central collineation is uniquely determined by its axis (co axis) and one specified nontrivial image point, it follows that $\mathscr{D}$ centralizes $\mathscr{B}$. Hence, if $y \in \mathscr{D} \cap \mathscr{B} - \langle 1 \rangle$ then the Sylow 2–subgroups of $\mathscr{D}$ would have order ≥ 4. So $\mathscr{D} \cap \mathscr{B} = \langle 1 \rangle$.

If $1 \neq h \in \mathscr{D}$ fixes π_0 pointwise then the collineation fixing π_0 pointwise has order $> q$ so that by result II(2), the net $\mathscr{N}$ (see notation in II(2)) is derivable.

Let π_1 be a Baer subplane of $\mathscr{N}$ incident with the zero vector $\mathbf{a}$. The infinite points of π_1 are exactly those of π_0. If σ is any elation in $\mathscr{D}$ then the axis of σ is in π_1 and σ permutes the infinite points of π_1. Hence, σ leaves π_1 invariant and since $\mathscr{D}$ is generated by elations, it follows that $\mathscr{D}$ must fix each of the q+1 Baer subplanes of $\mathscr{N}$ incident with $\mathbf{a}$. However, this means that h cannot fix π_0 pointwise.

Thus, $\mathscr{D}$ acts faithfully on π_0. Now π_0 is Desarguesian by result II(1) and since $\mathscr{D}$ is generated by elations of π_0, $\mathscr{D} \leq SL(2,q) \cong PSL(2,q)$ and $|\mathscr{D}| = 2 \cdot t$ where t is odd. Thus, $\mathscr{D}$ is dihedral and admits $\geq 1+q$ involutions. This proves (2.1).

(2.3) LEMMA. π is derivable with derivable net $\mathscr{N}$ (in the above notation).

PROOF. (2.2 and result I).

(2.4) LEMMA. Let σ be any elation in $\mathscr{D}$. Then for any $\tau \in \mathscr{B} - \langle 1 \rangle$, $\tau\sigma$ is a Baer involution. Furthermore, if $\rho \in \mathscr{B} - \langle 1 \rangle$, $\rho \neq \tau$ then the set of components of π not in $\mathscr{N}$ fixed by $\rho\sigma$ is disjoint from the set of components not in $\mathscr{N}$ fixed by $\tau\sigma$.

PROOF. If $\tau\sigma$ is an elation then $(\tau\sigma)\sigma \in \mathscr{D}$. But $\mathscr{D} \cap \mathscr{B} = \langle 1 \rangle$. Hence, $\tau\sigma$ is a Baer involution.

Let $\mathscr{L}$ be a component fixed by both $\tau\sigma$ and $\rho\sigma$. Then $(\tau\sigma)(\rho\sigma)$ also fixes $\mathscr{L}$ and

$(\tau\sigma)\rho\sigma = \tau\rho\sigma^2 = \tau\rho \in \mathscr{B}$ fixes $\mathscr{L}$. Thus $\mathscr{L}$ is a component of $\mathscr{N}$.

(2.5) LEMMA. Let σ be any elation in $\mathscr{D}$. Then each component of π not in $\mathscr{N}$ is fixed by exactly one Baer involution in $\sigma(\mathscr{B} - \langle 1 \rangle)$.

PROOF. By (2.4), there are q(q–1) distinct components fixed by some involution in $\sigma(\mathscr{B}-\langle 1 \rangle)$. Since there are exactly q(q–1) components not in $\mathscr{N}$, (2.5) is proved.

(2.6) LEMMA. Let $\mathscr{D} = \langle \sigma,\chi \rangle$ where σ,χ are distinct elations. Each component $\mathscr{L}$ of π not in $\mathscr{N}$ is fixed by $\sigma\chi$.

PROOF. By (2.5), there exists a Baer involution $\rho\sigma \in (\mathscr{B} - \langle 1 \rangle)\sigma$ which fixes $\mathscr{L}$ and similarly, there is a Baer involution $\tau\chi$ in $(\mathscr{B} - \langle 1 \rangle)\chi$ which fixes $\mathscr{L}$.

Thus $(\rho\sigma)(\tau\chi)$ also fixes $\mathscr{L}$. However, $(\rho\sigma)(\tau\chi) = (\rho\tau)(\sigma\chi)$ by (2.1). Further, $((\rho\tau)(\sigma\chi))^2 = (\rho\tau)^2(\sigma\chi)^2$ again by (2.1)) $= (\sigma\chi)^2$. Since $|\langle \sigma\chi \rangle| = q+1$ and q+1 is odd, then $\langle \sigma\chi \rangle = \langle (\sigma\chi)^2 \rangle$. Thus, $(\sigma\chi)^{2j}$ and thus $\sigma\chi$ fixes $\mathscr{L}$.

(2.7) LEMMA. Let $\bar{\pi}$ denote the translation plane obtained from π by deriving $\mathscr{N}$. Then $\mathscr{D}\mathscr{B}$ is a collineation group of $\bar{\pi}$.

PROOF. $\mathscr{D}\mathscr{B}$ leaves $\mathscr{N}$ invariant.

(2.8) LEMMA. Let $\mathscr{C}$ denote the cyclic stem of $\mathscr{D}$ of order q+1. Then $\mathscr{C}$ is a kernel homology group of $\bar{\pi}$.

PROOF. It was noted in the proof to (2.1) that $\mathscr{D}$ must fix each Baer subplane incident with a in $\mathscr{N}$. Hence, the stem $\mathscr{C}$ of $\mathscr{D}$ must fix each such Baer subplane. The components of $\bar{\pi}$ are the components of π not on $\mathscr{N}$ and the Baer subplanes of $\mathscr{N}$ which are incident with a. By (2.6), if $\mathscr{D} = \langle \sigma,\chi \rangle$ then $\mathscr{C} = \langle \sigma\chi \rangle$ so that $\mathscr{C}$ fixes each component of π not in $\mathscr{N}$. Thus, $\mathscr{C}$ must induce a kernel homology group in $\bar{\pi}$.

Let the kernel of $\bar{\pi}$ be isomorphic to $GF(2^r) \leq GF(q^2)$. Let $q = 2^m$ so that $r|2m$. then $1+q \mid 2^r-1$ by (2.7). Thus, $r > m$ so that $r = 2m$.

Thus, the kernel of $\bar{\pi}$ is isomorphic to $GF(q^2)$ so that $\bar{\pi}$ is Desarguesian. Thus, π must be Hall and we obtain the proof to theorem A.

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